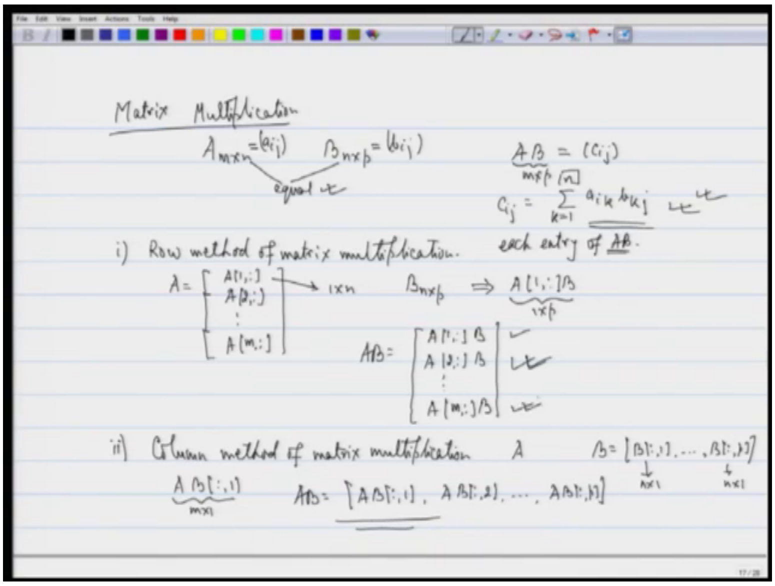
Linear Algebra

Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture - 04

Alright. So, let us recall what we did in the last class. We talked about matrix multiplication.

(Refer Slide Time: 00:27)



So, the idea of matrix multiplication was to look at things from a different prospective. So, what we have learnt in our school was a matrix multiplication, which was based on entry wise. So, the idea was that given a matrix A m cross n, B is say n cross p, so that you can multiply; here this n and this them are same equal here.

And therefore, you could define A B and write it has a matrix of the size m cross p, fine. Then we had this entries of this matrix has a i j, entry of this matrix has b i j, and once we have that, we wrote this has the entry of this matrix as c i j. And then c i j was nothing, but summation over k is equal to 1 to n; this n was the one that was here alright a i k b k j.

So, we have to remember this formula also this is very very important; but at the same time there are other multiplications also which you wanted to understand. So, let us look at other things once more. So, this was one matrix product that we learnt in our school, what is called entry wise; we are looking at each entry of A B here. So, we are looking at each entry of A B. Now, would like to understand the other one which was row method of multiplication; row method of matrix multiplication.

So, recall what the idea was; the idea was that, you have the matrix A which is. So, look at the rows of the matrix; the first row is, this is the first row of the matrix, this is the second row of the matrix and you have the mth row of the matrix, they are m rows of this matrix, fine.

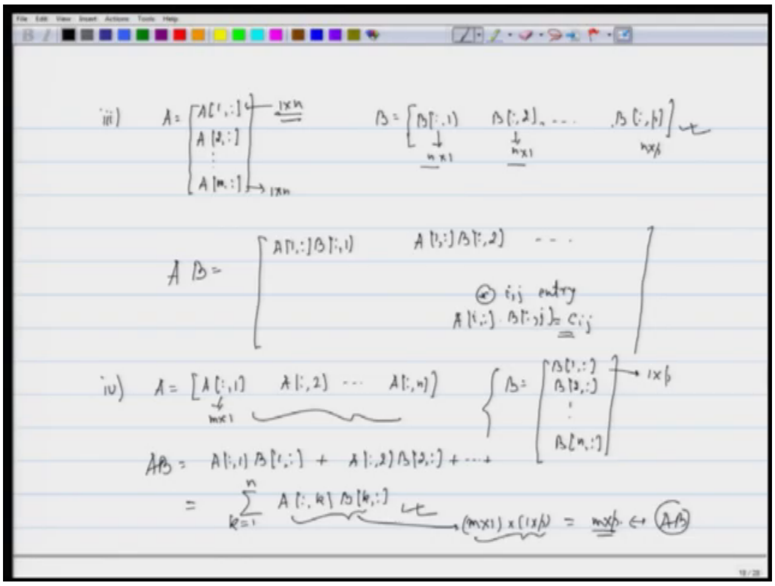
Each of these rows have size 1 cross n, B is of size n cross p; and this will imply that, I can talk of A of the first row times B and this is a 1 cross p matrix, fine. And therefore, when I want to talk of A B, I could write it has A of first row times B, A of second row times B and so on the mth row times B. This was one way of writing it.

The second method was looking at the column method of matrix multiplication. So, there the idea was that, we leave A as it is and write B as the first column of B, second column of B till the pth column of B, you can write like this. If I look at this B, this is of size n cross 1, each of them is n cross 1; and therefore we can talk of A times the first column of B and this is nothing, but A is m cross n, this is n cross 1.

So, this is matrix of size m cross 1. So, I could write A B as A times the first column of B, A times the second column of B and so on till A times the last column or the pth column of B.

So, A B was equal to this, A B was equal to this and this was the entry wise. So, what you have done is, there is the entry wise product; then computing the matrix product using the rows. So, you are looking at how do I get the first row of A B, the second row of A B and the mth row of A B; and this was getting the first column of A B second column of A B and the last column of A B.

(Refer Slide Time: 05:04)



Then there was another product that we learnt, what was the matrix product in the general setup, where what we had was, we wrote A as. So, here I write A as again first row, second row and the last row. So, you have to be careful; what is the size of this matrix? The size of this matrix is 1 cross m. We can look at B as first column, second column and the pth column. So, this is of size m cross 1, m cross 1 sorry n cross 1 not m; so this was n here, alright.

So, this was n 1 cross n, n cross 1, n cross 1 till n cross p and this was 1 cross n. So, if I see this I can multiply this with this, this with this and so on. So, the matrix product that we write here is A of this into B of this, A of this into B of this and so on; in general if I look at this entry which is the i j entry, then this corresponds to ith row times the jth column of B and this is what our entry c i j was, this was one way of writing it.

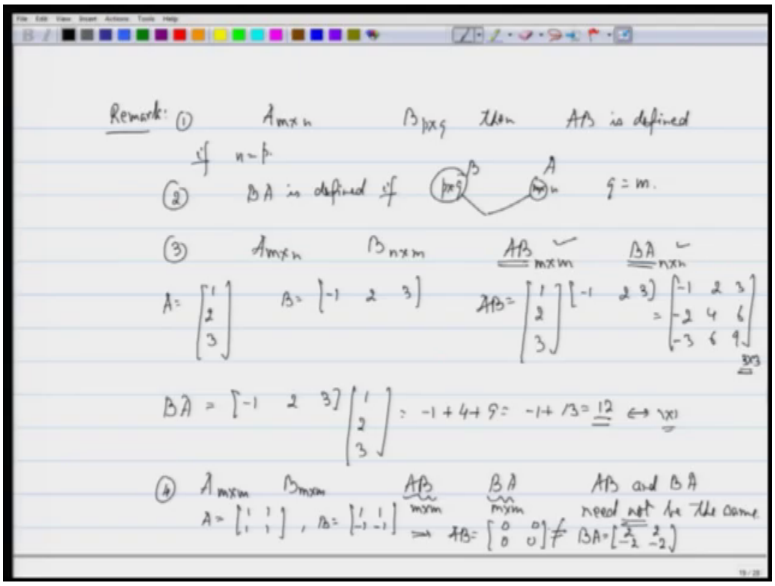
So, here we broke A in terms of the rows alright; we wrote it in terms of rows and B in terms of the columns, and we could get c i j, the first matrix product that will learnt in our school. Now, the last one that I wanted you to understand was, write A in terms of the separate one.

So, write A in terms of the columns, so A is the first column is; this is the first column, the second column and this is the last column of A. Write B in terms of the rows; so B is, this is the first row of B, the second row of B and the nth row of B, each of them is nothing, but 1 cross p alright and here each of them is m cross 1.

So, I can multiply A B. So, have multiply A B, what I get is; this times this. So, it is nothing, but the first one times the first one of B plus the second column of A with the second row of B and so on till the last one, which you wrote it as summation over k equal to 1 to n, the kth column of A times the kth row of B.

So, this is a matrix product; if you look at each of them, each of them is of size m cross 1 times 1 cross p and therefore, what I get is m cross p. So, each of them is a matrix of the proper size, of the size of A B. So, the size of A B and the size of this, you can see that they are same. So, these are the matrix product that we learnt, alright.

(Refer Slide Time: 09:11)



Now, let us look at some of the properties; I would like you to keep track of all of them, fine. So, remark. So, first thing is that, if A is m cross n and B is some p cross q; then A B is defined find if n is same as p, 2 B A is defined if. So, B is size p cross q, this is the size of B; the size of A is m cross n. So, what we need at this should be the same.

So, we need that q should be equal to m, fine. So, we have examples here where we have seen that. So, what we are saying is that it may happen that A B is defined, but B A may not be defined; B A is defined, A B is not defined. Even if both A and B are defined. So, if I take A as m cross n, B as n cross m; then A B is defined and the size of A B is m cross m, B A is also defined and the size of B A is n cross n.

So, even if they are defined, they are of different sizes. So, let us take an examples, I define A as say 1, 2, 3 and B as minus 1, 2, 3; then if I look at A B, the size of A B is 1, 2, 3 times minus 1, 2, 3, which is nothing, but minus 1, 2, 3 minus 2, 4, 6 minus 3, 6, 9. So, it is a 3 cross 3 matrix; but if I look at B A, B A is minus 1, 2, 3 times 1, 2, 3 and this is nothing, but minus 1 plus 4 plus 9 which is minus 1 plus 13 which is 12.

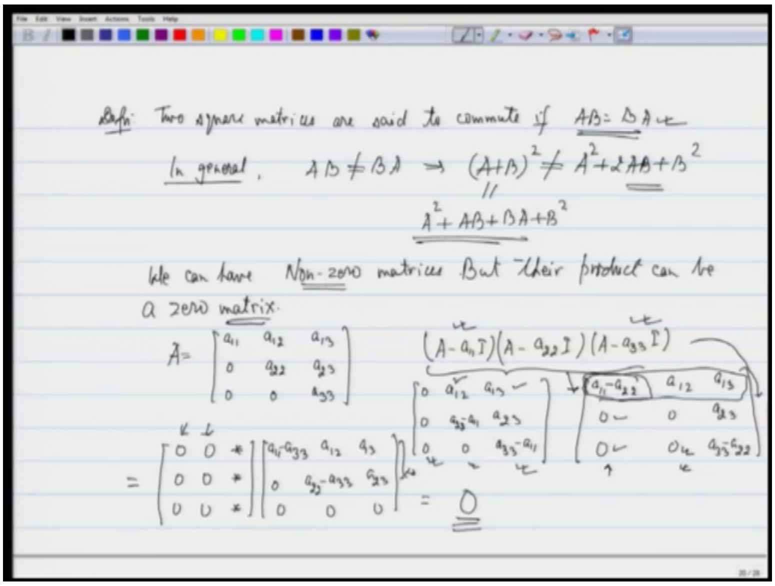
So, here it is a 1 cross 1 matrix and there it was a 3 cross 3 matrix, the sizes could be different, fine. It may happen that A B and B A both are the square matrices. So, if we A B and B A are both square matrices; then A B is defined and B A is also defined, both are of the size m cross m, but still it can happen that they are different.

So, A B and B A need not be the same. So, for example, let us look at an example here. So, example I will take A as 1 1 1 and B as 1, minus 1 1, minus 1; then this will imply that if I look at A B. So, let us look at the matrix product. So, I am looking at the I want to get the first row of A B, the first row of A B will be 1 times the first row of B times 1 times the second row of B and therefore it will give me 0, 0.

Similarly here also it will give me 0, 0; but where as if I look at B A B A is now this times this, should will give me 2 2, minus 2, minus 2. So, the sizes are same, everything is same, but the matrix product A B and B A they are not the same; one is the 0 matrix, the other has some rank that will come to afterwards. We have some non-zero entries, but at the same times we will see at a later stage that, when we come to Eigen values, Eigen vectors, that they have similar structure.

Same thing happens in the case of 3, where the sizes of A B and B A where different; but a still will see that lot of things are nice.

(Refer Slide Time: 13:31)



The next thing would like to look at. So, some more examples we will look at matrix product that will come to afterwards. So, some more definitions now definition; two A square matrix matrices are said to commute, if A times B is same times is same as B times A, fine. We need this basically to look at algebra from our point of view.

So, in general what happens is we have already seen that, in general A B is not equal to B A and this will imply that I cannot talk of A plus B square. So, this will not be equal to A square plus 2 A B plus B A square; even though this is equal to A square plus A B plus B A plus B square.

So, this part is fine, but this is not allowed. So, what we had learnt in school algebra was A plus B whole square is A square plus 2 A B plus B square; that is not true in general now, alright. So, we have to be careful when you do matrix product and not only that; we also saw in the previous slide that, I have got here A and B, they were non-zero matrices, but still the product A B could become zero, alright.

So, let me write that also has a point to see important that, we can have non-zero matrices. So, each into there is non-zero, we can have non zero matrices; but their product can be a zero matrix. We had looked at a similar example earlier also, when I was trying to look at matrix product. So, there for examples, so let me take an recall that example also which is very important example when you come to Eigen values, Eigen vectors.

So, I have a matrix A which is upper triangular, I can take it as lower triangular also; suppose it is entries are a 1 1, a 1 2, a 1 3; 0, a 2 2, a 2 3; 0, 0 a 3 3, so it is upper triangular matrix. I want to look at the product A minus a 1 1 times identity times A minus a 2 2 times identity and A times A minus a 3 3 times; I want to multiply this 3 matrices and see the product what this product is.

So, if you look at the first entry here, this part; this matrix is 0 here, a 1 2, a 1 3; 0, it will be a 2 2 minus a 1 1, a 2 3; 0, 0, a 3 3 minus a 1 1. This matrix is nothing, but a 1 1 minus a 2 2; then you have a 1 2, a 1 3; 0, 0, a 2 3; 0, 0, a 3 3, minus a 2 2 and I have this part that I am leaving as it is.

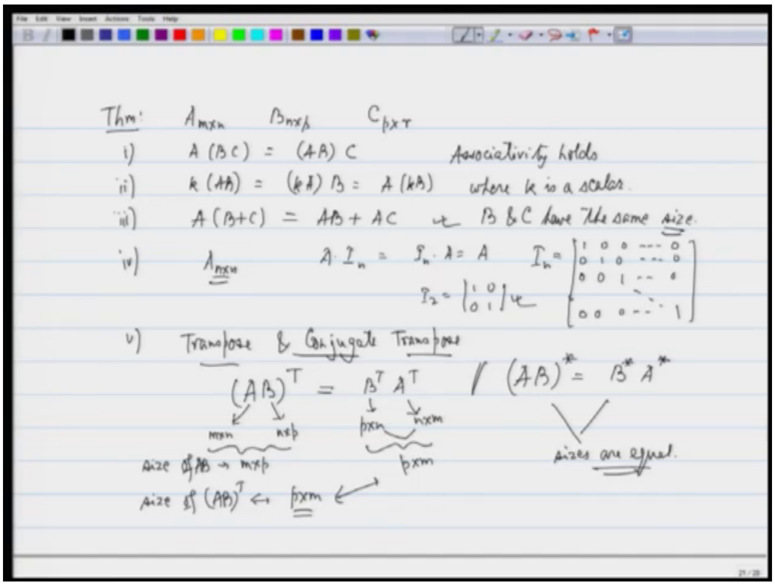
See if I look at this matrix product again, what we see is recall that; that the first column here is 0, so what happens in the first row is immaterial to us, because I am going to multiply. See if I look at this column times this; it is this entry is getting multiplied to the first column. So, that is 0, this entry is 0; so the this part will give me 0, this is 0 with this multiplication give me 0. So, the first column of this matrix will be 0.

Again here if I look at the second column of this product, it will be a 1 2 times 0 which is 0, 0 times; a 1 2 that is the second column and again 0 times the third column here which will again be 0. And here I will have some entries I do not know what they are; they will have some entries, fine.

So, this is the product of the first two blocks, first two matrices. Now, let us multiply with the third one; the third one is a minus a 3 3. So, it will be a 1 1 minus a 3 3, a 1 2, a 1 3; 0, a 2 2 minus a 3 3, a 2 3; 0, 0, 0. So, now, what you see is that, the two columns are 0 and therefore the first two rows of this part is not going to play any role as such, and therefore this whole product will be the 0 matrix, fine.

So, this is what you have to understand that, if you can see somewhere that there are zeros coming; you can play with these zeros with the understanding of row product and the column product, fine. Let us go to the next part, some theorems related to this and then I will again come back to some examples to make you have a better understanding of things.

(Refer Slide Time: 19:04)



So, theorem; so, I have got matrix A which is m cross n, B is m cross p, C is p cross r they are all other integers are coming from real numbers itself, we can take complex numbers rational whatever it is, fine. Then we have this results one associativity holds, so A times B C is same time as A B times C. So, what we are saying is associativity hold. So, we are saying that matrix product is associative.

Two we also have a scalar multiplication; so how does a scalar multiplication behave? So, k times A B is same as k A times B is also equal to A times k B, where k is a scalar. Then it also behaves nicely with addition; so A times B plus C that is distributive law hold is same as A B plus A C. So, the important thing here is, in this case, if you look at this case B and C are not meaningful unless C is of the same size as B.

So, here will assume that, B and C have the same size; otherwise the matrix product was not meaningful, in the previous once it was meaningful, fine. So, accordingly what will happen is that, there will be times when I will not say the size of the matrix, alright. But when I write the matrix product; you will have to assume that the sizes makes sense, so that matrix product makes sense, alright.

And also you have to be careful in the sense that, if you can find out that the sizes of the matrix match to define a matrix product; then you will surely get some nice result of the other, alright. So, just be careful about matrix product; wherever we can get the size are similar, you can define matrix product, you are bound to get something nice, alright.

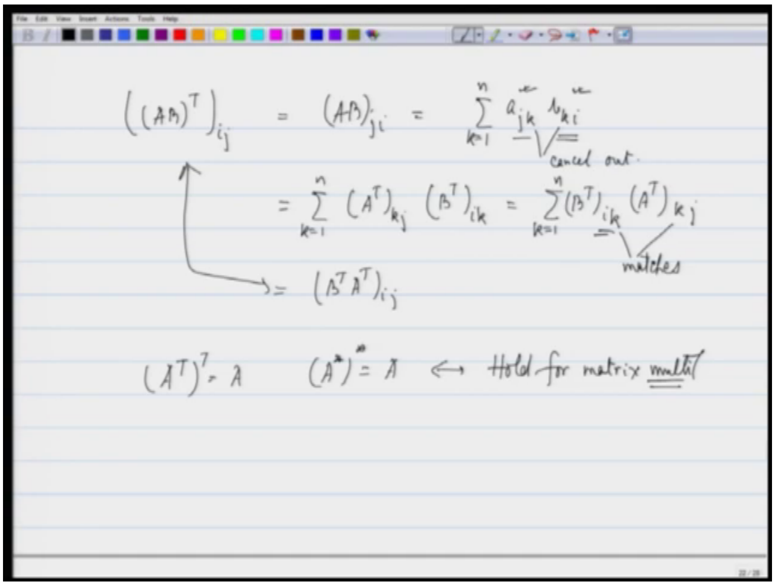
The fourth thing that would like you to understand is, now I want to take A as a square matrix m cross n; then for every A whatever A you take you have matrix identity matrix such that A times I is same as I times A is A. So, recall what was I, I was a matrix in which the entries where; so 1, 0, 0 so on 0, 1, 0 so on 0, 0, 1 and this 1 built up here and we had here 0, 0, 0 and so on, fine.

So, I 3, so I 2 was 1, 0, 0, 1 and so on, fine. So, keep track of that, you will need it. Now, so we have been able to relate the matrix product with respect to a scalar multiplication, with respect to addition; we are also learnt transpose and conjugate transpose.

So, let us relate this also. So, what would like to say is that, if I want to talk of A B transpose fine; then we will turn out to be B transpose A transpose, similarly conjugate transpose of A B will be equal to B star A star, alright. So, look at here A is of size m cross n, B was of size n cross p; together the product was m cross p, fine. Look at B transpose, alright. So, this was the size of A B, size of A B was this. So, therefore, the size of A B transpose is p cross m, fine.

Let us look at the size of B transpose; size of B transpose is p cross n, size of A transpose is n cross m. So, this n and m cancels out, you do get m cross p cross m. So, you can see that the size of the two sides are equal; similarly here also the sizes makes sense. So, we just have to check whether the entries are same or not. So, let us pull that part.

(Refer Slide Time: 23:58)



So, let us look at, suppose I have the matrix A B transpose; I want to look the at the i j entry of this. So, by definition i j entry of this will be equal to j i entry of A B; that was the way it was defined, the transpose was rows becomes columns, columns becomes rows. So, this is how what it was. I can write it has summation over k equal to 1 to n or some size whatever it is, alright; a of j k into b of k i; so this is important. So, this k and k cancels out remember it please, cancel out fine; but what is a j k and b k i?

So, I can write this as summation k equals to 1 to n a j k is nothing, but if I look at A transpose it is the k j entry. And what is b k i? Look at B transpose, it is the i k entry of that, alright. I can write this back as k equal to 1 to n B transpose i k into A transpose k j. So, now here k and k matches and therefore I can talk of this as B transpose A transpose i j. So, we can see that they are equal as such, alright. So, I would like you to try the same thing for the star also, prove it yourself, alright

So, once you have done that, you can also recall that; what we had was that A transpose was A and A star was A. So, it will hold for matrix multiplication also, for multiplication as well, alright. Now, some examples: so one of the examples that, I wanted to do, alright; so, I will just write that and maybe we discuss in the next class for a better understanding; how to use it. So, or I will just end it here itself I think and look at the examples in the next class, alright.

Thank you.