Linear Algebra

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Lecture – 39
Similarity of Matrices

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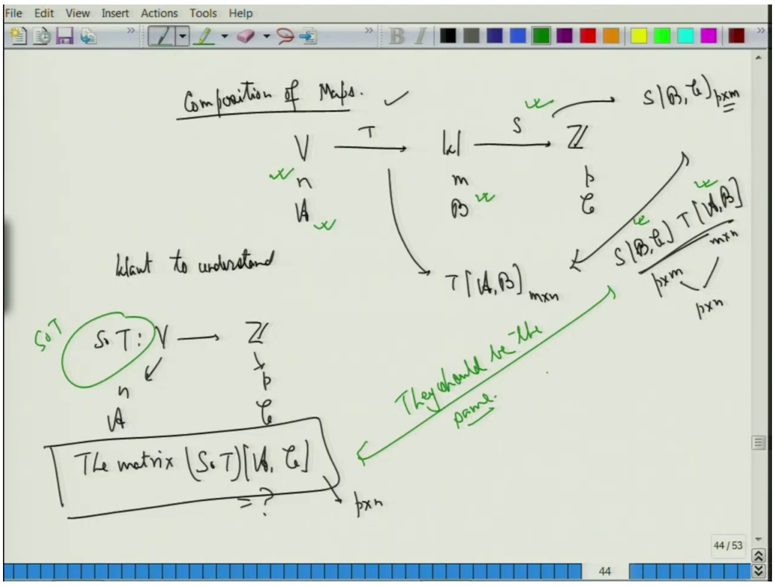

Alright, so now we are going to understand what it called similarity transformation. So, let us look at what are called composition of maps. So, I have 2 maps S and T. So, I have vector space V W and Z, I have got 3 vector space here the dimension is m or I was writing n I think n here m here and p here in different ways whatever it is.

I have a map here which is T, I have a map here which is S, I want to understand these we also have a basis here A, ordered basis B, ordered basis C. So, we want to understand the map SoT which is from V to Z alright. V has n and A here, this has p and C here; I want to look at the matrix SoT this A and C I want to understand what is this is that ok, this what I want to do.

What I know see this is given to me, so I know what is the matrix of T, T of AB is given to me, what is the size of T of AB? Recall since I am going to multiply by n here, so it will be an m cross n matrix fine so that it make sense.

Here it is going to be S from B to C so it will be p cross m matrix fine is that ok. So, you can see that if I want to talk of matrix multiplication here I need to do p cross m m cross n m has to cancel. So, it has to be S of B C times T of A B alright. This is the way it has to be because then only I can look at here p cross m, m cross n this cancels out I get p cross n and this is what it should be because this is p I have to multiply by n; so, this has to be a matrix of p cross n.

So, at least I can hope of thinking that they should be the same, forget about matrix multiplication at least the size wise I can see that it makes sense for me that I can go from these two different matrices to this. And observe here that S here even though it is on the right when I want to multiply it as a function it comes on the left it is SoT that is the function that comes here.

It is SoT. First apply T and then S; first apply T and then S is that ok. So therefore, you get S in the beginning and T at the later stage is that ok. So, SoT and SoT is here also together with you fine. So, how do I get it, let us try that out. So, what you have to recall here to understand everything is everything here is with respect to n or A, here it is with respect to B S alright. So, let us try to understand things in a nice way fine. So, just let me try it out here itself.

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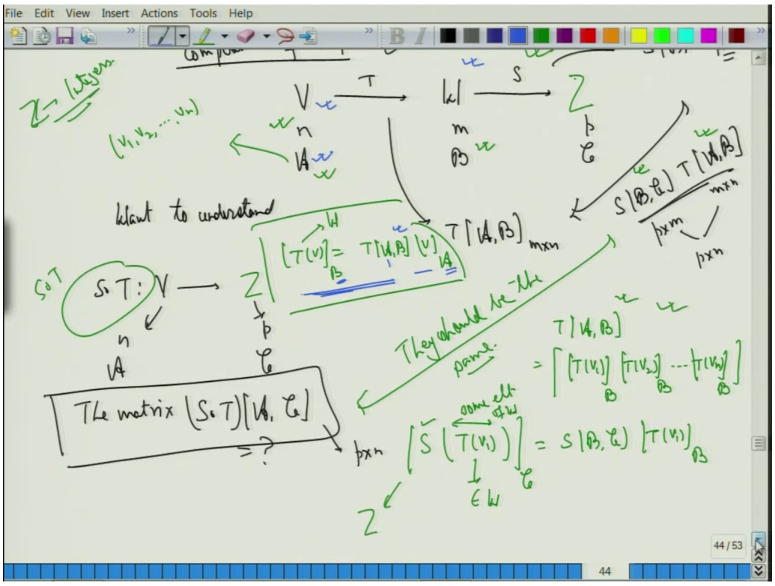

So, I have T of A B; if I want to look at T of A B, T of A B will look like T of V 1, T of V 2, T of V n. So, this has basis that is V 1, V 2, V n; I have this basis fine, each of them have to be evaluated with respect to B. Because B is the ordered basis of the co domain, I have to write like this fine.

Now, what is T V 1? I want to understand what is T V 1; T V 1 is an element of w everything is nice with us no problem. But I can also apply S to it, now if I apply S to this fine because T V 1 is an element of W this belongs to W. So, I can apply this fine, if I apply this then this is an element of Z now. So, this becomes an element of Z alright.

So, I wrote Z as a matrix. So, let me I think you will get confused. So, V W and say because Z is something else that we have been looking at Z just the Z alright, not double Z alright because double Z was for integers alright. So, let me just write Z here fine. So, here also let it be just Z here fine.

See if I am writing this here it means that I am supposed to look at see S here and this is some vector I do not know what that vector, but it is some vector; some element of W fine. So, what this part tells me this matrix part tells me that this is nothing but the matrix of S. So, this is basically equal to with respect to see if I want to look at it is S from go from B to C alright and then whatever this is T of V 1 you are supposed to do with respect to B is that ok.

So, recall here so here also if you want to see recall that T of V with respect to B is an element of this is an element of W. So, this is equal to T of A B times V, but V is with respect to the basis of that domain fine.

So, this is B we have to be careful here, but what you are writing should make sense for you alright. So, T of V is an element of W, since it is an element of W it has to be its coordinates have to be computed with respect to B. So, you write B here; V is an element of V. So, it has to be computed with respect to A so you compute it and now this A and this A needs to be canceled out to be left out with B alright fine.

Similar here also if you see here this B and this B has to be canceled out, you are evaluating at A and getting final answer at C is that ok. So, this is the way you have to understand. So, let us try to write it nicely with whatever understanding I have here.

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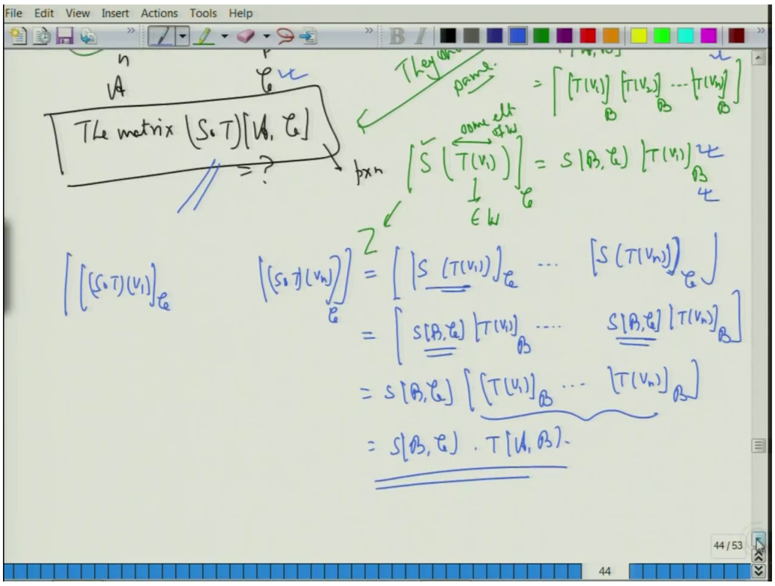

So, if I want to complete this matrix. So, this by definition is I have to look at SoT of V 1 fine and finally SoT of V m I have to compute this fine. So, this matrix is going to be this matrix, but this has to be evaluated with respect to C, because C is here. So, this has to be evaluated with respect to C is that ok. So, this by definition is equal to S of T V 1 this with respect to C, so S of T V n with respect to C.

This is equal to now look at this definition here for you I can replace this by S of B C times T of V 1, but with respect to the basis B here fine. So, I have this I will go on finally, here also it will be S of B C fine again T of V n will come into play and this is with respect to B this is what I have.

Now, this is a matrix this is a matrix by matrix product this thing will come outside S of B C will come out, I will be left out with T V 1 of B till T V n of B and recall that this basis is nothing but T A B matrix. So, this is equal to the matrix T of. So, I have already S of B C and this matrix is T of A B.

So, I have got the matrix is that ok. So, the matrix is very simple, I do not want you to remember it but and I am not going to ask a single question on this. But you should know that they come from matrix multiplication, everything made sense because of matrix multiplication alright or you can say that if you want to define it like this the matrix multiplication has to be proceeded accordingly fine.

So, the idea of matrix multiplication has indeed come from understanding composition of functions as such alright. So, from here I want to get one thing which will be useful for us afterwards is trying to understand what is called 2 things T of if I want to go from A to A.

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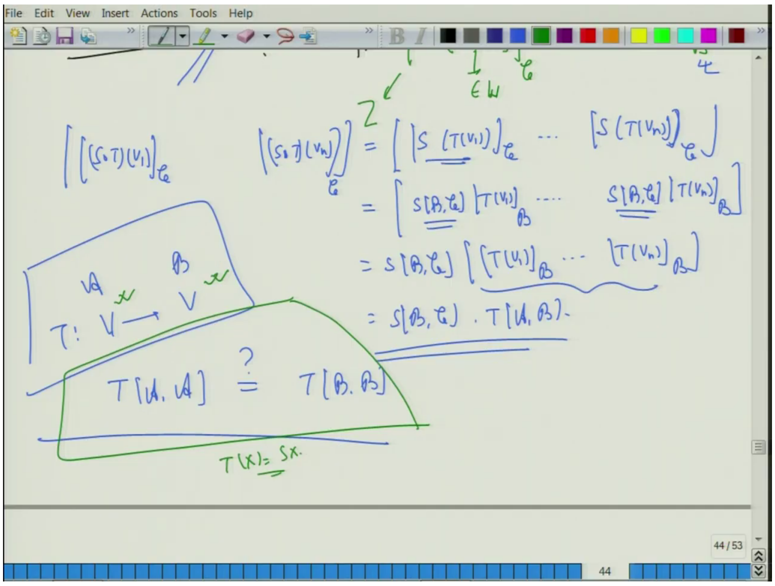

So, I have a map T from V to V there is a basis A here, there is a basis V here and I want to understand how are these two related alright. So, in the previous class we saw this but we saw it for I define T of S is equal to S x and from there we saw that there is B inverse A which comes into play and from there we went and understood these things alright. I want to understand it in a different way now using directly, because there it was R n and therefore I had matrix already with me.

But when I have just any V here I may not get R n, I can think of them as isomorphic to R n and then build a whole idea; but, in general if I want to do it how should I do it fine. So, again I am going to make a diagram the way I made a diagram here, alright I just made a diagram like this fine.

So, there will be similar diagram here that I am going to do there will be a lot of confusion just be here with me because, I am not going to ask you any question on that, but you should know how to do it. There will be some questions where you have to compute T of A A and T of B B that is all and check that everything is nice. But this diagram you need to understand because such diagrams are always there in mathematics and whenever you are studying some books you may have to come across that.

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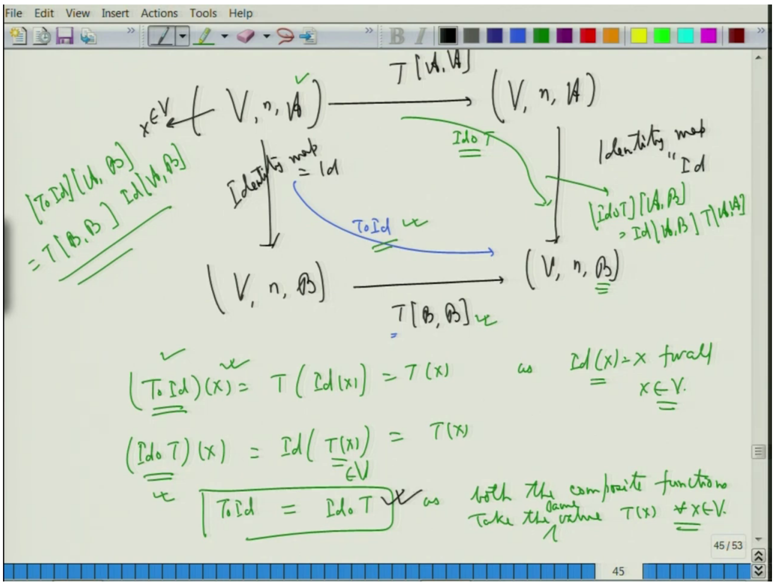

So, let us try to understand I have a V here fine this V comes to the dimension n and it comes with an ordered basis fine. Again V n is there I have A I have the map T here is that ok. So, this T will give me A and A with me fine; similarly now I have V dimension is the same, but I have an different ordered basis here with me; since I have a different ordered basis, so this map T will give me a matrix B here fine, I want to relate these two.

If I want to relate these two what I will do is that I will have a map which is the identity map, clear? Also the identity map here fine. So, that everything is I have been writing it as Id. So, let us try to understand that composition of functions make sense and then we look at how do I get the matrix out of it fine which will be helpful to us. So, take any element here say x belonging to V.

So, I am going to look at this map this composition of map, this composition is T is coming here. So, T composed with identity fine, this map will be identity is coming here so it will be identity composed with T. So, there are two different maps here fine. So, what is T o identity of x will be equal to T of identity of x, but the identity of x is x itself.

So, it is T of x itself as identity of x is x for all x belonging to V alright. If I want to look at identity composed with T of x, then this is equal to identity of T x, but what is T x? T x is an element of V itself. Since T x is an element of V itself and identity does not do anything to it, it will give me T x itself alright.

So, what we see is that these two functions alright composition of functions they behave same on every element of the domain alright. So, what we are seeing is that I have got two functions, one is T composed with identity, the other is identity composed with T; they behave the same on every element of the domain. Since they behave the same on every element of the domain, so they are same alright.

So, T o identity is equal to identity composed with T as both the functions both the composite functions take the value take the same value take the same value T x for all x belonging to V is that ok. So, they are same. So therefore, we should be able to look at their matrices alright.

So, if I am look at this matrix T o identity fine, this matrix is coming from starting with A, the domain is A and the final thing is B here. So, I need to compute this matrix is that ok. So, what we have learnt here recall in the previous when we went alright T is coming on the left, so it will be T which will come here alright and then identity will come here is that ok. Now look at identity, identity is a map from A to B.

So, it will go from A to B and this T is from B to B itself fine. So, this part gives me this matrix fine, what about this part identity composed with T this is also a map from A to B itself. So, A to B itself identity is coming on the left, so there will be identity here and there will be a T here, this is what we had learnt that if identity is on the left, the matrix of identity will come on the left.

So, this will be identity of identity is going from A to B. So, it will be from A to B and T is going from A to A. So, it is from A to A that you are looking at fine. So, what we have see here is that these two maps are same. So, their evaluation at domain and co domain will be the same.

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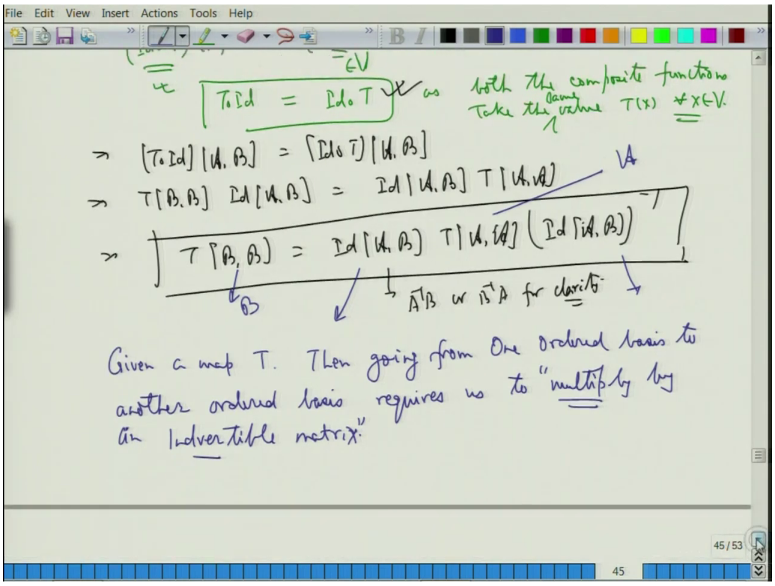

So, this will imply that that so this will imply that T o identity of this will be same as identity composed with T of this and this will imply that just write down the matrices here T o identity is T of B B times identity of A B, this is same as identity of A B times T of A A fine.

And therefore, what I get is that this implies T of B B is equal to just multiply here on the right. So, identity of A B, T of A A identity of A B whole inverse alright. So, again it is similar to what we had obtained earlier, here also identity will be the same here. So, you get the matrix B inverse A coming into play or A inverse B find out whether it is A inverse B or

B inverse A for clarity just try it out yourself do the computation and you can see that everything is nice alright.

So, what we are saying is that when I go from one basis. So, understand this A is one basis B is another basis alright. So, if I am going from one basis to another basis, so given a map T then going from one ordered basis to another ordered basis requires us to multiply by an invertible matrix alright.

So, I am not just multiplying, multiplying with a quote mark here, because I am multiplying on the left and on the right both the sides alright and on one side it is invertible the other side it is the inverse of that itself is that ok. So, there is this definition I would like to write for you.

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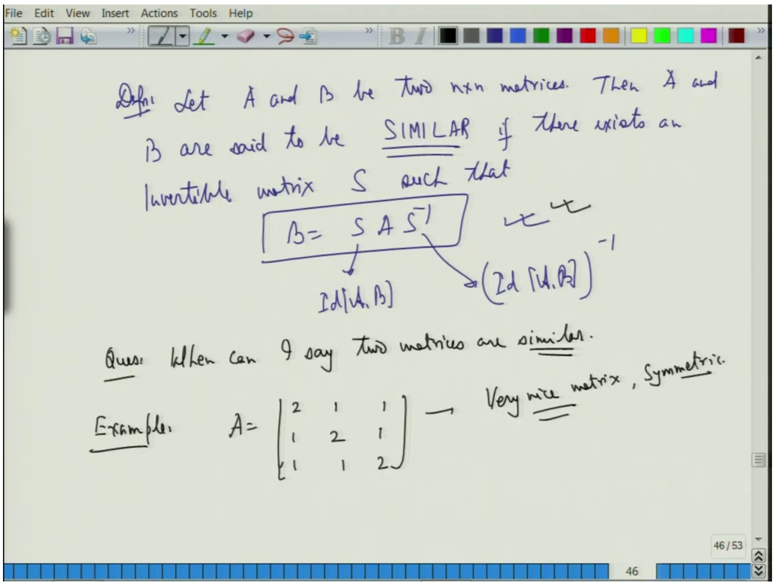

So, there if you look at T AA is of this is n cross n T of B B is also n cross n all of them are n cross n matrices. So, everything make sense the definition; so, let A and B be two n cross n matrices then A and B are said to be similar. If there exists an invertible matrix S, such that B is equal to S A S inverse alright.

So, it is similar to that here it was identity of A B and here it was A B inverse alright fine. So, it is the same thing; I have changed it in the language of matrices now because now ordered basis is not there. So, recall understand there will be at most one question on 2 cross 2 only T AA, T B B you may have to compute them fine and verify that everything is nice nothing more than that.

So, just take some examples practice it fine, what will be more important is looking at this that when are the 2 matrices similar. So, this will be the (Refer Time: 19:40) so question. When can I say 2 matrices are similar alright and as I said in the aim of these lectures that the main aim of the lecture was to understand this they will come through eigenvalues eigenvectors.

So, I would like you to understand them also we need to understand them basically because given a matrix we need to understand what are the transformation what are the ways we can do, so that I get a matrix in the nice form. So, let me take one example here to show you the importance of this example alright.

So, let me take a matrix A as I am going to take a very easy example 2 1 1 1 2 1 1 1 2 alright this is a very nice matrix symmetric, everything is nice about this you can try that out yourself, I will not talk about what are the nice things about it very nice matrix fine yes. At least you can see that it is symmetric fine there are lot of nice things about this matrix I will not get into it.

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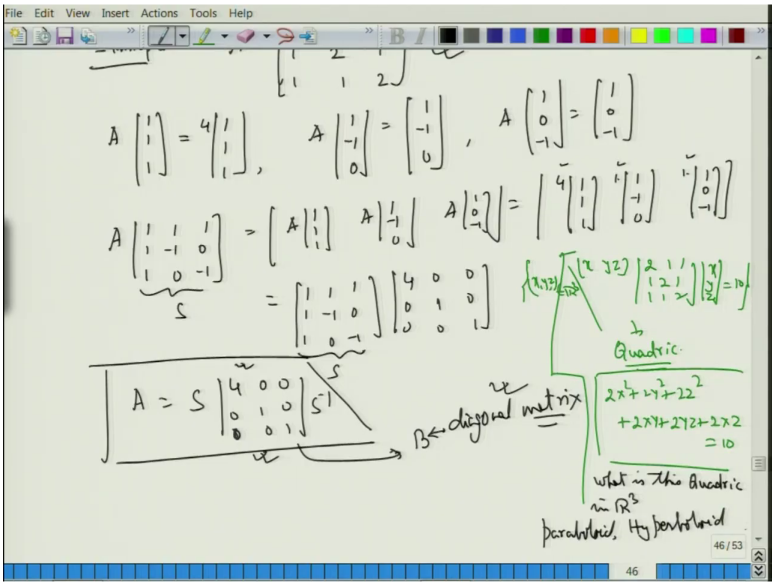

What I would like you to see that A times multiply it by 1 1 1 alright, you get it as 2 plus 1 plus 1 that is 4 times 1 1 1 you can see that nicely. A times this 1 minus 1 0 look at this, this will give you 2 minus 1 is 1, 1 minus 2 is minus 1, 1 minus 1 is 0 fine. Similarly A times 1 0 minus 1 will give you 1 0 minus 1; I would like you to check it yourself fine.

So therefore, what I see here is that a time this matrix a times 1 1 1, 1 minus 1 0, 1 0 minus 1 alright, this is same as A times 1 1 1, A times 1 minus 1 0, A times 1 0 minus 1 which is same as 4 times 1 1 1, 1 minus 1 0. So, it is 1 is here and 1 times 1 0 minus 1.

So, I can write it as see 4 is there, 1 is there, 1 is here alright. So, they are being multiplied to the columns since I am multiplying to the column it has to be on the right. So, this thing I can just write it as it is 1 1 1, 1 minus 1 0, 1 0 minus 1, 4 will come because I am multiplying 4 to the first column; 1 is being multiplied to the second column, again 1 is multiplied to the third column.

So, A time this is same as this is that ok. see if I am writing this matrix as S for me fine then I do get that A is this is same as this S times 4 0 0, 0 1 0, 0 0 1 times S inverse fine. So, even though this matrix A the starting matrix A was very nice it was symmetrical. What I like to say is that the matrix that I have got, now this matrix B which is nothing but a diagonal matrix is even better.

In the sense that I know that I just have to multiply by 4 1 and 1 fine and all the calculation whether it is determinant trace or anything that you can think of solving system of equations everything is very easy as far as this matrix this diagonal matrix is concerned alright.

So, the idea will be when you come to eigenvalues eigenvectors to come back to these ideas and try to get matrices which has this nice property that I can make them diagonal fine. Further one thing more I would like to point out here, that if I want to understand so let us see I want to understand x y z times 2 1 1, 1 2 1, 1 1 2 times x y z is equal to say 10; I want to look at the system, all x y z in R 3 such that this happens alright.

So, you can see that this will give me some sort of quadric alright. For example, it will give me 2 x square plus 2 y square plus 2 z square plus 2 x y plus 2 y z plus 2 x z is equal to 10 alright. So, 2 can cancel out and then I can proceed. What does this represent? So, the question is that what is this quadric? What is this quadric in R 3? Is it a paraboloid? Is it a hyperboloid? Is it a sphere? Alright or is it a ellipsoid? So, what is this? That is the question alright.

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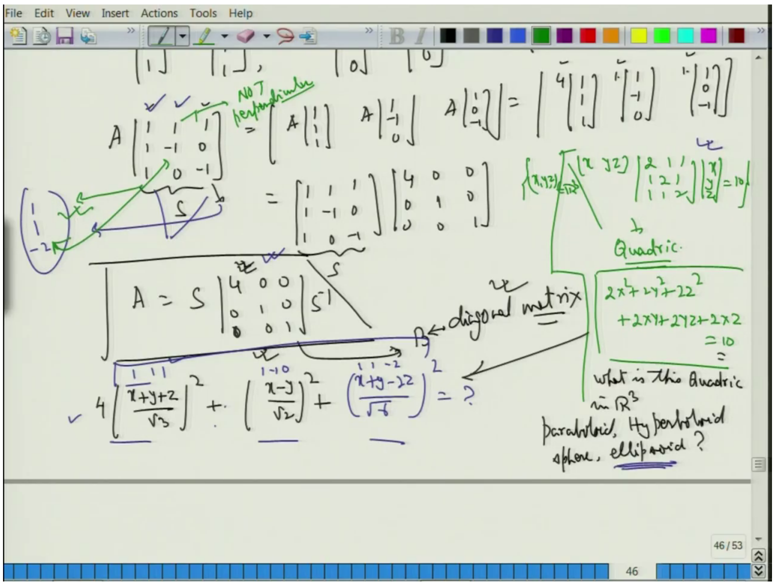

So, where I would like you to see here that there is a 4 here, 1 here and 1 here it means that and because of this 1 1 and this matrix, I can easily write that this is same as forgot about the constant that I will have to compute.

But otherwise you can try to see that after some calculation it will turn out to be 4 times x plus y plus z upon root 3 whole square plus something coming from here which is x minus y upon root 2 whole square plus I will do something to this, I will have to make some changes here and then change it to something like 1 minus 1 sorry 1 1. So, that this is perpendicular to this and minus 2 if I do then this is perpendicular this is.

So, I will have to do this, so it will be x plus y minus 2 z upon root under 6 whole square will be equal to this alright. So, this tells me that I have got some things here this 1, so this vector is 1 1 1 this vector corresponds to 1 minus 1 0 this corresponds to the vector 1 1 minus 2, you can see that these three vectors are perpendicular to each other alright.

And, hence they will form some sort of in place of x y z axis three axis I have another set of three axis which are perpendicular and therefore I have got the axis of the ellipsoid with me alright. Since, there is a 4 here 1 here 1 here with positive signs I am getting ellipsoid, if I would have got 1 negative, I would got a paraboloid sorry hyperboloid with 1 sheet. If I had 2 minuses here I would have got hyperboloid with 2 sheets and so on is that ok.

So, we will come back to these studies when you come to eigenvalues eigenvectors, there we will try to understand when we can get matrices and then this part of quadrics we will look at also 2 dimension. So, that we have a better understanding of things, that will again come when we come towards the end of the class fine.

So, this finishes our linear transformation for the time being. What we will be doing it in the next class is looking at what are called inner product spaces and build up on those idea, so that we can talk a perpendicularity alright. So, here if you see I had to use the word perpendicular here because 1 1 1 is perpendicular to 1 1 minus 1.

So, in this figure this is perpendicular to this this is also perpendicular to this, but these two are not perpendicular these two vectors are not perpendicular alright. So, what we want is a way to do perpendicularity. So, I have got this vector here if you see this vector is perpendicular to this one this is also perpendicular to this one alright fine.

So, the next class we are going to understand how when can I make some things perpendicular. So, we will start with not the dot product. So, we will have a definition of inner product, but that will come from dot product itself. We will try to do everything in the general setup, but whenever we come to R n or matrices or some things which are useful to us it will be the standard dot product alright.

So, you do not have to worry about it, but we should know that how do we compute inner product, how do you define new inner products alright. So, that is all for now.

Thank you.