Linear Algebra
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Lecture - 38
Matrix of Linear Transformations Continued...

So, in the previous class we had learnt a few things, I would like to recapitulate one after the other.

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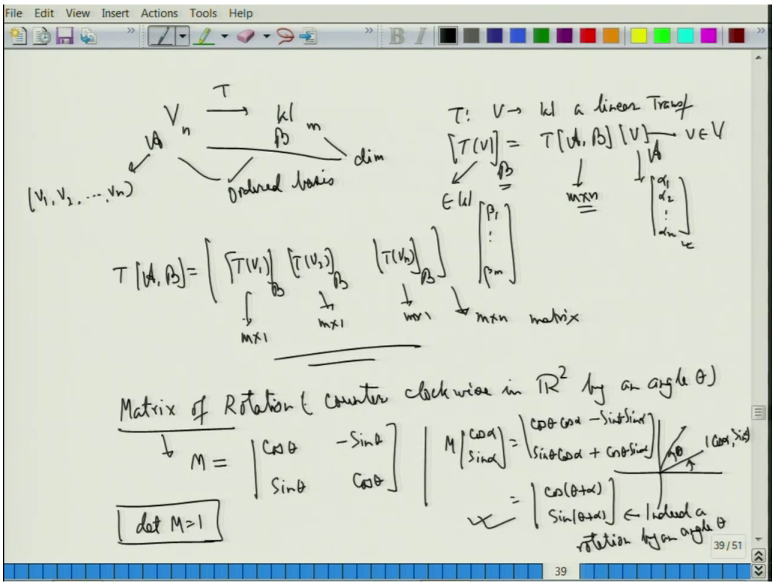

So, the one thing that had done is that; that given any vector space V and a vector space W in V you have dimension n you have an ordered basis A here the dimension is m ordered basis is B so, these are ordered basis. And the dimensions are m and n whatever those dimensions are, fine if T is a linear transformation from here to there. So, T is from V to W a linear transformation, alright so, if T is a linear transformation.

We have T of V is equal to some element of W, but if I want to write it in terms of B. So, T V is an element of W so, I can expand it in terms of W elements of basis of W. So, this is going to be since it is dimension m so, this vector is going to look like some beta 1 to beta m and we wrote it as T of A, B here and then V; V is an element of so, this is an element of capital V. So, there is a basis A here and this has n component so, we have got alpha 1 alpha 2 alpha n.

So, I am going from n vectors to m vectors so, this vector this matrix has to be of the size just look at it matrix multiplication m cross n, fine that was the idea that we had. And here if I look at T of A, B T of A, B was achieved as. So, look at this has a basis so, we wrote it as a V 1, V 2, V n ordered basis. Then we have to compute T of V 1 T of V 2 T of V n we have to compute them. And then we have to write them with respect to the basis B, because they are elements of W. Is that ok?

So, once I write them as an elements of this so, each of them is a m cross 1 vector. So, total we have a m cross n vector, alright. So, we have matrix of size m cross n, fine m cross n matrix. So, this was the idea of T A, B we have used this T A, B to write the matrix of rotation you wrote it counter clock wise rotation; counter clockwise in R 2 by an angle theta. And it gave us the matrix as the matrix m cross 1 in matrix was cos theta minus sin theta cos theta, alright.

So, I would like you to verify here that determinant of M is one that is very important so, determinant of M is 1, means that directions are taken care of all these what we have told about determinant that, when it is positive means that directions are it is meaningful object, fine. I would also like you to see that this is indeed this because if I look at any point here suppose, there is this vector which is cos alpha sin alpha will let us multiply and see what happens to M of cos alpha and sin alpha.

Then this by definition is just multiply it first component is cos theta cos alpha minus sin theta sin alpha and denominator is sin theta cos alpha plus cos theta sin alpha, which is nothing, but cos of theta plus alpha and sin of theta plus alpha so, it is indeed a rotation, alright; indeed a rotation by a angle theta, alright. Earlier, the vector was of cos alpha sin alpha, means there was already rotation of alpha.

Now, I am rotating by theta and I am getting another vector, fine and this is what it says. Now, so the idea of a studying such things is also that one has to take care of for example, you have want to look at R 3, alright and I have a robot I want to move that robot from one point to another point from this point to this point or from this point to this point or from any point to any point. Then I should be able to understand what are called rigid motions.

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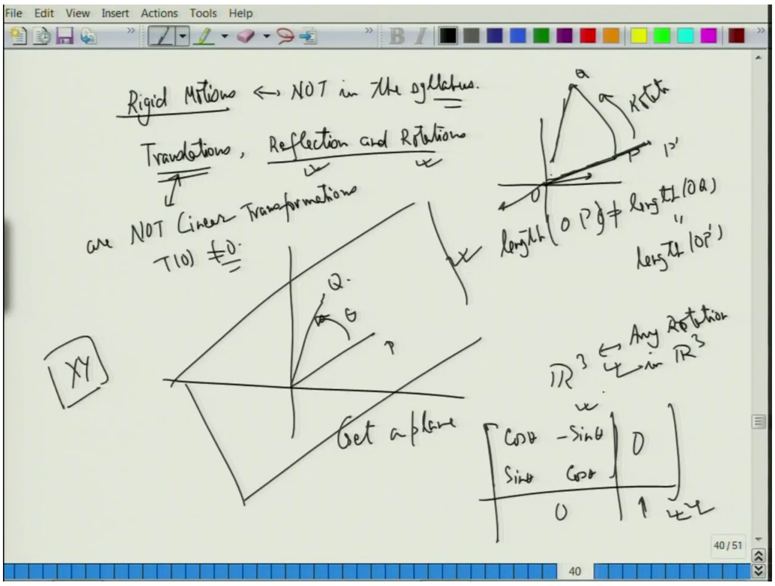

So, we need to understand what are called rigid motions. Not in the syllabus, but you need to understand that rigid motion means you are trying to move one object from to the other object. So, if you look at R 3 the movements that you would like to have is what are called translations. And then it turns out to be reflection and rotations.

So, using these 3 you can show that all the rigid motions can be done, that whatever you want your object, whether it is a pulley or whether it is your what do you say? Any object that you want to move motor or robot or anything for that matter or truck. These 3 things will help you to get the things, fine.

So, for example, I have an object which is lying here say so, this is my origin and this is my object at p, I want to move like this alright. So, this length is not the same this length is so, I want to move it to Q so, length is not the same so, OP length of OP is not equal to length of OQ. So, what I can do is that I can move this first translate further so, that the length becomes, ok. So, that the length of OP prime is same and then I can rotate it and then rotate or I can first rotate and then increase the length, alright.

So, that increasing the movement from one P to P prime is called a translation. So, such translations are NOT are NOT linear transformations. Why they are not linear transformations? Basically because T of 0 is not 0, alright. You are not able to fix because, here since you are moving it away or going to move it near whatever it is this 0 will get disturbed so, translation is not there as a rigid it is a rigid motion, but it is not a linear transformation.

But reflection and rotations are and therefore, if you want to move something in R 3 you do not want to move so, if I want to move from here to hear, I can just move my hand like this. I do not want to start from here come back to the original place origin and then go like this I do not want to do that, fine. If I want to move from here to here or here to here whatever I want to do I should be able to do it in one goal. If I am able to do it one goal like this, alright, so it is just the rotation that I am looking at.

So, when I want to rotate it means that there is a plane in which these 2 points are so, origin is at one point this is at another point, third point is this so, using the these three points I can get a plane in R 3. Once I have a plane I need to rotate it in that plane itself. Is that, ok?

So, since I want to rotate in that plane itself it means that in R 3 if I want to look at there will be plane which I will get here for example, this is the plane as such it is a X Y plane, but in R 3 I will have a plane 0 is there, then the point where P is there point Q is there so, these 3 points will give me a plane.

So, I will get a plane like this get a plane which will be something like this and there is an object here and I want to rotate by angle sum theta to get to Q is that ok, that is the idea that I have. So, you can see that at some at stage I am in a plane R 2 and therefore, the idea of cos theta minus sin theta sin theta cos theta will come in to play.

So, you can show that any motion in R 3 any rotation any rotation in R 3 has the form 0 here 0 here 1 here, alright. So, what you are saying is that there is a plane so, 1 here basically means, that if you look at the normal of this plane what about the plane I have drawn here whatever.

So, we are moving from here to here so, look at the normal there is no change in the normal as such, whatever is happening is happening on the plane the normal does not get disturbed, alright the normal going to remains as it is, fine. Similarly, we can also what are called reflection so, let me go to the reflection now.

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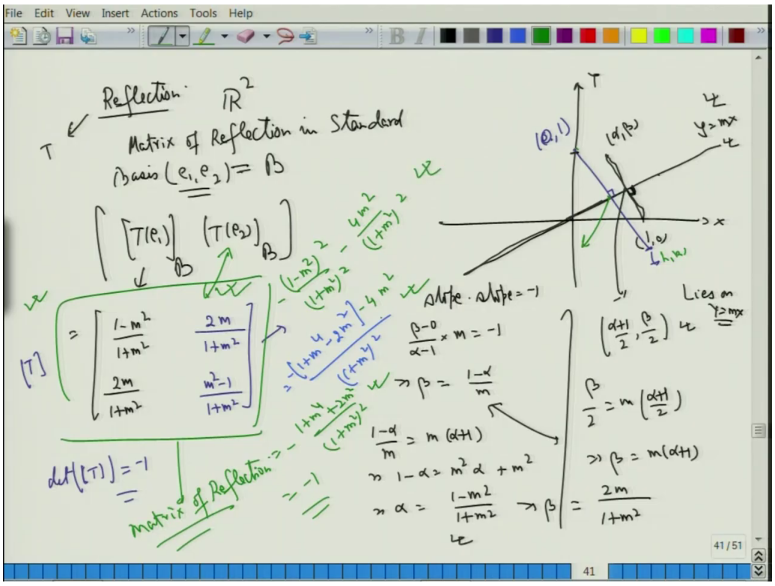

So, reflection so, I have something here so, I have the line here Y is equal to mx it is passing through the origin, fine. So, if I want to get a matrix of this so, I will again do a standard basis in R 2 itself. So, I am looking at X and Y and I want the matrix of reflection in a standard basis e 1 e 2, fine.

So, what I am supposed to do? If I am writing T as the reflection, then I need to compute what is T of e 1, I need to compute T of e 2 and again I did in this basis itself this one is basis B. So, this is the matrix that I will have fine. So, I would like you to see that if I have the point here 1 comma 0, I draw a perpendicular suppose this is the point which is the reflection so, this is the alpha comma beta.

So, what I will get is this point the coordinates of this point will be alpha plus 1 upon 2 beta upon 2 be careful I may do mistakes so, just look at it so, I will get this point. Now, this lies on this line so, beta upon 2 will be equal to m times alpha plus 1 upon 2. This will imply that beta is equal to m times alpha plus 1 also see this is the reflection so, this line is this is perpendicular the slope of this will be perpendicular slope of this their product will be minus 1 slope of the new one is slope. So, this part was lies on Y equal to mx.

Now, slope in to slope is equal to minus 1. So, slope of this is beta minus 0 upon alpha minus 1 this times the slope of this is m should be minus 1; implies beta is equal to 1 minus alpha upon m. So, solve this two so, these two together implies that 1 minus alpha upon m is equal to m times alpha plus 1; implies 1 minus alpha a is m square alpha plus m square; implies alpha is equal to 1 minus m square by 1 plus m square.

And this will imply that beta is equal to so, m and m will be there so, m in to 1 plus so, I hope it is 2 m upon 1 plus m square, please verify it, alright. So, what we will get is this matrix will look like 1 minus m square by 1 plus m square.

Fine, this is the alpha part and the beta part will be 2 m upon 1 plus m square so, you have got correspondent T of e 1, for T of e 2 you again look at this point e 2. So, 0 comma 1 look at this perpendicular here, you get a point here. The coordinates of this point will turn out to be I will just verify it so, that everything is nice so, I do not know, I do not want to waste my time I just want to look at things here 2 m upon 1 plus m square I hope.

And here it will be m square minus 1 upon 1 plus m square. So, what I know is that determinant of this matrix T determinant of this matrix so, I am writing T here. So, determinant of T should be minus 1 just check, whether it is, or not so, let you verify it.

So, this is 1 minus m square whole square upon 1 plus m square m square with the minus sign; minus 4 m square so, this should be minus I think just a minute so, cos theta should be minus here. So, because it is so, this is equal to 1 plus m square whole square 1 plus 1 plus m to the power 4 minus 2 m square, alright. I want plus 4 so, it should be a minus here; it should be a minus here, alright.

So, I get here is this plus 4 m square divided by 1 plus m square whole square so, it will be plus 4 m square so, this will give me minus of 1 plus m to the power of 4 plus 2 m square by 1 plus m square whole square, which is minus 1, fine.

So, this is the correct one there should be a minus here. Is that ok? So, this is the matrix of reflection. I would like you to check that T of e 2 is indeed this, alright just the way I have calculated you look at this part suppose, this point is h comma k get the point here and then use the slope as well as lies to get the values yourself, fine.

Now, what is more important is once you have understood this I would like you to understand that I can make. So, let me go here, what I am going to say here is that I can also think of something here.

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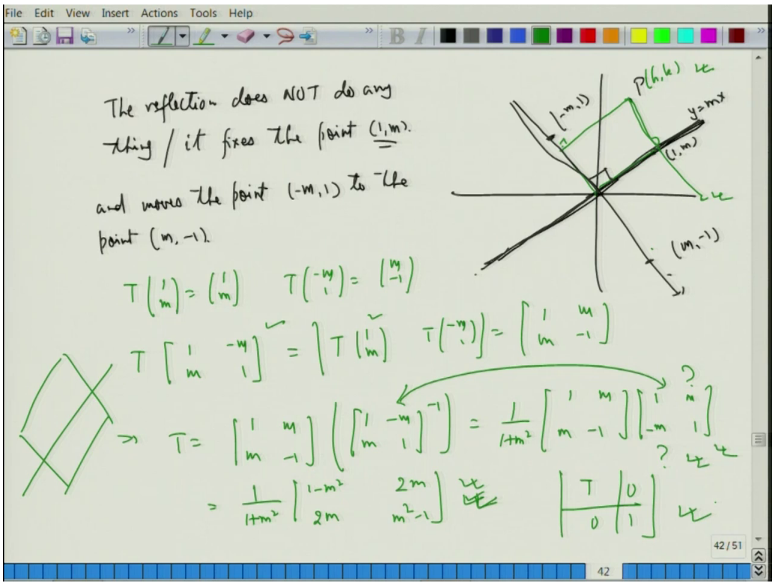

This is the line that I am looking at Y is equal to mx there is a point here, which is 1 so, Y is equal to mx. So, if x is 1 y is m, fine so, this is 1 plane or 1 line passing through origin, which is a sub space there is another one, which is perpendicular to it. Since, it is perpendicular there will be a point here, which is m comma minus 1, fine. So, here it should not be minus 1, because I am looking at the height; height is always 1. So, it will be this side so, minus m comma 1 here, fine.

So, what do I know is that if I look at this line the reflection fixes this point does not do anything to this point. So, the reflection does not do anything or it fixes the point 1 comma m, fine. And what does it do to this point minus m comma 1? It translates in to the point m comma minus 1 this and moves the point minus m comma 1 to the point m comma minus 1, alright this is what the reflection does it fixes the every point here, and here fine it just do does.

So, if I want to look at any point here P say h comma k what it will do is I can just write P in terms of projection here a projection here, fine. This part will get so, if I want to look at the translation here this will be just look at this part and whatever is this component it will get added, alright.

So, accordingly you can get a point here, because any h k can be written as linear combination of this vector and this vector is that ok, they are perpendicular vectors. So, they are linearly independent and therefore, you can get it. So, what we are saying is that if T is a reflection then T of the point 1 comma m is 1 comma m itself and T of minus m comma 1 is same as m comma minus 1, fine.

So, therefore, what we are saying is that look at T. If I look at this matrix 1 m and minus m comma 1, fine what it does? This is T of 1 m T of minus m comma 1. So, this part remains as it is 1 m, this part becomes m comma minus 1 fine. So, this will imply that T is equal to 1 m m minus 1 into the inverse of this 1 m minus m 1 inverse.

So, what is the inverse of this? So, look at this the determinant of this matrix is 1 plus m square. So, 1 upon 1 plus m square comes in to play, fine. Then it is 1 m m minus 1 times inverse of this, what is the inverse of this? Just look at it, so the idea is that it should be identity, fine if you multiply it with this. So, what should it be here should it be m or minus m, where it should be what? So, is there a way to find it out? So, look at the things this what you should get.

And then you can look at it is 1 plus 1 upon m square and you can use the adjoint method to get the inverse or just verify that this into this should be identity. So, it is 1 plus m square here, m minus m is 0 m minus m is 0 m square plus 1 so, therefore, this is the inverse. Is that ok? Now, if I multiply these two, I get 1 minus m square m plus m is 2 m so, I wrote it wrongly I think m here and plus m 2 m itself. So, please I will have to go back m square and minus 1 m square minus 1, alright.

So, it means that I wrote it wrongly there so, let me go back here and see what was the mistake so, it says that it should be this here itself minus 4 m square, should be here so, it should be minus 4 m square itself. I have used the green pen; so it should be minus itself here. So, alright it should be minus 4 m square so, minus here so, it is correct here yeah so, this part is correct so, that was the mistake I took it wrongly, fine. So, correct yourself this is how you learn that you do mistakes that you come back and do the corrections by different ideas.

So, here I had calculated using without; without calculating this the image of 0 comma 1 I tried myself there was a mistake in this calculation at this stage and therefore, that mistake cropped up, but once I look at from other direction I will able to get that what should it be, fine. So, again if I want to in R 3, fine if I want to reflect anything in R 3, what will happen is that I have a line here, now there will be plane, fine.

So, if I look at this line I want to have a point here which you should get reflected here, fine. If I want to get reflected here this line is there so, this reflection will give me this part and what happens if I take anything like this here, fine there is no change there itself, because it is remain as it is.

So, what will happen is that I get this whatever this T is for me and I will get 0, here 0 here 0 here, alright. So, again I will get something like this so, I would like you to understand these are small small ideas which are very important for us, fine. The next thing I would like you to understand is the last thing that we had done in the previous class and that was to do with.

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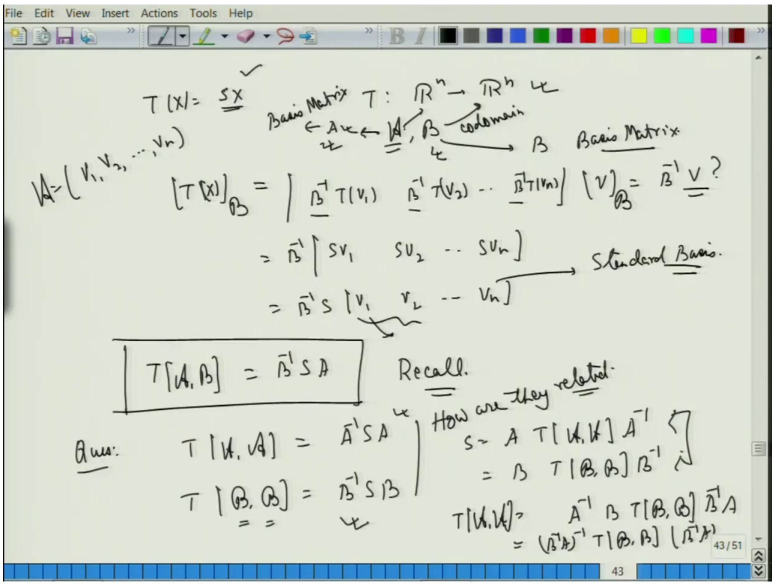

So, suppose I have matrix T so, I have a map T of X was S of X this is what the was and we had basis of R m. So, T was a map from R n to R n and I had a basis A here so, we wanted to write A where s A and as well as the basis B wanted to see. So, A is for the domain B for codomain I think. And we wanted to write T of X in the basis of this. So, this was equal to what you have to do was you have A as the matrix here basis matrix you have a B as the basis matrix.

So, what we had learnt was that since it is R n so, any V with respect to this B will be equal to B inverse V itself, alright. So, I would like to use that idea so, here it will be B inverse T of X, fine. So, T of X we wanted in terms of this so, that I have already taken care of B. Now, I want to write this part in terms of A so, want T of X. So, not like this I should write it correctly otherwise, there will be a problem for you.

So, this is equal to B inverse of T of I do not know how should write V 1 B inverse T of V 2 B inverse T of V n, where this has basis V 1, V 2 V n. Then you took out V here from everywhere V inverse B inverse B inverse is outside. And T of V 1 is S of V 1 S of V 2 S of V 1 so, this was same as B inverse of S of V 1 V 2 V n, but these are in a standard basis.

So, we wanted to write it in terms of A, alright and therefore, the idea of A inverse will come in to play, fine. And therefore, what we saw was that this matrix T of A B, this matrix turns out to be B inverse S A, fine.

So, you can see here again each have to be written in terms of A. So, A will come and this is what we had this was your T of A, B. So, recall this part recall, fine. Now so, questions that I had that I had asked T of A, A by definition is if I go from here it will be A inverse S A T of B, B will be equal to B inverse S A how are they related?

So, they are related with the idea that I can write S as look at this part S is A times T of A, A of A inverse. This is also equal to from here, sorry here it is B well it is B, B here so, S is equal to B times T of B, B of B inverse. So, using these two what we see here is that T of A, A will be equal to this will be equal to multiply so, I have already B here T of B, B B inverse.

So, I am writing this so, A inverse is here so, it will go as A and A is on the left so, it will go as A inverse. So, this is same as look at this is as the inverse of this part. So, B inverse A whole inverse T of B, B and then B inverse A, fine. So, this is the way we wrote and recall that B inverse A was the matrix of identity. So, I have written I think I had written that.

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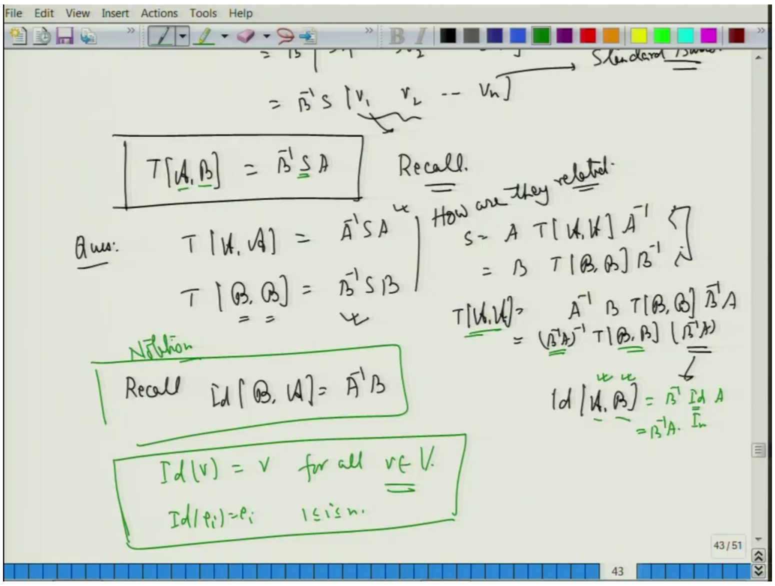

So, recall there was this map identity of I wrote it as B, A as A inverse B, here looking at B inverse A and therefore, this is identity of from A to B, alright. So, again be careful this was the notation, that I had used notation, alright. So, let us match it here S is here, the identity map and therefore, you get B inverse A going from A to B A to B, S is the identity map for me so, S is replaced by the identity. So, here it is B inverse identity, which is I sub n of A, which is same as B inverse A, fine.

So, what was the identity map, recall identity of this vector space V was V for all v belonging to capital V, alright. So, this is what we had so, here it is R n itself for us so, it is so, I d of e i is e i 1 less than equal to i less than equal to n, fine. So, it is the same map that it does not do anything only thing that we are doing here is that we are going from basis vector A to the basis vector B we get this, fine. So, what I would like you to check here is that going from A, A to B, B requires us to multiply this, alright.

So, would like to understand this nicely using what are called similarity transformation; it is going to be a bit confusing. So, be careful understand it that how do I go from one to the other, alright and then we will build up on it so, that is all for now.

Thank you.