Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 37
Matrix of a Linear transformation Continued...

So, in the previous class, we computed, so what we had was we had a vector space V, we had a vector space W, alright.

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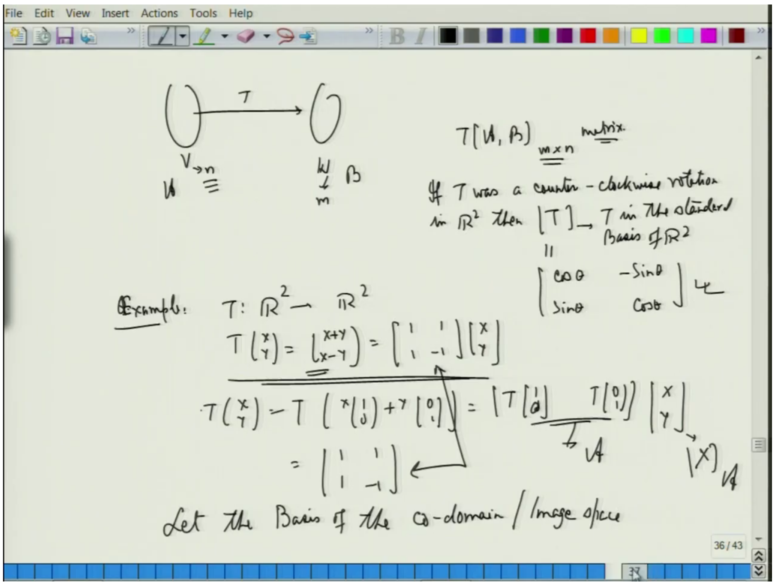

.And then there was a map from here to here, there was a basis A here, B here finite basis. So, dimension here was n, dimension here was m, then we have the matrix T of A, B Alright; this is n. So, therefore, it was n here and m here. So, we have m cross n matrix that we had alright.

And we had also computed that if T was a counter clockwise rotation clockwise rotation in R 2, then T with respect to a standard basis. This is the way of writing T in the standard basis of R 2. This looked like cos theta minus sin theta sin theta cos theta this we computed, alright.

So, let us take of some more examples to have better understanding of these things alright. So, example 2, another example, so example, define T from R 2 to R 2 by T of X Y is equal to X plus Y, X minus Y. So, in the standard basis, this is nothing but 1 1 1 minus 1 X Y, because you are defining in things in terms of a standard basis itself, because writing X Y here which is X Y is nothing but, so T of; X Y is nothing but T of X times 1 0 plus Y times 0 1, which is T of 1 0 T of 0 1 times X Y.

So, this was the capital X. If I want to write it as with the basis A here, and this has to be written with respect to again the basis A itself the standard basis, and T of 1 0 if I see here T of 1 0 by definition is X is 1 Y, 0 it is 1 1. If I write 0 1, Y is 1, X is 0, so I get 1 and minus 1, and this is what this matrix is fine. So, this was same as this fine. Now, I want to change the basis. So, first let me change the basis of the images space, alright. So, let the basis of the from that of the co-domain oblique image space be 1 0 and 1 1, alright.

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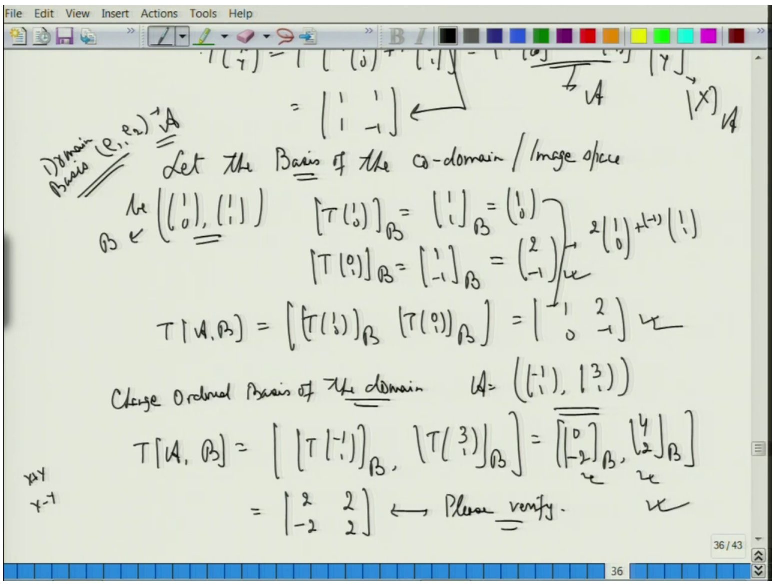

So, this is the basis that I am looking at this is my B. So, I am changing only the basis of the co-domain, I am not doing with anything with the domain. Domain is R 2 with standard basis alright, domain basis is e 1, e 2 alright the standard basis fine. So, here what I have to do is that I have to change only these two; this part remains as it is. So, for me T of 1 0; I have to do it with respect to B which is same as 1 1 with respect to B, which turns out to be if you relate this, this is 0 times 1 0 plus 1 times 1 1; so it is 1 0 for me.

Similarly, if I want to compute T of 0 1 with respect to B, this is equal to where is that 1 minus 1 with respect to B, so this by is equal to we have to solve it out; so I want you to solve it and check that this is nothing but 2 comma minus 1. So, just verify it, it is 2 times 1 1 plus minus 1 times; sorry, 2 times 1 0 plus minus 1 times 1 1. Alright, just verify it. Is that ok. Minus 1, so 2 minus 1 is 1 and minus 1 is 1 yeah, so this is what it is.

So, therefore, this T of A, B, where this is my A the standard basis is nothing but T of 1 0 with respect to B, T of 0 1 with respect to B is equal to first will come from here, second will come from here; so it is 2 comma minus 1 is that ok. So, T of A, B is this. Now, I want to change ordered basis of that domain.

So, change ordered basis of the domain I am taking A as ordered basis as minus 1 comma 1 and 3 comma 1; I am taking this as my ordered basis fine. And I want to compute T of A, B. So, T of A, B, by definition again, it is T of the first vector, first vector is minus 1 comma 1 this with respect to B.

T with the second vector 3 1 with respect to B. So, I am to supposed to look at T of minus 1 comma 1 is so this is the definition minus 1 comma 1, you have to do it there. So, what I have written here is T of this is nothing but 0 comma minus 2 with respect to B and 4 comma 2 with respect to B, which I have written as 2 minus 2, 2 and 2 fine.

So, please check whether this is correct or not, please verify, please verify. So, for me B is this itself, so let me just check it 2 and minus 2, I get here 0 that is ok; and now it is 2 and 2, so 2 plus 2 is 4, and 2 that is 0 that is correct. So, this part is correct.

Let me check whether 0 and minus 2 is correct or not, so it was X plus Y and the definition was X minus Y; so if minus 1 plus 1 is 0 minus 1 this minus 2. And now if I look at 3 1, 3 plus 1 is 4 and 3 minus 1 that is 2 that is correct. So, this is correct I think, alright. So, this is the way you are supposed to compute the matrix of the linear transform. What you are supposed to do again, let me do it.

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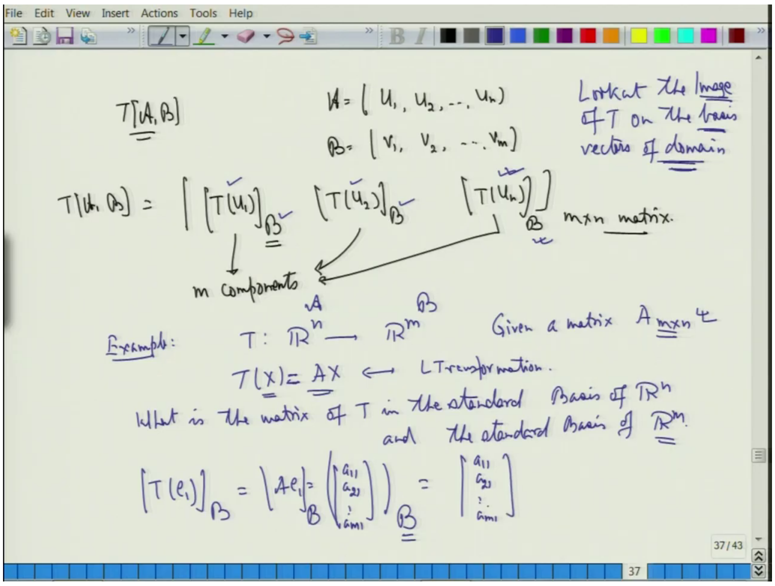

So, if you have asked to do, if you are asked to compute T of A B alright, so A is given to you which is a standard basis, so, you will be given some u 1, u 2, u n; B will be given to you as some v 1, v 2, v m. What you are supposed to do? Look at T of u 1, T of u 2 T of u n. So, this matrix AB is nothing but this matrix, but with this in terms of vectors not in terms of coordinates. So, I will have to compute the coordinates with respect to B I will have to compute this fine.

See, I am computing with respect to B, it means that this will have m components this also has m components, this also has m components. There are m column vectors here. So, this is going to be an m cross n matrix alright fine. So, look at the image of T, so look at the image of T; look at the image of T on the basis vectors of domain alright.

Why I am asking this? Try to repeat it once more for u is that recall that any linear transformation T was known was understood if I knew the image on the basis vectors alright. So, I have to look at the domain I have to look at the image on the basis vector. So, I am doing the same thing here, I am looking at the image of the basis vectors of the domain. So, I am looking at that.

Once I have done that, if once I have looked at the image fine, then what happens? Their certain vectors, but those vectors they will give you T of u 1, T of u 2 T of u n, but I want a matrix. So, if I want a matrix, so I will have to find the coordinates of those vectors T of u 1, T of u 2, T of u n with respect to the basis that is given for the co-domain.

So, T of u 1, T of u 2, T of T u n is computed. Now, I want a vector. So, I have to coordinate find the coordinates of the vector T of u 1 with respect to B coordinate of the vector T u 2 with respect to v, coordinate of the vector T u n with respect to B is that ok, that is the way to proceed.

So, this is what we have to be careful that this is the way you have to compute. And there will be at least one question which will ask you to find the matrix of the linear transform, may be it will be A and A, B and B things like that, but there will be something in that. So, we will have to be careful about it fine.

Now, the next question is example, suppose I have T from R n to R m fine, this is I am looking at. And we are given. So, given an given a matrix A which is of size of m cross n. And you are defining T from R to R by T of X is equal to A X fine this is the way I am defining. So, this is the linear transformation that I am looking at fine. Question is what is the matrix of T in the a standard basis of R n and a standard basis of R m, fine.

So, as you had seen here let us go back. So, the standard basis it remains as it is fine. So, again we will have the same thing here. So, let us try to see it. So, I need to compute what is T of e 1; I have to compute T of e 1 with respect to the basis B of co-domain. So, I have to have got a basis B here, a basis A here.

So, T of e 1 will be nothing but a times e 1 which is nothing but a 1 1 a 2 1 a m 1 fine, because A is an m cross n matrix this is what it is. So, I have to get B and with respect to the basis B here with respect to the basis B here. Now, B is the standard basis, so it is a still the same thing itself a 1 1, a 2 1, a m 1 fine.

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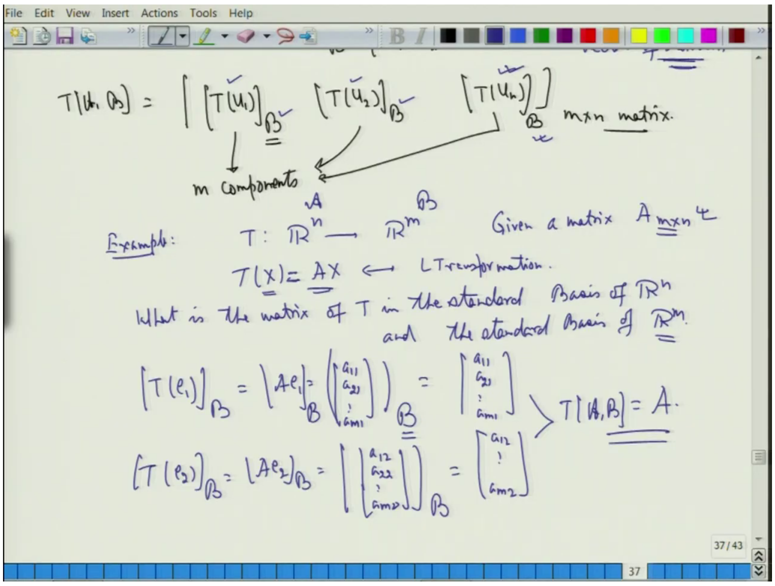

Similarly, T of e 2 with respect to B will be A of e 2 with respect to B which is again a 1 2, a 2 2, a m 2 and this I have to compute with respect to B. Again a standard basis itself should will remain as it is a m 2 itself. And therefore, what we see is that in terms of the standard basis here this will be A itself alright. There would not be any change because this is what we are getting here fine.

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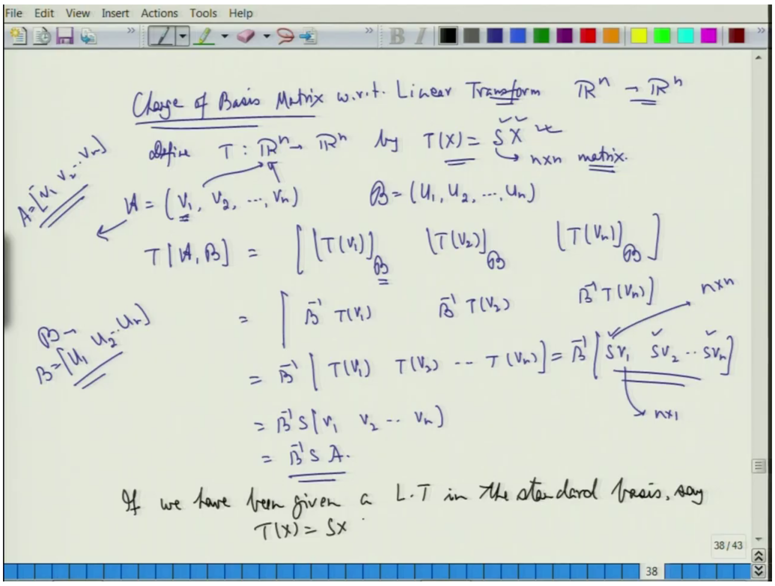

And how do I change it? So, let us understand that how to change. So, we computed different things. So, change of basis matrix with respect to linear transform. So, I am doing this only for R n and R m. So, let you understand what is going on. Is that ok? So, you have to be careful let me write to only from R n to R m, so that I can take inwards all right; otherwise I will not be able to take inwards. So, I have got A here.

So, let me just write. So, define T from R n to R n by T of is equal to S times X. So, S is an n cross n matrix, and I am doing things there fine. I have a basis A here which is v 1, v 2 v n. I have another basis B which is u 1, u 2, u n. And I want to compute T of A, B. How do I compute T of AB? Fine.

So, what am I supposed to do? Let us understand that. If I want to compute this, I am supposed to look at T of v 1 with respect to B, T of v 2 with respect to B, T of v n with respect to B, this is what the matrix of linear transformation is alright that is the definition.

But you have to compute the image of T on the basis vectors. So, T on v 1 the basis vectors, T on the basis vectors v 2, T on the basis vector v n. But at the same time each of them have to be computed with respect to the basis B fine, maybe there is this. It does not nicely here. So, this is the way it is supposed to look at fine. So, you compute this.

Now, what is T of v 1 and all of them? So, this by definition is equal to and I am looking at it with respect to B, I have the matrix B here and T of v 1. So, from B, I need to go to the matrix B which was u 1, u 2, u n alright fine. So, this was this. Then B inverse of T of v 2, B inverse of T of v n which is same as B inverse times T of v 1, T of v 2, T of v n fine.

Now, what is T of v 1? How do I compute? I know this for the standard basis. So, how do I get T of v 1? So, T of v 1 by definition here look at this is S times v 1, S times v 2 and so on. So, it is B inverse S of v 1 S of v 2 s of v n fine. How do I handle this now? That is the question fine. Now, what is S? S is a matrix. v 1 was what? S is n cross 1; v 1 is also an element of R n, each of them is an element of R n. So, this is n cross 1. So, I can multiply them out fine.

So, this make sense for me. So, it will again B inverse times S of v 1 v 2 v n which is same as B inverse of s times A, because for this matrix the basis matrix is nothing but v 1, v 2, v n alright. So, we have got just got this part. So, what we are trying to say is that if we have been given a linear transformation in the a standard basis say T of X is equal to S of X.

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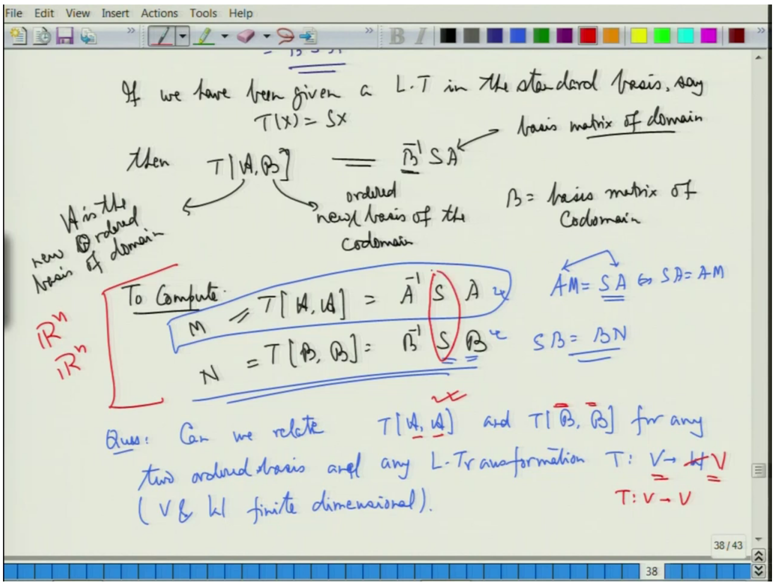

Then T of A, B, where A is the new ordered basis of the co-domain new basis new ordered basis of domain new basis of the co-domain; I am forgotten to write ordered-ordered, but as I said you have to keep track of ordered here fine, then this is equal to nothing but B inverse S A fine. B inverse is what?

So, B is your basis matrix of co-domain and A is basis matrix of is that ok. So, this is way they are related fine. So, if you want A, A, to B, B what will happen? So, let us try to understand this, what I am trying to say.

So, if I want to compute. So, to compute T of A what I will have? This will get replaced here by A inverse S A; if I want to get T of B B, it will be B inverse S B fine. So, if I think of these, for example, if I think of this as a matrix say M this as a matrix N all right, then what we are doing here is I want you to understand this usually we used very frequently here. Look at this I am looking at A inverse here. So, I can multiply A on the right side I get A M is equal to S A alright. So, A is getting multiplied in two different directions.

We are started with S. So, I got SA. So, what we are saying is that S A is equal to A M this is what we are saying. Similarly, here S is given to me. So, I am writing here if I look at this part, I am looking at here as S times B here as this B will go this side it will be B N fine. So, understand it nicely. Even though we are saying we are looking at M and N differently what we are saying is that somehow they are related here fine.

Now, question is, question can we relate; can we relate T A, A and T B, B? Here you can compute you can see here nicely that s is fixed, so can you relate this and this for any two ordered basis and any linear transformation T from V to W, V and W finite dimensional?

See important thing is that here I am looking at in this example that I have looked at till now this is about R n to R n, and therefore, could do things fine. Here I have gone from V to W, but there dimension could be the same. So, if this is I am looking at A and A, B and B, so this has to be removed. It has to be put V itself because same ordered basis is here, same ordered basis here. So, I need the V to V itself. So, T has to be from V to V itself. And what is the relationship between them? That will come to the next class.

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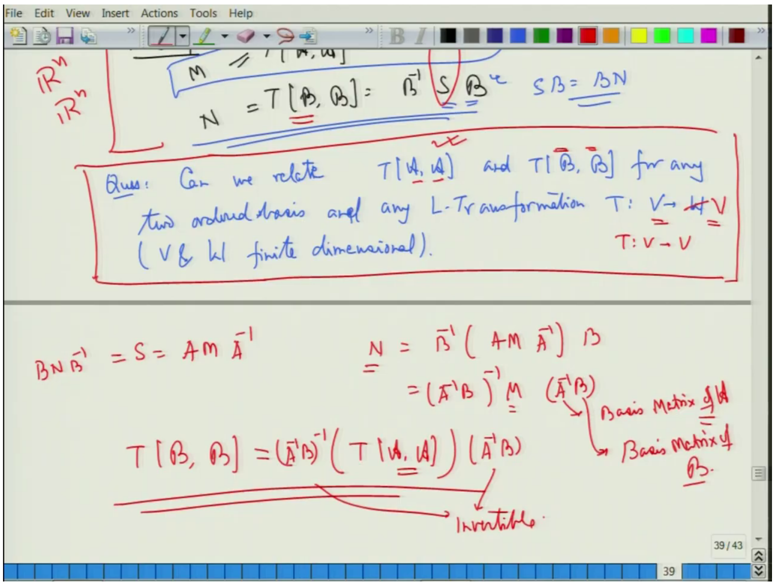

What would like to say here is that at least if you look at this example S is fixed, so what we are doing here is. So, let me write that here. So, S, if I look at S is equal to A M A inverse this is also equal to look at here B N B inverse; N B inverse. So, I can write N here as I input B inverse here A M A inverse times B which is same as M is here A inverse of B here, here it is A inverse of B whole inverse alright.

So, what we are saying is M came from here T of B, B is equal to alright M is T of A, A. So, I have got T of A, A here fine. And this is being multiplied by ordered basis. So, this is a basis matrix of A; B was basis matrix of B alright. So, I am multiplying here by A inverse B which was A, A here. So, this A and this A in subsets are cancelling out. You left out with this B and that B, is that ok. So, this what you have to be careful about.

And we will be having this theme again and again at a later stage. So, I would like to understand this. So, this was for R n to R n alright. But I would like to do it for the general set up also as I said I want to do it for the general set up. So, if there is clarity for you. And at the same time this will also lead us to the understanding of what are called similar matrices. Here also we are saying similar matrices that if I look at this these two are invertible and inverse of each other fine A inverse B and A inverse B inverse.

So, we will come to that, and we have also understand what are called composition of matrices to proceed further alright.

Thank you for today.