Linear Algebra

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Lecture – 34
Ordered Basis of a Finite Dimensional Vector Space

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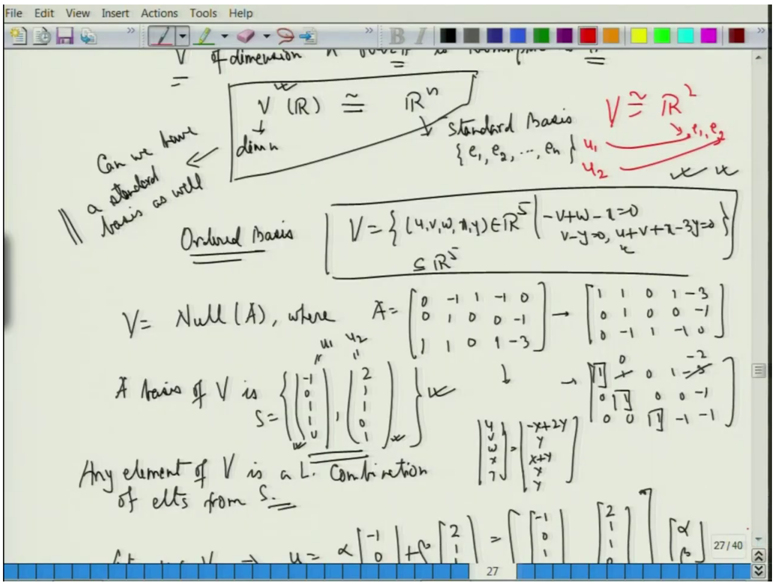

Alright, so in the previous lecture what we saw was that every vector space V of dimension n over F is isomorphic to F n, alright. So, for example, what we are trying to say here is that if I have vector space V over R then this looks like, and its dimension is n, then it looks like R n itself, fine. Now, we are saying it looks like R n, but in R n we have something nice we have what is called an a standard basis, fine.

So, we have a standard basis e 1, e 2, e n fine, can we have, so the question arises can we have a standard basis as well, fine. That is one thing. There could be choices in the sense that if I am looking at V, I can look at V differently somebody else can look at V differently and therefore, we may get different basis here, fine.

So, we have to understand what do you mean by a standard basis that is the one thing. So, depending on the question things are going to be different fine and how do I do this to get the standard basis? So, there is a notion of what is called an ordered basis. So, let us take an example to make you understand, what is what do you mean by ordered basis?

So, let me take V as, so I written it as u, v, w, x, y belonging to R 5, suppose that minus v plus w minus x is 0, v minus y is 0, and v plus u plus v plus x minus this, this is equal to so, I have looked at this as my V. And, this is a subset of R 5 you already know nothing is special.

So, it is nothing but the null space also V is nothing, but the null space of A, where A is nothing, but look at the component coming from V, coming from V is minus 1 1 1 there is u also, so let me write first u here. So, u is this, there is a u here v is minus 1, 1 1, u, v, w, w is 1, 0, 0, then you have x, x is minus 1 0 1 and y is 0 minus 1 minus 3, fine.

So, what you are supposed to do is, look at the RREF of this. So, RREF of this you can write a 1 1 0 1 minus 3 fine, then you have already 0 1 0 0 minus 1 0 minus 1 1 minus 1 0. And, from here you will get it as 1 1 0 1 minus 3, 0 1 0 0 minus 1, 0 0 just add it these 2 last 2 you get 1 here minus 1 here minus 1 here and you can use this to make this one 0.

So, this is your pivots, this minus this will give you 0 here, then this minus this will remain this and this will give you minus 3 plus 1 is minus 2. So, accordingly rewrite the things. So, you can see that in this example the solution is this, so the solution is u, v, w, x, y.

So, which is same as look at this part here it says minus x plus 2 y, this part says it is y, this part says it is x plus y, then you have x and y here. So, it gives you minus 1 0 1 1 0 and 2 1 1 0 1, alright. So, this is the basis that I get, fine. So, you can compute it yourself. So, what I would like you to understand here is that, any element here is a linear combination of this. So, any element of V is a linear combination of elements from S, fine. So, as we have started in the system of equation we said that, my first column corresponds the variable x 1, second to the variable x 2 and so on.

So, if I can say that at the back of my mind at somewhere I keep a track that, this is my first vector, this is my second vector, then I can say that I have already fixed my vectors u 1 and u 2, and since I have fixed it I can understand it in a different way, fine. So, what I am trying to say here is so, let me just proceed further. So, I cannot take up those examples, because those examples are wrong. So, take any vector there, alright.

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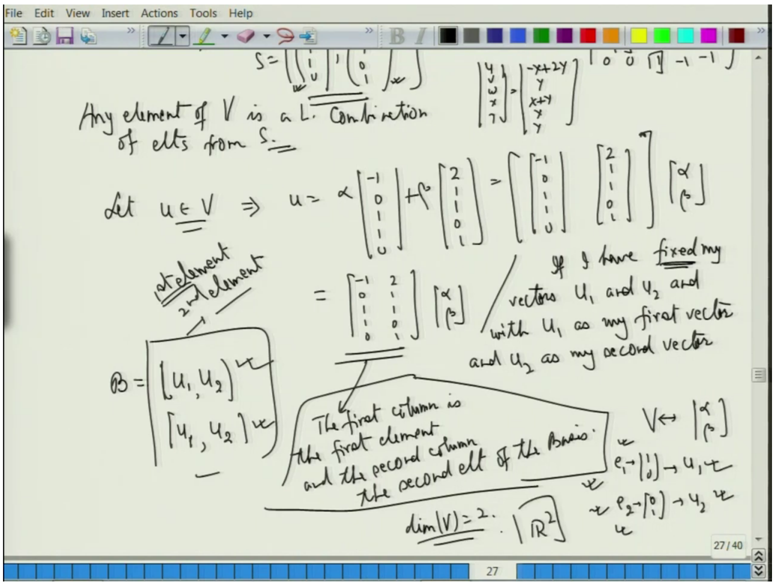

So, whatever vector you want to take so, take let u belonging to V. Since, u belongs to V, so I know that u will be equal to so, this will imply that u is equal to alpha times minus 1 0 1 1 0 plus beta times 2 1 1 0 1, this is what I know, fine. So, I can write this as minus 1 0 1 1 0 as the first vector, 2 1 1 0 1 as a second vector, and then multiply it by alpha beta, I can write like this.

So, which is same thing as looking at in some sense, since I am looking at here subset of R 5, so I can write like this, fine. So, therefore, so, what I am trying to from here say that, so what we are trying to say here is that, if I have fixed my vectors u 1 and u 2, and with u 1 as my first vector, and u 2 as my second vector, alright. So, I have fixed here that is important fixed.

See, the important thing is that when I write S alright, when I write S, then it is a set. Once it is a set so, any element can appeared any place, there is no guarantee that u 1 has to be the first element or this has to be the first element and this has to be second element that I cannot guarantee, because it is a set, fine.

So, in place of writing like this what you do is you write B and put it as u 1, a small bracket u 2, or in many books they will write it as u 1, u 2 fine. So, depending on which book you are looking at they will have different ways of writing it. So, somewhere they will use the small bracket, somewhere they will use this middle bracket and so on. But, we write it like this to say that, this is what is called the 1st element and what is called the 2nd element and so on, is that ok?

So, when I look it as a set any element can appear anywhere, but when I am writing it in a small bracket or middle bracket, then I am saying that, what is my first element, what is my second element and so on. So, when I write like this, it means that the first column is the first element and the second column, the second element of the basis, fine.

So, once I have fixed this idea that, what is my first, what is my second, I can just think of V any element of V as just alpha beta fine. And, with this idea I can say that the vector e 1, which corresponds to 1 0 is basically u 1 for me, and e 2 which corresponds to 0 1 is the vector u 2 with me, is that ok?

And, therefore, I am writing everything in terms of these vectors u 1 and u 2 or the standard e 1 and e 2 the notation has changed u 1 was having 5 components, u 1 is supposed to have only two components, because dimension of V is 2. So, dimension of V is 2 in this example. Therefore, I have just 2 vectors e 1 and e 2 each of them has only 2 components. And, therefore, what we are saying is that V, actually looks like R 2 itself, alright.

And, this is what we learned, when we said that dimension of V is n alright this is what we said in the beginning here itself that, when the dimension is n, then I have got R n it is isomorphic to R n. So, we are saying here in a different language that V is isomorphic to R 2, with R 2 here is e 1 e 2. So, e 1 here is u 1 which corresponds to e 1 for me. And, u 2 corresponds to e 2 with me, is that ok?

And, every element I am able to write with respect to that fine. So, I would like you to understand this that, there is a way of saying that what is my first element, what is my second element. If, I change the order, alright. So, if I change the order for example, so, if I change the order things will change here accordingly, fine. So, this is what you have to be careful about.

You first fix your basis that this is my first element this is my second element and so on and then do your work. So, this way there are two things that happens, you can easily see that V is indeed isomorphic to R n, in the sense that I have got e 1 e 2 e n coming into play, fine. Another advantage that I have is that when I am looking at a problem the vector, the size of the vectors could be very high, but the actual calculation may require only few vectors as such, alright.

So, if I have to look at only few vectors, fine. So, I would just have to look at the linear span of those two vectors, hence the size of the place where if the problem that I am going to look at may have only a small size. And, if the size is small, I can reduce my work and look at a small number of vectors and do work and everything will be dimension will be as, fine.

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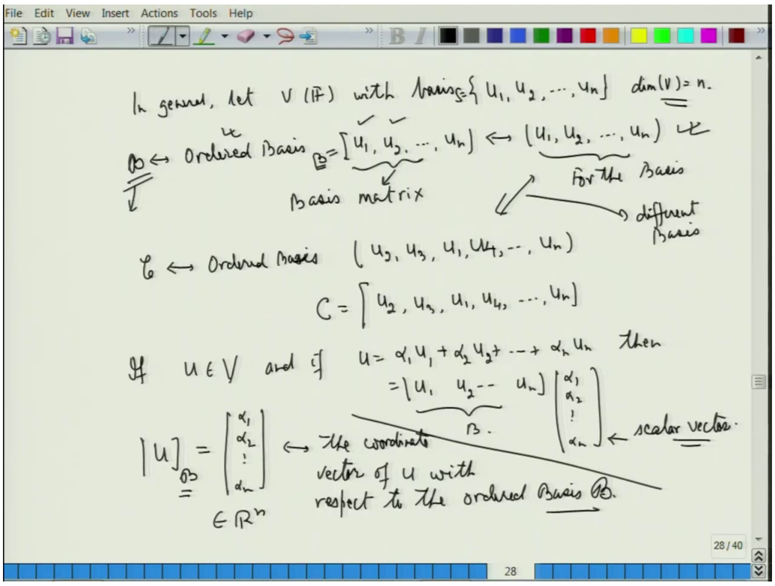

So, let me write the definition here what I am trying to say that. So, in general, let V over F be a vector space with basis S is equal to u 1 u 2 u n. So, the dimension is n dimension of V is n, fine. So, there is a notion of what is called B which is called the ordered basis. So, I can take my ordered basis as u 1, u 2, u n as I said in some books they take this notation. I will be using this notation u 1, u 2, u n.

So, this for the basis; for the basis and this for the basis matrix fine, and I will write it as a small as this B. So, there is a, what is called calligraphic B here and this is just a capital B fine. So, calligraphic B means ordered basis as I have already written here, it will be in a small brackets and corresponding matrix will be in terms of middle bracket and I would like to differentiate between the two to have a better clarity, fine.

So, here I have put my u 1 as the first u 2 as my second, I can also take C n another ordered basis again calibrate graphic here, ordered basis as say u 2, u 3, u 1 and so on so u 4, so on till u n, I can write like this. Accordingly, my C matrix will be u 2, u 3, u 1, u 4 and u n, fine.

So, these two are different basis, this is what you have to be careful about, because I am looking in terms of ordered basis, not the basis as an ordered basis and ordered basis they are different. And, there is a notion of what is called so, if u is any element of V and if u is equal to alpha 1 u 1 plus alpha 2 u 2 plus alpha n u n, then this corresponds to, I do not know why I am writing so, many mistakes here.

So, this is as I said if you recall this is the way we write here this times alpha 1, alpha 2, alpha n, we wrote the vectors like this, and these were the scalars that we are looking at, the scalar vector fine. So, if look at here this is nothing, but B itself. So, what we do is that we write u and then we put as this bracket and then put this calligraphic B, to say that this is nothing, but alpha 1, alpha 2, alpha n, alright. And, we call it the coordinate vector of u with respect to the ordered basis B, is that ok?

So, this is what you have to be careful about, since at the back of my mind I already have fixed this ordered basis, I do not have to write that again and again. I can just look at u write it with respect to B and say that it is nothing, but an element of R n now. Earlier I did not know what my V was, from where V came that is immaterial to me, what is important is that, what is an ordered basis. Once, I have an ordered basis then I have an element of R n associated with it that is more important. Is that ok?

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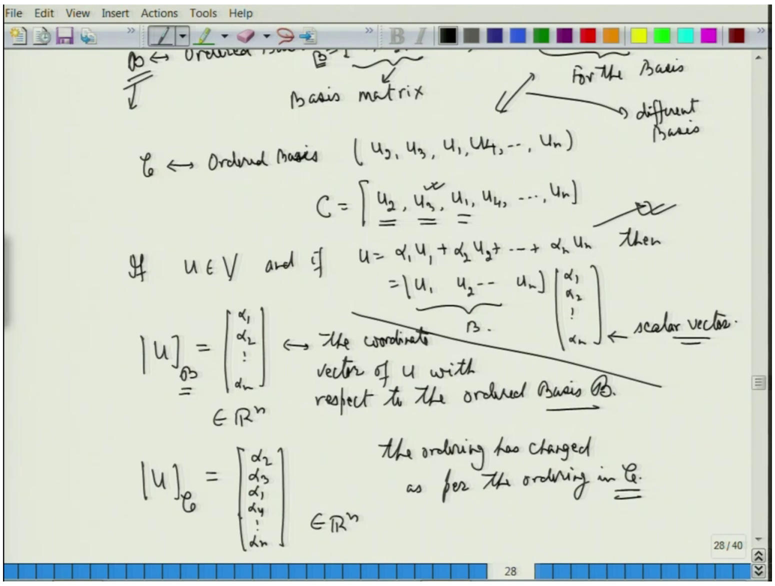

Similarly, if I want to look at u with respect to C alright, C is an ordered basis. So, we want to look with respect to C, what I am supposed to do is C had u 2 u 3 and u 1 with me u 2 u 3 u 1. So, what is the component of u 2? In this expression the component of u 2 is alpha 2. So, alpha 2 will come first then it is u 3.

So, what is the component of u 3, it is alpha 3, then alpha 1, alpha 4, and so on till alpha n, fine. So, again it is a still an element of R n, but the ordering has changed, the ordering has changed as per the ordering in C, fine. So, whatever ordered you are fixing based on that everything changes that is what you have to be careful about.

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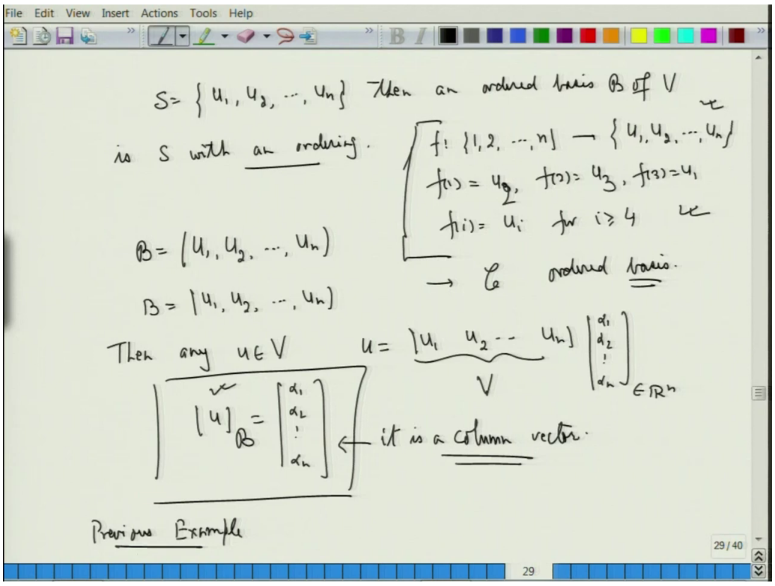

So, in some sense what we are saying is that take a basis as u 1, u 2, u n, then an ordered basis B of V alright is S with an ordering, S with an ordering, is that ok? So, there is a function f from the set 1 to n to the set u 1, u 2, u n and then I am saying that, f of 1 is suppose I am writing it as u 3, f of 2 as u 2 sorry, u 2 here u 3 here f of 3 as u 1, and f of i is equal to u i, for i greater than equal to 4, then this part, this idea, gives me the ordered basis C, fine.

So, there is a basis and there is an ordered both are related here in case of ordered basis, is that ok? So, this is what you would need to understand that, there is a small difference between the two and you have to be careful about we will see the use of this at lot of places from now on maybe till the end of the lecture, we will see that ordered basis keeps on playing roles again and again.

In sometime during inoperative to may not understand much, but afterwards again we will come back and we will see that it has implications, alright. So, let me write the definitions again whatever we wrote. So, starting with an ordered basis B, which was say sum u 1, u 2, u n, I have the matrix B which is u 1, u 2, u n. Then, any u belonging to V is of the type u 1, u 2, u n time sum alpha 1, alpha 2, alpha n. This is an element of R n, this was coming from V cross so, this is coming from V as; so, let me not write V cross V cross V.

So, this comes from V fine, and you write u of B as alpha 1, alpha 2, alpha n, fine, this is the way we write. So, again remind yourself that, from the very beginning of the class we have been saying that our vectors are column vectors. So, this is what I am saying that my u, even if the u could look like anything else, but it is a vector it is a column vector alright, it is a column vector.

And, therefore, this is the idea that we have been following from the very beginning that we want every vectors to be written in terms of columns, fine. So, for example, just look at the previous example the basis was wrong, so I do not know, but if you want to right say, it is a previous example, so previous example let us pick a vector, which will give me this plus this so no so, let me write it is 2 1 1, alright.

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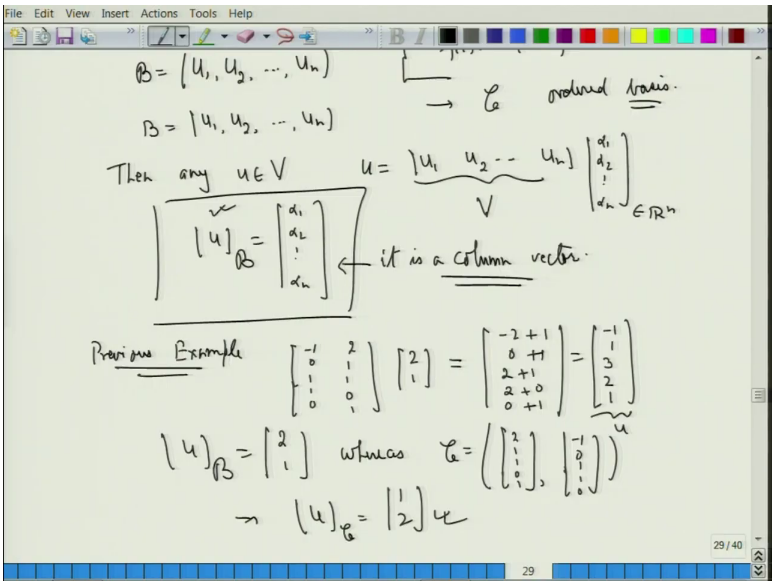

So, if I have got this vector. So, there I had u 1 as minus 1 0 1 1 0, 2 1 1 0 1 times if I want to do 2 and 1 here, fine. This vector is going to look like minus 2 minus 2 plus 1 0 plus 1 2 plus 1 2 plus 0 0 plus 1, I hope I am correct, see I do lot of mistakes. So, be careful when you do copy my things, alright.

So, what we are saying is, if I am writing this as u, then u with respect to B looks likes 2 1; whereas, if I take C as this vector 2 1 1 0 1 and minus 1 0 1 1 0. If, I take C as this vector alright, this will imply that u with respect to C is 1 2, alright. So, in some sense x has become y axis and y axis has become x axis, alright. So, you have to be careful about doing things, alright.

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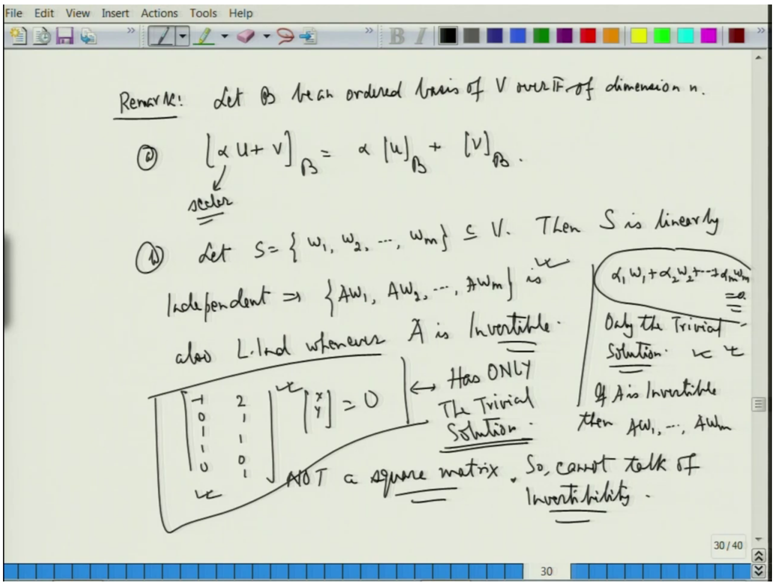

So, remark; let B be an ordered basis of V over F of dimension n, then these things are important; a, If, I take any so, look at any linear combination of u and v, this with respect to B will be equal to so, alpha is a scalar. And, u and v are elements of v itself, that will be alpha times u with respect to B plus v with respect to B, alright. To b, let S is equal to w 1, w 2, w m, be a subset of V.

Then, S is linearly independent, alright. So, let us recall things, when is S linearly independent? S is linearly independent, if I am supposed to solve a system alpha 1 w 1 plus alpha 2 w 2 plus alpha m, w m is equal to 0, I need to solve this system. And, my solution should be only the trivial solution fine, this is what I need.

So, let us recall one of the result, that we had that if I have multiply this by A linear transformation if I multiply by A fine, then what we know is that if A is invertible, if A is invertible, then Aw 1 to Aw m, alright. S linear independent will imply that Aw 1, Aw 2, Aw m, this is also linearly independent, whenever A is invertible, this is what we had seen, alright.

But, here if I look at in the previous example this vector that I had minus 1 0 1 1 0, 2 1 1 0 1 this is not a square matrix; not a square matrix fine, this is not a square matrix. So, cannot talk of; so, cannot talk of invertibility. So, I cannot talk of invertibility, but a still alright, this behaves as invertible, alright. So, this behaves as an invertible matrix, because if I want to solve this. If, I want to solve this x y here 0 alright, then this has only the trivial solution, fine.

So, again even though this matrix is not invertible, but it has only 2 columns and the 2 columns basically of a pivot or the rank of this matrix is 2, and hence it behaves as an invertible matrix, because this result that when I want to look at alpha 1 w 1 to so on, this system alpha 1 to this, this system has only the trivial solution.

And, therefore, these are linearly independent. So, in some sense linearly independence and invertibility they are playing a similar role. So, what we are saying is then, S is linearly independent will imply that this is linearly independent, fine.

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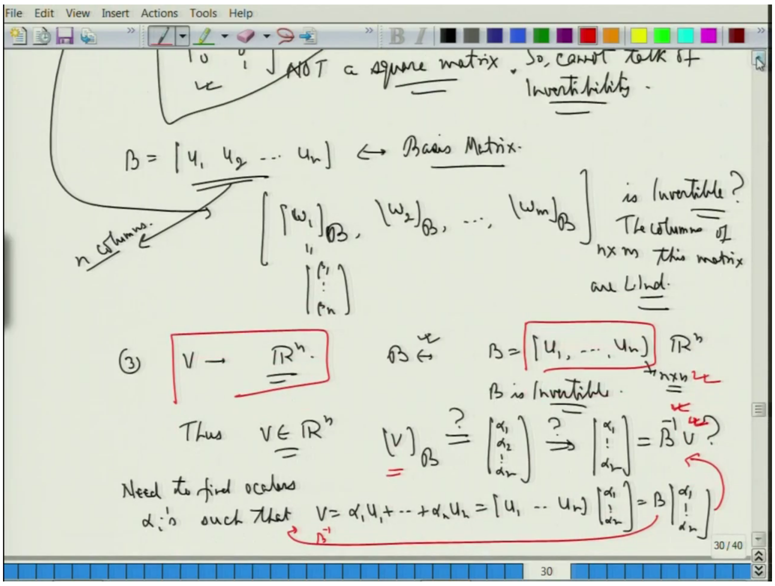

Now, if this linearly independent what is it mean? I am thinking of A as the basis vector. So, if I think of A as my basis vector say u 1, u 2 so, here I wrote it as B I think, so u 1, u 2 till u n basis matrix, fine. Then, I can multiply this w 1, w 2, w m with the basis vector, things like that and I will get something now.

So, from here I can go to w 1 with respect to B, w 2 with respect to B, till w m with respect to B, this is a matrix, alright. What will be the size of this matrix? There are m columns here and there are n vectors here, so each B has n columns here.

So, any vector will look like some beta 1 till beta n it will look like this. So, this is an n cross m matrix and we are saying that, this is invertible with a question mark, because it is not a square matrix. But, what we are saying is that invertible in the sense that, the columns of this matrix, so of this matrix are linearly independent, is that ok?

So, this is what you have to be careful about in particular, so in particular if I look at the third part that I would like to say here is that, if I start with in place of V if I start with R n itself, alright. And, I have a matrix I have B as a basis corresponding to B I have the this matrix, u 1 to u n everything here is now in R n, alright. So, this is an n cross n matrix.

So, since it is coming from a basis so, it is invertible, B is invertible. Thus, it will imply that, thus V if I want to write any V belonging to R n I want to write V as, so I want to write V with respect to B and if I am writing this as alpha 1, alpha 2, alpha n. Then, can I say will that imply that alpha 1 to alpha n is equal to B inverse V, that is the question that I am asking alright, will that be equal.

So, understand it. So, what I need is that I need to find, so need to find a scalars alpha i is such that V is equal to alpha 1, u 1 plus alpha n u n, fine. This is same as we wrote earlier u 1 to u n, times alpha 1 to alpha n, and which is same as B times alpha 1 to alpha n, which you can see that this will imply that alpha 1 to alpha n is nothing, but this B will come here as B inverse and I will get this is equal to this.

So, this thing here when I am looking at R n, I get an n cross n matrix invertibility comes then any element is nothing, but B inverse times V, is that ok? So, in general I cannot do that, because I do not know the size of the matrix vector, basis matrix as it happens in the previous case.

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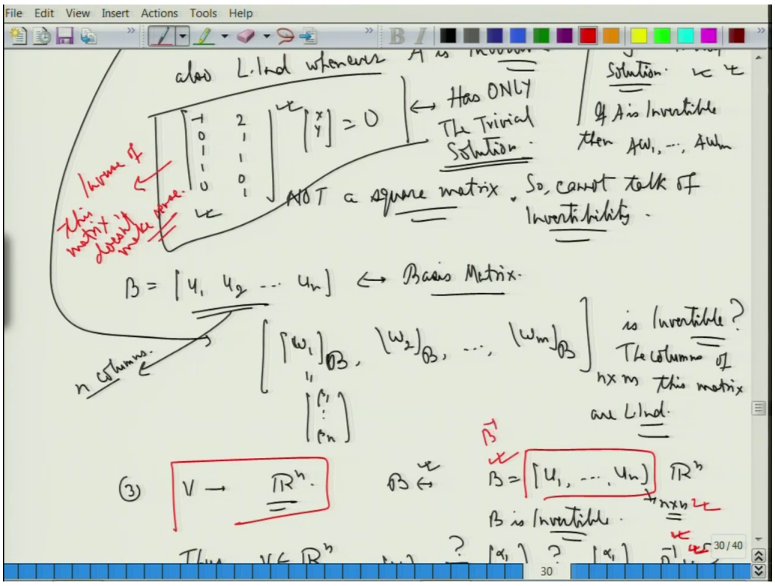

I could not talk of inverse of this matrix; of this matrix, does not make sense, fine. But, still I could solve it, alright. So, here when everything is nice I get a n cross n matrix from a small B from this capital B I can go to B inverse, in general I cannot go. So, you have to be careful, when you can go and when you cannot go, alright. So, we will take up this idea again in the next class.

Thank you for now.