Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture - 33
Isomorphism of Vector Spaces

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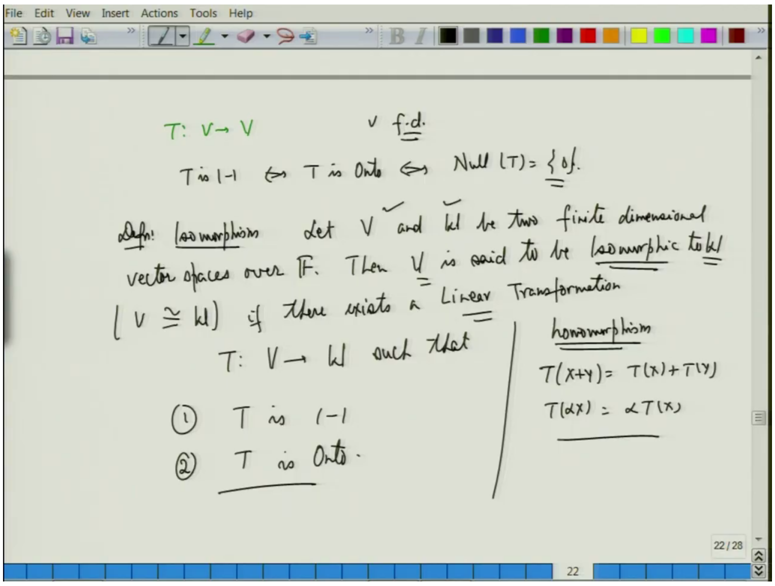

Alright, so we saw the implication of rank nullity theorem for T from v to V alright. We saw that v finite dimensional is very important that T is 1 1 if and only if T is onto if and only if null space of T is the 0 vector fine. Now, we want talk of what is called isomorphism. So, earlier we had in our school days what is called bijective 2 sets of.

In bijection, if there is a map from 1 to the other which is 1 1 and onto. So, we have a notion of here what is called definition of isomorphism. So, let V and W be 2 you can define it for infinite also, but I am look at only finite dimensional 2 finite dimensional vector spaces over v f alright.

Then V is said to be isomorphic to W alright or v is isomorphic to W thus notation isomorphic W if there exists a linear transformation T from V to W such that 1 T is 1 1 to T is onto is that ok. So, you can see that in some sense I have used these ideas somewhere.

So, we are saying that 2 vector spaces are isomorphic alright if I can get a map linear transformation T from 1 set to the other. So, the T is 1 1 as well as onto alright. So, the idea of isomorphism basically means I have not use the word what is called a homomorphism, but I have said it is the linear transformation.

So, there is a notion of what is called homomorphism. I have not used it, but basic idea it is of linear transformation itself T of X plus Y should be equal to T X plus T Y because they are 2 operations here T of alpha X is equal to alpha T X alright. So, they are already for linear transformation hence a things are ok. So, I will not prove it, but a look at some examples to show that what I am trying to say examples alright.

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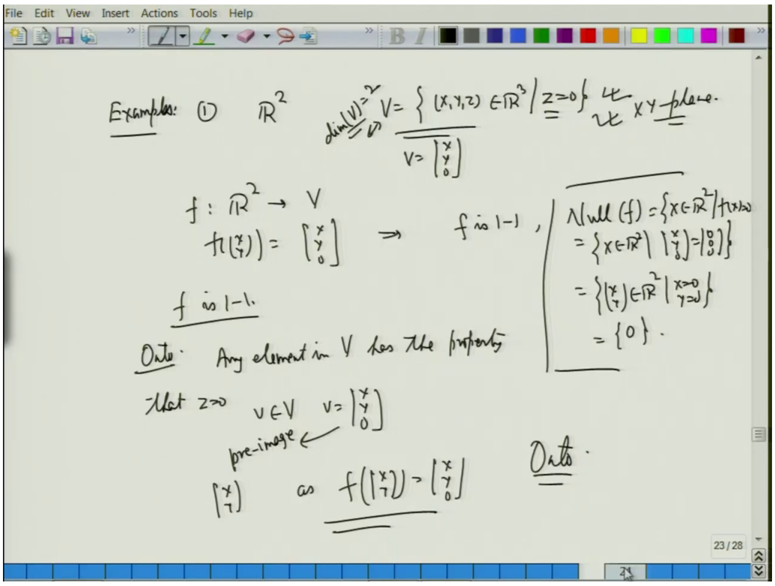

So, first example is let us look at this R 3 or R 2 and I want to look at this vector space V which is all X, Y, Z belonging to R 3 such that Z is 0 alright look at this. So, here if I look at since Z is 0 any vector here V has the property that it looks like X Y 0 fine. So, I can define in my map f from R 2 to V by f of X Y as X Y 0 alright.

So, this will imply that f is 1 1 because you can see that the kernel or the null space of this map null space of f is f of X such that X belonging to R 2 such that f of X is 0 alright which is same as X belonging to R 2 such that X Y 0 is 0 0 0 which is same as X is equal to 0 or X equal to 0.

X Y belonging to R 2 such that X is 0 Y is 0 which is the 0 vector itself is that. So, the null space is 0. Therefore, the map is 1 1 alright. So, f is 1 1 what about onto take any vector here. So, any element in v has the property that what is the property that z is 0. So, any V if I have taken V then V is X Y 0 alright it is pre image pre image is X Y itself fine as f of X Y is equal to X Y 0 alright. So, it is a onto fine. So, the behavior of these 2 are same that we already know that this gives us the XY plane.

So, they are same as such there is nothing a special we have been we have a already known it. So, I have not given anything a special here fine. But, what I would like you to understand here is that they get related. So, here the dimension of this space is also 2 alright.

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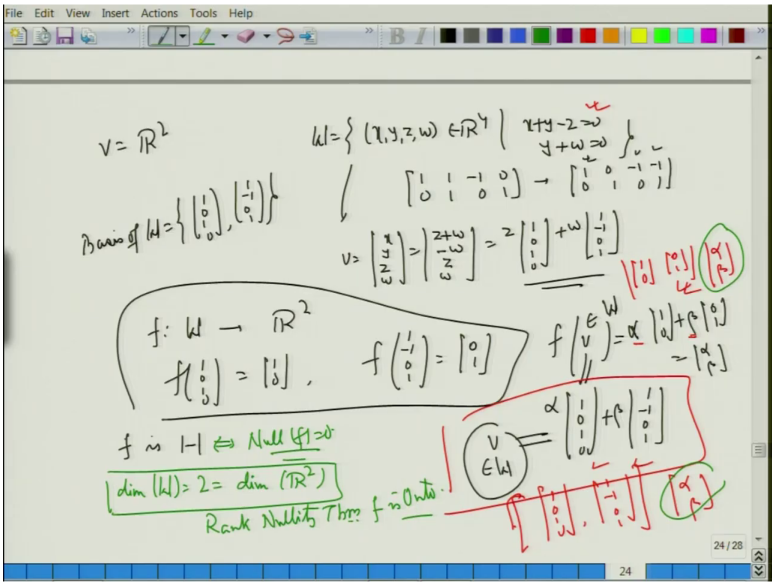

So, some more examples to; have a better understanding. So, again R 2 here V is R 2 and another space W is X, Y, Z, W belonging to R 4 such that X plus Y minus Z is 0 and Y plus W is 0.

Suppose I am looking at this fine. So, I will if a look at this it tells me hear that the matrix that I am looking at is 1 1 minus 1 0 0 1 0 1 I am looking at the null space of this. So, this I can write it as 0 1 0 1 1 minus 1 is 0 minus 1 minus 1. I hope this is rref I think this minus this is 0 minus 1 minus 1 alright.

So, therefore, the solution of this that I want to look at any V here is of the type X, Y, Z, W. X from here is nothing, but look at these 2 points this is nothing, but Z plus W fine and Y is minus W Z and W. So, this gives me the vector Z times 1 0 1 0 plus W times 1 minus 1 0 1 fine. Said that these 2 are elements there itself.

So, 1 and 1 is 0 and that is there is 0 0 and 1 minus 1 yeah it is 0 fine. So, these are the basis of W. So, the basis of W is 1 0 1 0 and 1 minus 1 0 1 fine this is the basis. So, I can define my map f from W to R 2 by f of 1 0 1 0 as 1 0 and f of 1 minus 1 0 1 as 0 1 fine then you can see that f is 1 1.

So, I am going to define f using this. So, what I am saying is that I want to define f of for any V belonging to W I have to define this. So, any V here is some alpha times 1 0 1 0 plus beta times 1 minus 1 0 1 any V is of this form. So, I define this as f of V as alpha times 1 0 plus beta times 0 1 which is alpha beta is that ok.

So, for every V belonging to W I write V as this. Is that ok? That is important I am writing this first using linear combination from here basis and then I am taking the maps. And therefore, this is a linear transform because I am looking at the linear transform here.

I am looking at 1 0 0 1 times alpha beta and here it was looking at the vector 1 0 1 0 1 minus 1 0 1 this matrix times alpha beta alright. So, the basically it is this alpha beta alright this alpha beta this alpha beta which is playing the role here is that ok. So, you have to be careful here. So, you can see that f is 1 1 because of this part because if we are saying that 1 1 means the images has to be 0. So, this part has to be 0 this what will happen alright. So, we will imply that.

So, look at null space of f. So, where is null space? So, f is 0 when this is 0 alpha is 0 beta is 0. So, you get a 0 vector here. Is that? Dimensions of the 2 are same. Dimension of W is 2 which is same as dimension of R 2 and therefore, by the rank nullity theorem f is onto fine or you can should do it directly also using this idea of the function itself I have given you the actual function fine. So, you can do that yourself fine.

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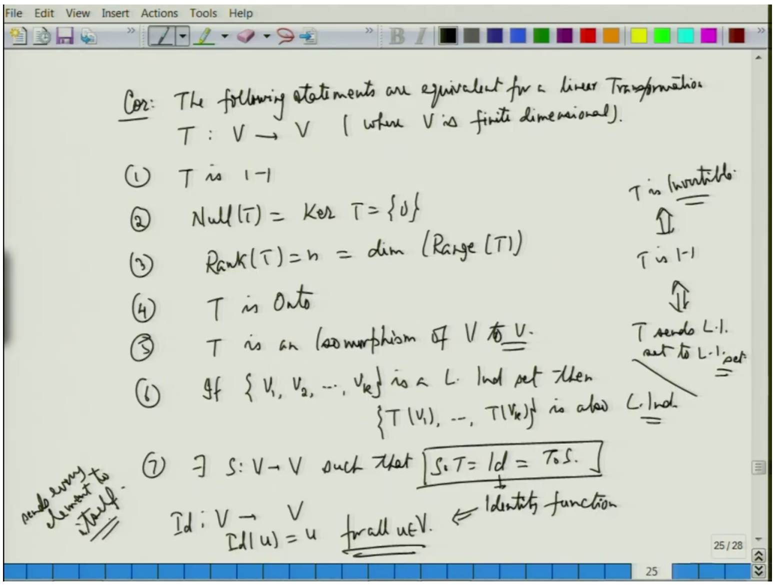

So, as a corollary of things I will just write something's here that I would like you to remember and understand corollary. The following statements are equivalent the following.

The statements are equivalent for a linear transformation T from v to V where v is finite dimensional; alright 1 T is 1 1. 2 null space of T which is same as kernel of T is 0. 3 rank of T is n which is same as dimension of range of T. Fourth T is onto which is same thing as saying that alright I have already said dimension of (Refer Time: 11:39).

T is onto, T is an isomorphism of V, V to v itself alright V to v itself. Different basis 6 if. V 1, V 2 V k is a linearly independent set, then T of V 1 till T of V k alright is also linearly independent thus important. So, it takes. So, T is 1 1 means T sends linearly independent set to linearly independent set is that that is important. So, if you remember what we had proved earlier was that if V 1 to V k is linearly dependent, then T sends dependent to dependent.

Now, what we are saying here is that whenever T sends linearly independent set to linearly independent set implies and is implied by T is 1 1. In the some in some notion what you are saying is that T is invertible alright. So, these things gets related alright inverse in terms of matrix we had similar things. So, we need to understand these things. So, when I say the T is invertible I have not given you the idea of invertibility as I said.

So, what we are saying is that there exist S from again v to V itself such that So T is the identity function identity function identity function which is same as T composed with S. I think I am not defined the identity function for you I forgot define it. So, identity function basically means that.

So, identity function from v to V is identity of any u is u itself for all u belonging to V alright fine. So, what we are saying is that I have an identity function which sends every element to itself that is all I am saying sends every element to itself is that.

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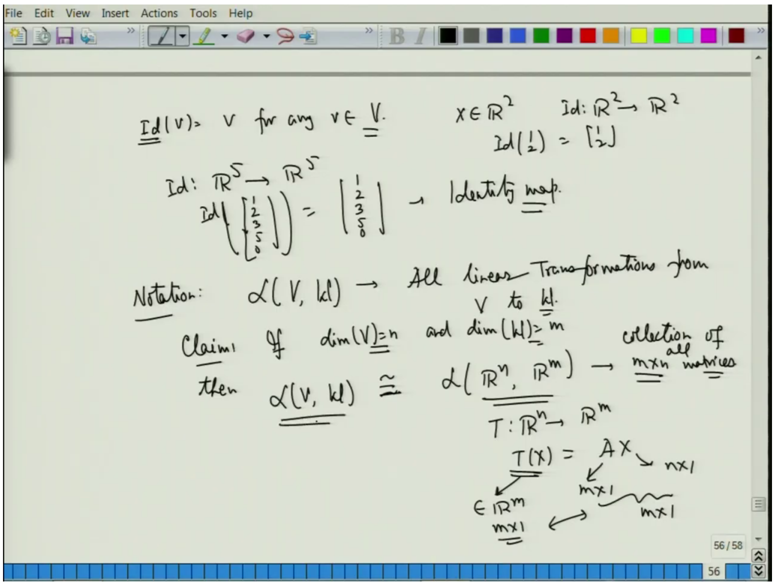

So, we have defined what is an identity map. Identity map is identity of v is V for any V belonging to the vector space. So, we will depend on which vector space you are looking at where definition of this identity map that is all nothing else.

So, recall that if you are looking at X belonging to R 2 and you have map id from R 2 to R 2 then id of 1 comma 2 is 1 comma 2 itself is that ok. If I have got identity map from say R 5 to R 5 then identity of say 1 2 3 5 0 is basically equal to 1 2 3 5 0. So, it is the identity map identity map; alright nothing else now we are also.

So, I think I have forgotten the notation related with notation related with L of V, W. So, this was all linear transformations from the vector space V to W fine. So, would like to say that. So, claim if dimension of V is n and dimension of W is m, then L of this in some sense corresponds to. So, V as dimension n, so it is corresponds to R n here this corresponds to R m and therefore, I get a matrix which is of size m cross n. So, it is the collection of all matrices collection of all m cross n matrices.

Why all n cross m matrices? Because if I want to define any linear transformation T from R n to R m T of X is equal to A times X this x is n cross 1. So, this has to be m cross 1. So, that the total product defined; to be equal to 1 cross m and T X is an element of. So, this belongs to R n. So, this has to be m cross n. So, the matrix multiplication makes sense and therefore, this is what we need alright.

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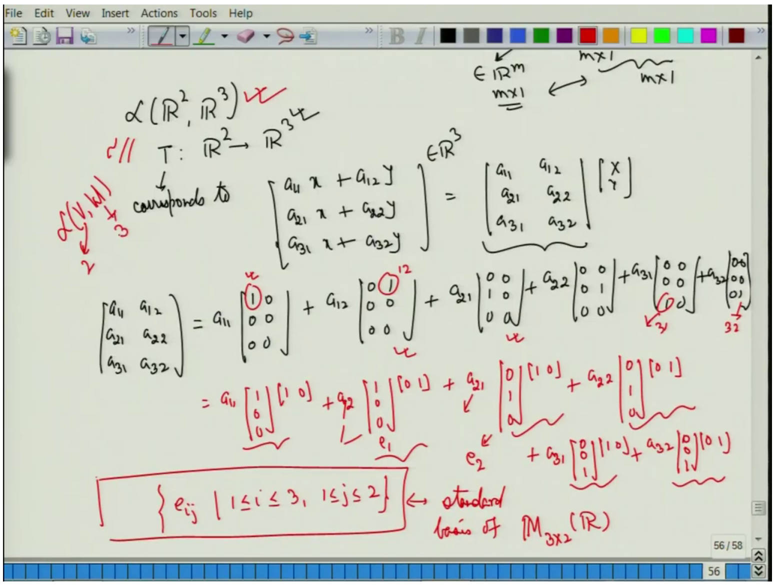

So, let us look at the idea here what I am trying to do here. So, what we want to say is that if I want to look at R of R 2 to R 3 fine then I am looking at linear transformation collection of all linear transformation from R 2 to R 3. So, as shown here T will corresponds to corresponds to corresponds to something like. So, I have to take X and Y here. So, there will be 2 things, but

I want 3 components here. So, I will have something like a 11 x plus a 12 y a 21 x plus a 22 y a 31 x plus a 32 y.

This is how I get an element of R 3. This is an element of R 3. I can break this up as a 11 a 12 a 21 a 22 a 31 a 32 times X Y fine and therefore, T is basically corresponding to this part fine. So, what I would like you to see here is that this matrix a. Which is a 11 a 12 a 21 a 22 a 31 a 32 this corresponds to this following that it is a 11 times a matrix which is 1 0 0 0 0 0 plus a 12 times 0 1 0 0 0 0 plus a 21 times 0 1 0 0 0 0 plus a 22 0 0 0 0 1 0 plus a 31 0 0 1 0 0 0 plus a 32 times 0 0 0 0 0 1 alright.

So, important here that important here is that this 1 is the 11 entry, this entry is the 12 entry, this entry is the 31 entry and this is the entry 32, thus the way to remember it. So, what we are trying to do here is if you look at it is nothing, but a 11 times.

So, this is the first matrix just look at this is the matrix that I am looking at 1 0 0 and this I am multiplying by 1 0. Is that ok? So, I get here this part here it is plus a 1 2 times 0 this is still the same thing 1 0 0. But, now I am multiplying it by 0 1 here fine. So, this 1 corresponds to this 1 that is this is e 1 for you things like that.

So, if I am looking at a to 1 2 is here. So, need to look at a 2. This is your e 2 for you this is your e 2, but it is in 3 dimension because in r 3 and then I am looking at 1 here. So, 1 means 1 0 that I have. So, this product will give you this part. Similarly, a 22 again this is 2 itself. So, it will be 2 here e 2 here and e 2 here. Similarly, a 31 times e 3 this is e 3 times 1 plus a 32 times 0 0 1 and 0 1 here is that. So, what we can see here is that these matrices that I am writing here this; matrices if you recall they were nothing, but e ij they were.

So, 1 is less than equal to I less than equal to 3 and 1 is less than equal to j less than equal to 2. These were the standard basis of R of. So, this is a matrix of size 3 cross 2. So, not R also the matrix of size 3 cross 2 3 cross 2 matrices over R or over C whatever you want to see.

So, whatever saying here is that that every matrix here or any linear transforming here basically looks like a matrix and that matrix can be written in terms of this basis ordered basis. And therefore, this is isomorphic to L of any vector space V and any after this; where V has dimension 2 and W as dimension 3. Is that? That is all.

Thank you.