Linear Algebra
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Lecture – 32
Rank-Nullity Theorem and Applications

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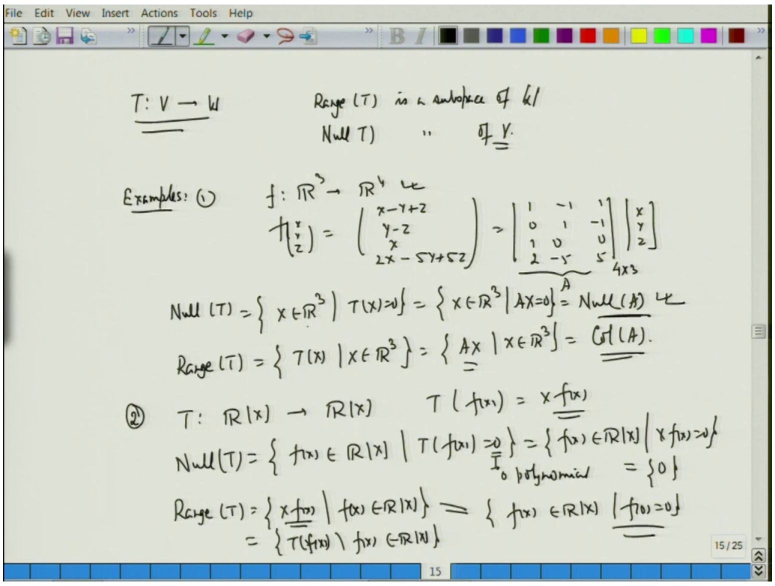

Alright; so recall that I had asked you to find out that given a T from V to W, I would asked you to look at that range of T alright is the subspace of W sub space of W and null space of T is the sub space of V alright; T is a linear transform that is given to me.

So, let us some take some examples to compute them. I hope you have computed, you have tried it out yourself. So, let me take examples here to construct them so that you understand what exactly we are doing. Examples 1: alright, suppose I have given you f from R 3 to R 4 given by f of X Y Z as X minus Y plus Z, Y minus Z, X 2X minus 5Y plus 5Z. I give you this.

So, this is same thing as looking at this matrix 1 minus 1 1 0 0 1 minus 1 0 1 0 0 0 2 sorry, can you just a minute 2 minus 5 5 there is only three components here times X Y Z, alright. So, I have got a matrix which is 4 cross 3 and therefore, I am able to go from R 3 to R 4, is that ok? So, if you want to find the null space of T null space of T.

So, if I write this matrix as A null space of T is nothing, but all X belongs to R 3 such that T of X is 0 which is same as all X belonging to R 3 such that AX is 0 alright. So, this is same as null space of A and we are already computed it. So, you do not have to worry about. Range of T is same as TX such that X belongs to R 3; this by definition if I look at here is a X such that X belongs to R 3 and my matrix multiplication tells me that AX is nothing, but column space of A.

So, we had already done these two in the previous class alright, one of the previous classes where we looked at the fundamental theorem of linear algebra and there were the fundamental spaces fine. So, I would like you to try them out yourself and get your answers, fine?

Example 2: define T from R X to R X by T of f of X is equal to X f x fine and define this. What is null space of this? So, null space of T is look at this all polynomial f of x belonging to R X such that T of X. So, T of f of x is 0. So, it is 0 as a polynomial 0 polynomial fine.

So, this is same as all f x belonging to R X such that X f x is 0 and this is true only of the 0 polynomial. So, it is just the 0 polynomial for you, fine. What about the range of T is X f x such that f of x belongs to R X I did not write the first line that is T X part. So, as such it is suppose to be T of f of x f of x belonging to R X fine, but here if I see whatever I do X is a factor of this polynomial.

So, when I say the X is a factor by remainder theorem X minus 0 is there. So, it is nothing, but collection of all polynomials f such that f is 0 f of 0 is 0. So, it is all polynomial f such that f 0 is 0. So, as theorem between the other way around f x belonging to R X such that f 0 is 0. So, this is this collection, fine.

So, you can compute them for different examples and learn yourself. So, now, let me prove this very important result, fine.

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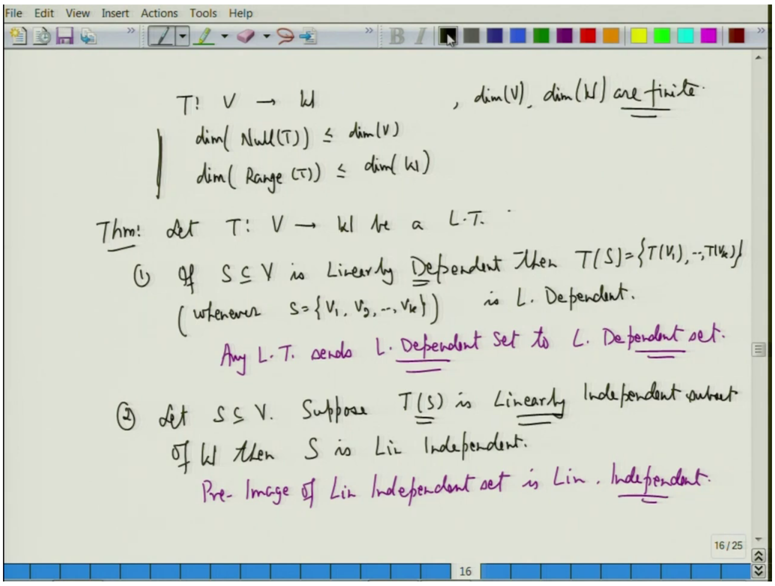

I would also like you to understand this which I said, but I did write it down that if T is from V to W then dimension of null space of T is less than equal to dimension of V. So, I am assuming that dimension of V and dimension of W are finite, fine.

I am assuming they are finite here and therefore, I can talk of these things dimension of null space of T is less than equal to dimension of V. And, dimension of range of T is less than equal to dimension of W. I cannot go beyond W anyway fine because the it has to go the images have to be inside the co-domain itself fine. So, you can prove yourself this theorem is very very important this was trying to say. So, let us look at this definition.

So, let T be from V to W be a linear transformation V and W are vector spaces, fine. I will take them to be finite dimensional for once and for all always finite dimensional and their vector spaces over the same set the scalar set, is that ok? Then what happens is the first thing if S is a subset of V is linearly dependent then T of S so, which is same as the set T of V 1 till V of k whenever S is equal to V 1, V 2, V k then this is linearly dependent.

So, what I am saying is that any linear transformation so, we are saying that any linear transformation sends linearly dependent set to linearly dependent set, is that ok? I do not have a choice here, fine sets dependent to dependent fine. 2nd so, let S be a subset of V. Suppose T of S is linearly independent subset of W alright T S is an element of the range is element. So, I am looking at linearly independent subset of W then S is linearly independent.

So, what we are saying here is that pre image of linearly independent set independent set is linearly independent, fine. This is very important. So, any linear transformation sends dependent to dependent and if I got something in the co-domain as linearly independent, it brings back linearly independent, is that ok?

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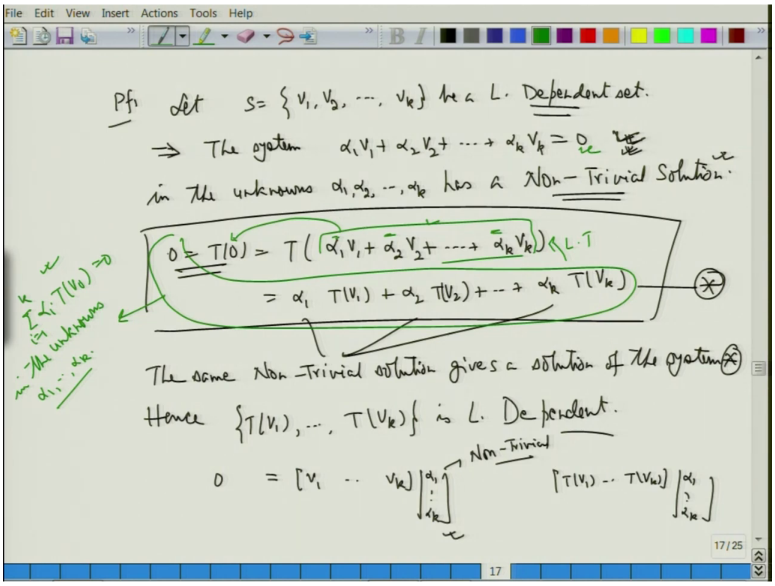

So, let us prove the two things. Proof, fine. So, let S is equal to V 1, V 2, V k be a linearly dependent set fine. Now, it sends this linearly dependent it means what? This implies the system alpha 1 V 1 plus alpha 2 V 2 plus alpha k V k the system this in the unknowns alpha 1 alpha 2 alpha k has a non-trivial solution alright, fine.

So, this condition was very important for linear dependent there was the definition of linear dependence that something is linearly dependent means the system this system will have a non-trivial solution, fine. Or since this has a non trivial solution just applied T 2 it both the side so, I will get 0 is equal to T of 0.

Every zero vector is set to the zero vector itself under any linear transformation which is same as T of alpha 1 V 1 plus alpha 2 V 2 plus alpha k V k and by linearity I can write it as alpha times T of V 1 plus alpha 2 times T of V 2 plus alpha k times T of V k alright. So, I am looking at this part now.

So, what we know is that there is a non-trivial solution; the same non-trivial solution here gives you 0, alright. The same non-trivial solution gives a solution of the system star, alright. Hence, T of V 1 till T of V k is linearly dependent. So, let us understand again.

I have a linearly dependent set. It means that if I look at the system alright the system that I am supposed to form to understand whether independent or dependent. So, that has the non-trivial solution. I know so, alpha 1, alpha 2, alpha k some of them are nonzero I know. They are some of them are nonzero; from there I go to the next step.

So, from this system I went I want to look at another system in which the vectors have changed unknowns are still the same; alpha 1, alpha 2, alpha k are is still the unknowns, the vectors have changed. So, in the new system I see that the same alpha I which were the solution at the earlier stage is the still solution to give me 0, fine and hence everything is fine was thus, is that ok?

So, so, the idea here is again see very nicely that we are just writing earlier we had 0 was V 1 to V k alpha 1 to alpha k. This was our 0, we already had a solution here for nonzero non-trivial solution. Similarly, again we are doing T of V 1 till T of V k and this ok. The important thing was that T of 0 was 0 itself. So, the same thing we will give you, fine.

So, how do you go back? So, understand that when you want to solve a system you look at this part, fine. These are system you would like to construct to show that T of V 1 T of V 2 T of V k is linearly dependent. So, you have to consider the system summation alpha i T of V i i equal to 1 to k equal to 0 in the unknowns alpha 1 to alpha k is what you need to consider fine.

This will lead you to this part fine and I know that. This has a solution because of this previous part. So, this will give me 0 here. So, this vector will give me 0 here and therefore, I will get that, is that ok? So, you are not doing it directly you are going it indirectly. So, understand it nicely I wrote it like this, but the idea behind is that you look at the system, proceed, use linear transformation to go back here.

From here I know that there is a choice of alpha i's, there is a choice of alpha i's which gives me 0. Put that choice alphas to get 0 and therefore, you get a 0, is that ok? The same alpha i's help you.

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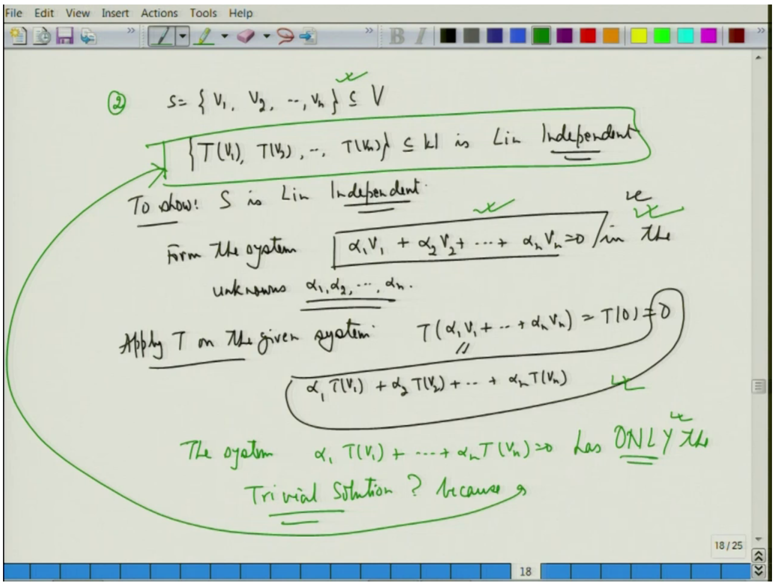

The 2nd one; the 2nd theorem was that if so, I have been given that suppose see some set which is V 1, V 2, V n is a subset of V and I have been given that T of V 1, T of V 2, T of V n this a subset of W is linearly independent. This is given to me I have to show S is linearly independent fine.

So, again since I have to look at linear independence dependence I will have to look at the system of equation. So, again look at the system of equation. So, form the system alpha 1 V 1 plus alpha 2 V 2 plus alpha n V n is equal to 0 in the unknowns alpha 1, alpha 2, alpha n. So, look at the system, form the system alright. So, here I do not know the values of alpha is I am suppose to find those values alright, but now I can apply T to it, fine.

So, apply T on the given vector on the given system alright. So, if I apply T on the given system what I get is that T of alpha 1 V 1 till alpha n V n is T of 0 which is 0; 0 is sent to 0, but this is also equal to alpha times T of V 1 alpha 1 time this plus alpha two times T of V 2 plus alpha n T of V n alright. So, from this system now I am going to this system alright.

In the previous example, we are started with this system and went here, the other way around, alright. Here now I am looking at I am a starting with this system and then going to this system, is that ok? Now, if I look at this system. So, the system alpha 1 T of V 1 till alpha n T of V n is equal to 0 has only the trivial solution. Why? Why does it happen only the trivial solution?

Because we have been given that this is linearly independent because of this linear independence here, is that ok? So, this set is linearly independent therefore, this has only the trivial solution. So, therefore, this will have only the trivial solution and therefore, we have that this set is linearly independent, is that ok? So, that is what it is.

Understand it nicely, spend some time. There is nothing special, but at least you should understand that any linear transformation picks dependent sets to dependent sets sends dependent to dependent and it takes its pre-image. So, if something in the co-domain is independent the pre image will also be independent, alright.

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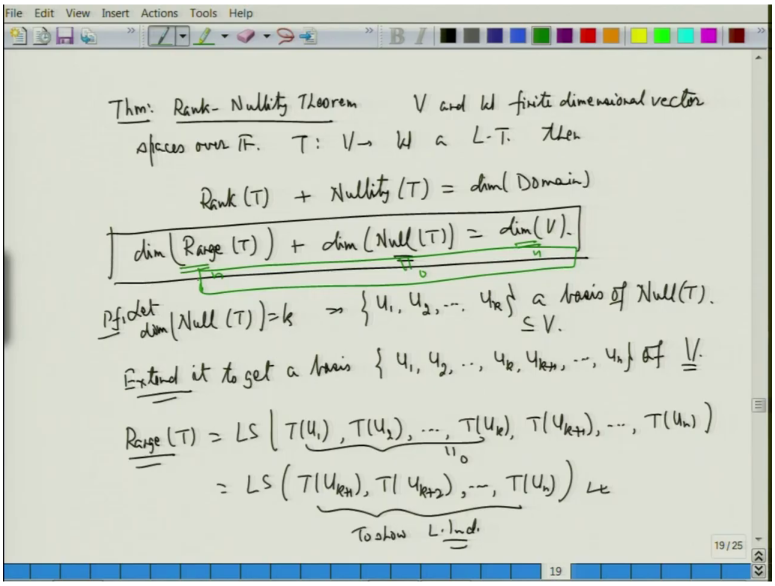

With this idea you have this theorem I will not prove it because the same idea was used in the previous part also what is called the rank – nullity theorem. So, V and W finite dimensional vector spaces over F T from V to W a linear transformation, then rank of T plus nullity of T is equal to dimension of domain which is same thing as saying that range of T look at this dimension of this plus dimension of null space of T is equal to dimension of V, is that ok?

So, we proved it earlier the same proof. So, the idea that I will just write the proof here alright. So, look at nullity of T null space. So, let dimension of this is equal to k implies you have u 1, u 2, u k basis of null space of T, fine. This is a subset of V. So, I can extend it to form a basis of so, extend it extend it. So, what we know is that every linearly independent set can be extended to form a basis.

Here is finite dimensional that is very very important. So, we are saying that extend it to get a basis u 1, u 2, u k u k plus 1 to u n get a basis this of V, is that ok? What we know is that range of T is generated by linear a span of T of u 1, T of u 2, T of u k, T of u k plus 1 so on till T of u n. This is important range of T is generated by the linear span of the images, alright. This is why the first idea was fine.

So, this is same as linear span of T of u k plus 1 T of u k plus two till T of u n the rest of them these are the 0 vectors. So, they do not give us anything and when again go back use those some ideas that you get to prove that this is linearly independent set to show linear independence, try that out yourself.

Just follow those things, you will get the answer. I do not want to waste my time there fine or you can look at the notes, the transcripts where the note the proofs will be given alright. So, try it out yourself. I am not going to do that. The next idea is what is called when do we say.

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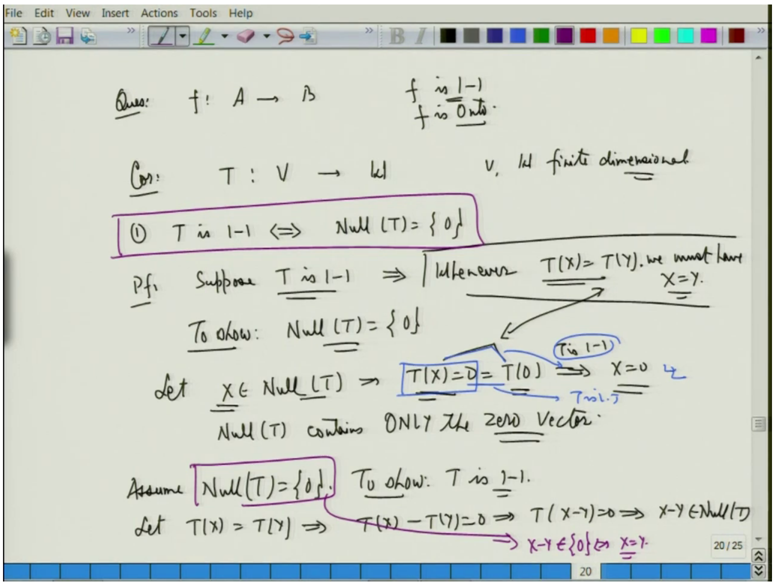

So, question so, if you remember we used to have a function f from any set A to any set B, when do we say that f is 1 – 1 and f is onto, alright? fine. So, similarly here also you can use the rank nullity theorem to say some things.

So, let us look at some corollary of that corollary. So, again I am assuming things are finite dimensional feed to W right is that ok. So, T is from V to W V, W finite dimensional finite dimensional. So, 1 T is 1 – 1 if and only if null space of T is the zero vector. So, let us look at the proof of this proof. Suppose, T is 1 – 1; so what was 1 – 1? 1 – 1 means that whenever f X is equal to f of Y then I must have X is equal to Y you alright.

So, the idea here was so, suppose. So, T is 1 – 1 means whenever T of X is equal to T of Y you must have X is equal to Y alright fine. So, let us look at the T X is equal T Y. So, let us rewrite it. So, since it is a linear transformation, therefore I would like to use that idea. So, T

X minus T of Y is 0 this implies that, sorry. So, this is what it is. So, I have to show. So, null place of T or the kernel of T is the 0 vector, I have to show this, alright.

So, let X belongs to null space of T this implies T of X is 0 that is the definition. So, I am assuming that T is 1 – 1. So, this is what the definition of 1 – 1 means that whenever T X is equal to T of Y we must have X is equal to Y, fine. We want to show that T is 1 – 1 implies null space of T is 0. So, take any X belong to the null space of T by definition of null space T of X is 0, but I know that 0 is always equal to T of 0 alight. The zero vector of the domain is send to zero vector.

So, what we see here is that this part is telling me that T of X is equal to T of 0 and therefore, here I can apply T is 1 – 1 to get that X is equal to 0 alright look at this definition, but here whenever T X is equal to T of 0 X must be equal to 0 fine. So, you have shown that null space of T contains only the zero vector alright you have taken any X is the null space.

So, by definition of null space this is you know because the definition of null space this is be there because of T is linear transformation and now this part gives you that which T is 1 – 1 gives you that X is 0, is that ok? The other way around now so, assume null space of T is 0 assume null space of T is 0 to show T is 1 – 1.

So, assume so, let T of X is equal to T of Y, I need to show that X is equal to Y. So, this part gives me T of X minus T of Y is 0. This you gives me the T of X minus Y is 0, this gives me that X minus Y belongs to null space of T. But, null space of T is just the zero vector alright. So, this together will imply that X minus Y belongs to the set and which is same thing as saying that X is equal to Y, fine.

So, we have shown that T is 1 – 1 if and only if null space of T is 0, fine.

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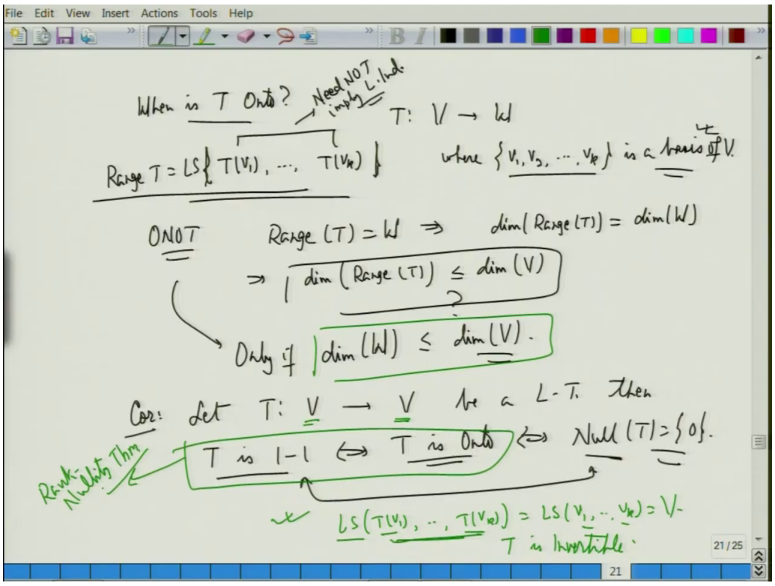

The next thing we would like to look at what happens to onto, when can I say it is onto. So, when is T onto? So, I have T from V to W, fine. What you know is that if I look at T fine I need to look at T of V 1 till T of V k this where V 1, V 2, V k this is a basis of V. So, where if I basis here I have to look at this. So, this gives me range of T as linear span of this here, is that?

If I want onto so, onto will imply that range of T should be equal to W dimension of range of T should be equal to dimension of W or implies that can I say from here. So, if I look at here. So, I can say the dimension of range of T is less than equal to dimension of can I say this. What we have seen is that V 1 to V k are linearly independent need not imply that this is linearly independent; need not imply need not imply linear independence alright.

If we does not linearly independence, the dimension of this will be less than equal to k and what was k? k was the dimension of V alright. So, therefore, this is true and have a hence I can think of onto. So, onto comes into play only if dimension of W is less than equal to dimension of V, is that ok? As such I cannot guarantee anything I can only say this fine.

So, there is this corollary. Well, I will just write the corollary that let T from V to V be a linear transformation then T is 1 – 1 if and only if T is onto if and only if null space of T is the zero vector alright. So, look at this, we have already shown that T is 1 – 1 implies this part. I have already shown this part. This part we will to use this idea of dimension.

So, for us T is onto will imply that though both are V itself. So, this will imply that linear span of T V 1 till T V k should be equal to linear span of V 1 to V k which is your V itself. So, what we are saying is that from V 1 to V k I am going to T V 1 to T V k. So, in some sense you are saying that T is invertible.

I have not yet defined what do I mean by a function being invertible, that you already know when is the function f is 1 – 1 and onto, fine. As soon as you have said that these are linearly independent you get that null space is 0 because of the rank nullity theorem who says that if this has dimension n, this has dimension n.

So, this will have dimension 0, is that ok? Fine. So, the rank nullity theorem tells you that this will itself be valid alright. So, this part looking at this part is coming from the rank nullity theorem is that ok.

So, I will end here. In the next lecture, we will look at what are called isomorphisms and so on and relate these ideas, fine. So, that is all for now.