Linear Algebra
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Lecture – 31
Results on Linear Transformations

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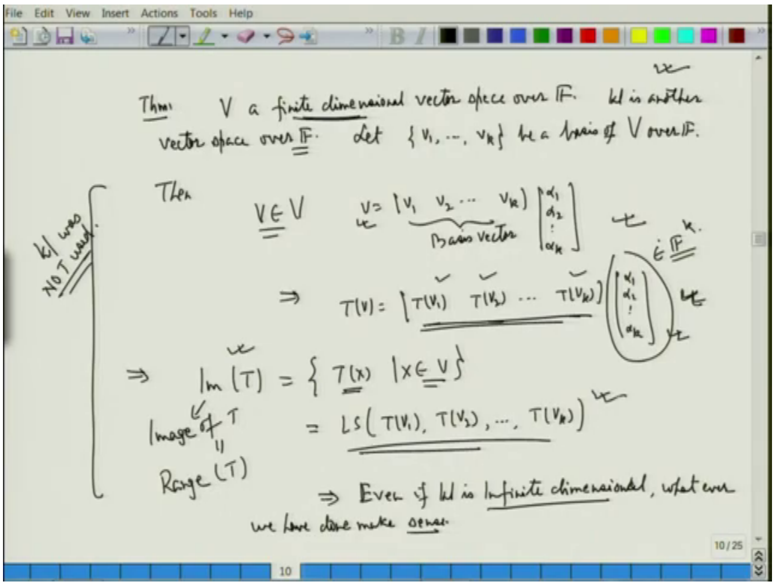

Alright, so in the previous class we saw that the most important result that we saw was alright theorem that V a finite dimensional vector space over V over F, alright. Suppose, V is a finite element vector space V over V and T is a linear transformation from V to W. Did I write that V to W? Alright, so V finite element vector space W; W is another vector space over F, alright. I am not saying that W is finite dimensional, I am just saying that V is finite dimensional, alright. So, that V as a basis, fine.

So, let v 1 to v k be a basis of V over F, then what we had was then I can get any v, take any v belong into capital V; I can write v as v 1, v 2, v k basis vectors, and some scalar which is alpha 1 alpha 2 alpha k; is that ok? This is my v. Once I have this as my v, then this implies that T of v is nothing but T of v 1, T of v 2, T of v k, fine; but this alpha 1, alpha 2 to alpha k remains the same.

There is no change there, this is what we had seen, fine. What we had said was that the image of v is known as soon as we know the image of the basis vector. So, these are the image of the basis vectors and therefore, it is just those alphas coming into play, nothing else, fine.

So, this is very important you need to understand. So, what we are saying from here is that, so from here I want to imply that if I want to look at image of T. What is image of T? So, this stands for image of T. This is also called a range of the function, range of T is all T x such that x belongs to the v, fine.

So, therefore, if I look at this idea, what we see is that, it is nothing but linear combination of these vectors. These alphas are there, they are just the scalars. So, you are looking at linear combination of this. So, what we are seeing here is that, it is linear combination of T of v 1, T of v 2, T of v k. Is that ok? So, this is what is important.

So, T of x, I am looking at, here x belongs to v. What we see here is that each of these T of x is known as soon as I know the images here of the basis vector T v 1, T v 2, T v k and then, I am just multiplying by certain scalars, which are elements from, so if I look at each one of them, they are elements from F; but overall, they belong to F of k, fine. If alpha is where real numbers, I am looking at R to the power k and I am just multiplying by that and nothing else.

So, what we are saying is that image of T comes from just looking at the linear span of T v 1, T v 2, T v k. So, this is the way that idea also is helpful to us. Now, as a corollary of this idea that T is known as soon as you know on the image.

So, image of T is this implies quite a lot of things. This will imply that even if W is infinite dimensional implies, even if W is infinite dimensional and even if this is finite dimensional everything make sense. So, whatever we have done is make sense. We have done make sense.

The whole of this argument W was not used at all; W was not used. This is what is important; nowhere W was used, fine. And then, we will come to a part, we will use this idea in finite dimensional or is the dimension of W is greater than or dimension of V, then you cannot have an onto function and things like that because of this part, alright. So, we will come to that afterwards.

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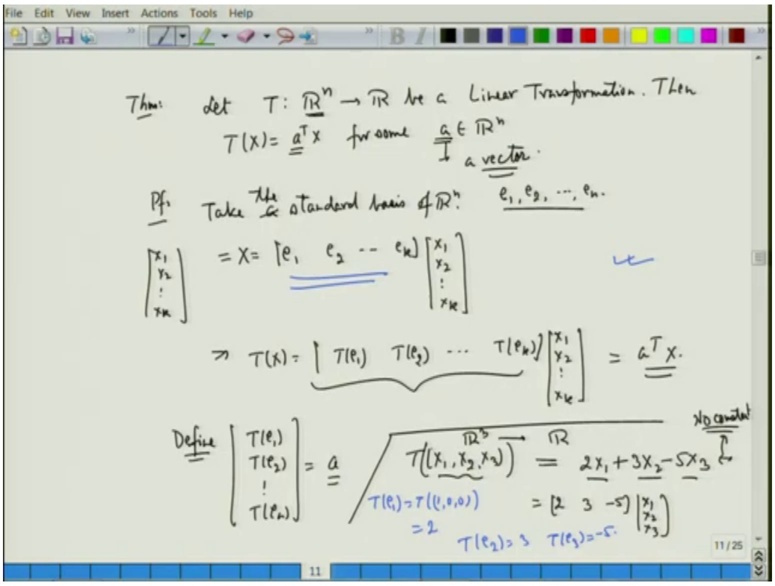

So, let us look at the first theorem which is important theorem. Let T from R n to R be a linear transformation; then, T of x is equal to; so, this is a vector a; a transpose x for some a belonging to R, n alright. So, this is a vector, not a scalar quantity, is that ok, fine. So, let us try to prove this. So, what we are saying is that I am starting with R n, I take a standard basis of R n. So, take a standard or take the standard basis, the standard basis of R n. So, what are the standard basis? e 1, e 2, e n.

So, I can write x as e 1, e 2, e k and if x is equal to say x 1, x 2, x k; then, this is same as this times x 1, x 2, x k, there is no change there, fine because this was a standard basis and this will imply that T of x by definition that we have already seen is T of e 1, T of e 2, T of e k times x 1, x 2, x k, fine. I want to get this a. Can you see that, this is your a, alright. So, define alright define T of e 1; so, define T of e 1, T of e 2, T of e n. This vector as my vector a. If I do this, I get this as a transpose x as a result. Is that ok?

So, you can see that direct idea gives you the answer, fine. So, for example, as if you want to look at an example, I defined T of say x 1, x 2, x 3 suppose I have got R 3 here for me and I am going from T from R 3 to R, it means what? Each of them has to come only once in some sense, so it has to look like some; so, if I look at this linear transformation, it will look like some 2x 1 plus 3x 2 minus 5x 3, things like that. I cannot put any constant; no constant.

Why no constant? Because linearity will be lost. This is what we are seeing that you cannot have linearity, if you have a constant term for linear transformation. We also cannot have a square, cube, a square root, sin x, cos x, so those things are not allowed, only linear terms are allowed fine in linear transformation and this is nothing but we are looking at 2, 3, minus 5 times x 1, x 2, x 3, fine.

So, this is so if you look at here in this example, what is T of e 1? T of e 1 means x 1 is 1 alright, this means that you are looking at 1, 0, 0, so which is nothing but 2. Similarity, T of e 2 is 3 and T of e 3 is minus 5. So, you can see that this theorem is valid, fine. So, now, let us look at so for e, when I am looking at a standard basis everything is nice. It is very easy to compute it. How do I compute when I do not have a standard basis? Alright, so let me take an example to look at that part.

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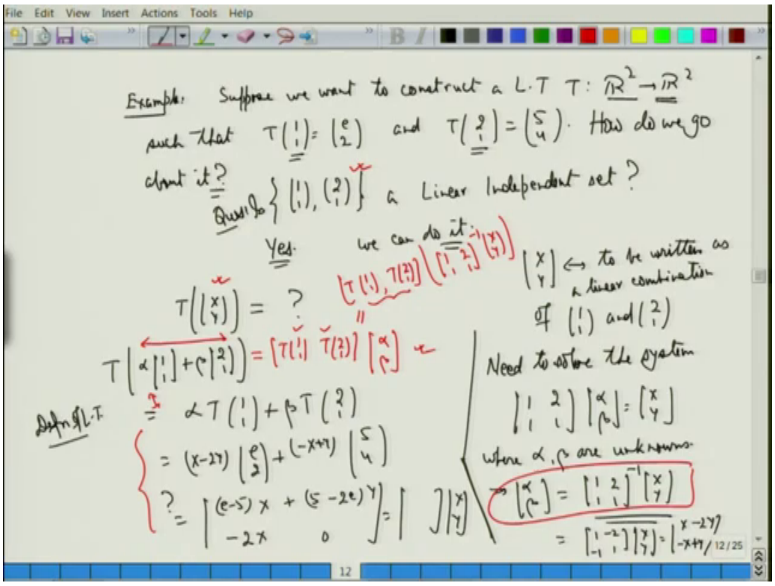

Example; so, suppose I want to construct, question is suppose we want to construct a linear transformation T from R 2 to R 2, such that T of 1 1 is e and 2 and T of 1 minus 1, T of say 2 1 is say 5 comma 4, I want to construct suppose you want to construct this fine, how do we go about it?

So, how do we go about it? Fine, so the first thing, we need to see is look at these two vectors; the vectors of the domain that I am talking of here. The vectors are 1 1 and 2 1. So, question is first thing is first question that you need to answer is, is this a linearly independent set? Alright, the answer is yes, fine. Since the answer is yes, I can do it; so, we can do it.

We can do it that is the first thing that we have to understand. Now, so, I have to define this. I am defining from R 2 to R 2, I need to define it on a general vector x and y. So, what should be T of x will look like? What should this be? Fine, so what we know is that x and y is any vector, I have to write it as this to be written as a linear combination of 1 1 and 2 1.

So, if I want to write it as a linear combination of 1 1 and 2 1, I need to solve, I need to solve the system 1 1, 2 1 times c alpha beta is equal to x y; where alpha, beta are unknowns, fine. I need to solve it and which is same thing as saying that this implies that alpha beta is equal to 1 1, 2 1, this inverse times x y, fine. Is that ok?

So, let us compute it what it is. So, the inverse of this is I hope I am correct, because I always do mistakes. So, determinant of this matrix is 1 minus 2 that is minus 1. So, I should get minus 1 coming into play. So, maybe minus 2 here or minus 1 here itself and minus 2, I think let us, 1 2 into 1 minus 1 is; so, 1 multiply by 2. So, 1 so minus 2 multiplied by 2. So, minus 2 and 2, this is 0, this is minus 1.

The first one is fine, alright; the second one, 1 1 into this will give me 1 minus 2 and minus 1 1 which will give me 0 here, y, so this is ok, this is the inverse. So, no problem. So, this times x y which is nothing but x minus 2 y, x minus 2 y and minus x plus y, fine. So, alpha is this.

So, what we are saying is that this is equal to T times x y is nothing but alpha. So, alpha is so it is alpha times 1 1 plus beta times 2 1. So, by definition of linear transformation; definition of linear transformation, this is alpha times T of 1 1 plus beta times T of 2 1. Alpha here is x minus 2 y times 1 1; beta here is minus x plus y times T of 2 1 sorry, what was T of 1 1? T of 1 1, so it is I have write T of 1 1.

T of 1 1 is e comma 2 and T of 2 1 is 5 comma 4. So, I get this as x is coming with e, so what are the coefficients of x? e minus 5 times x, y is 5 minus 2 is so, plus 5 minus 2 e times y, then here again something, so x is 2 minus 4. So, minus 2 x I hope. So, please check I always do mistakes, alright.

So, minus 4 y and plus 4 y is 0. Is that ok? So, plus 4 y and minus 4 y is 0. So, this is something again a matrix times x y. This is what I want you to see that you can compute it that is one thing, fine. So, I would like to do it in a different way now. So, what I would like to; so, that there is an understanding here.

So, what exactly we have done if you see, I wrote x y as alpha times the first vector, beta times the second vector, fine and I am doing this. So, what I am doing is if I look at this part this is nothing but T of 1 1, T of 2 1 I am writing this and then, alpha beta. This is what I am writing here, fine.

Just look at this part for I wrote this as a definition of linear transform, but basically what I have done is this, and that is what I have been emphasizing that I want to look at images on the basis vectors. These are the basis vectors alright and what is alpha beta? Alpha beta is supposed to be this.

So, I can expand it I have done that part, but it is also this. So, what I am saying is that I have nothing but this is also equal to T of 1 1, T of 2 1, this is a matrix and then, I am multiplying by 1 1 2 1 inverse x 1, so I am doing this fine, that is all I am doing here.

I am writing alpha beta here with this, fine and this if I go here if I write further here, I can think of this nicely. Again, this 1 1 2 1 this. So, what I am doing is alright, so let me just leave it as it is, I think that will be better. There is too congested here, fine. So, I want you to understand this in this example, we could get it because of this being a linearly independent set. Let me look at another example before I come to the main theme, alright.

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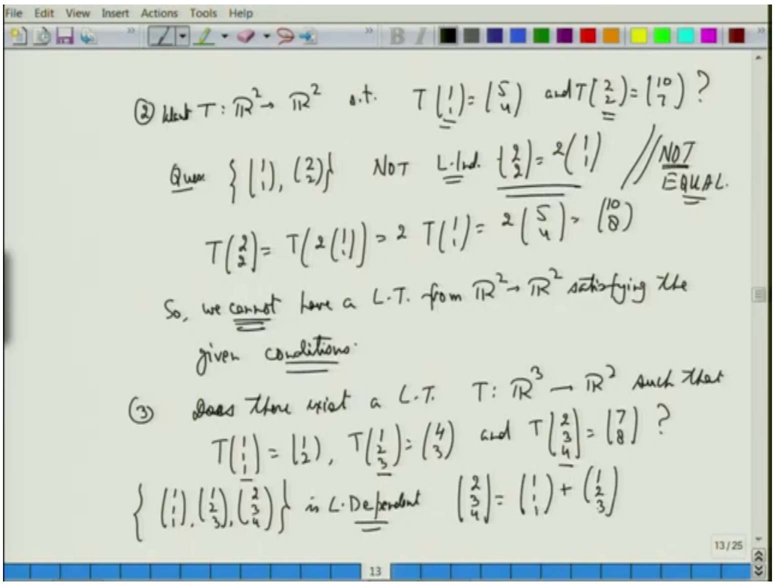

Second theme is, so in example 2, want T from R 2 to R 2 such that T of 1 1 is 5 comma 4 and T of 2 2 is 10 comma 7, fine. I want a linear transformation T with this property all right. So, again as I said questioning first the question that you need to answer is look at this domain vectors, are they linearly independent. So, if I look at 1 2 and 2 2 not linearly independent and not only that we are also able to say that 2 2 is 2 times 1 1.

So, by definition of linear transformation, what we need is that T of T 2 2 should be equal to T times 2 of 1 1; should be equal to 2 times T of 1 1, this is what I am saying which should be 2 times 5 4 which is 10 and 8. So, these two are not equal, fine. Since, they are not equal, you cannot have a linear.

So, we cannot have a linear transformation from R 2 to R 2 satisfying the given conditions. Is that ok? So, as I said you have to first look at things and then proceed further. One more example, alright or I will do some more examples so that we understand better, 3 alright.

So, does there exist a linear transformation T from R 3 to R 2, such that T of 1 1 1 is 1 2, T of 1 2 3 is 4 3 and T of 2 2 2 or T of let us make something different, so that there is a so 2 3 4 is equal to 7 8, alright. So, let us look at this part, can I do such a thing? So, as I said, we need to look at the domain vectors. So, look at the domain vectors 1 1 1, 1 2 3, 2 3 4. Look at this set, now here I see that 1 plus 1 is 2, 2 plus 1 is 3, 3 plus 1 is 4, so is linearly dependent, not only that we see that 2 3 4 is equal to 1 1 1 plus 1 2 3, is that ok?

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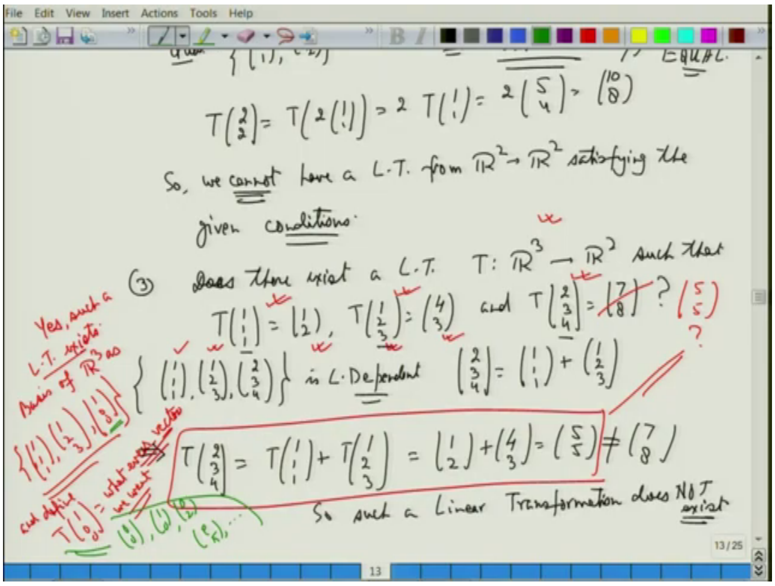

So, this implies alright, the T of 2 3 4 should be equal to T of 1 1 1 plus T of 1 2 3, which is same as 1 2 plus 4 3. So, it should be 5 5 alright, which is not equal to 7 8. So, such a linear transformation does not exist. Is that ok? So, this is what you have to understand that whenever you want linear transformation, what we understand is that linear transformation is known as soon as we know it on the image, fine.

So, image of what? The basis vectors, alright. So, I want to need to go to the basis vectors and see what is happening as such fine, that is very important for us; otherwise, we will have a problem.

So, at each stage, we need to see the image on the basis and then proceed. So, here for example, now in here in for example, if I forget what 7 8 and I say comma 5 comma 5, then what happens? So, does such linear transformation exist? The answer is yes, such a linear transformation exists, because if I look at these, then these conditions are valid for me, I just check that I get 5 5 here, which is what it is and therefore, such a linear transformation exist.

So, but how do I define it? So, I know that on 1 1 1, 1 2 3 these are two linearly independent vectors and they have the image; but, R 3 is defined using 3 vectors, it has basis consisting of 3 vectors.

So, I can just take anything which is linearly independent of these two, fine. For example, I can take my third vector. So, I can take the basis as basis of R 3 as 1 1 1, 1 2 3 alright and I hope I think 1 0 0 is linearly independent I think fine, because I cannot get 1 0 0, I can easily see fine and therefore, I can take this and define T of 1 0 0 as whatever vector I think, whatever vector we want. Because, the main condition was these three conditions have to be valid. So, what we have done is, we have verified that these three conditions are valid for us.

So, I am not going to change the image of 1 1 1 because, that is given to me. I cannot change the image of 1 2 3 because that is also given to me. So, what we do? We go to the basis, create one more vector to get the basis. Now, that vector can be sent to anything. We can send it to the vector 0 0 of R 2; we can send it to 1 0, 0 2, e pi whatever you can think of, this element can be mapped to whatever we want, and still will get all the things that are required for us. Is that ok? So, this is what you have to understand, fine.

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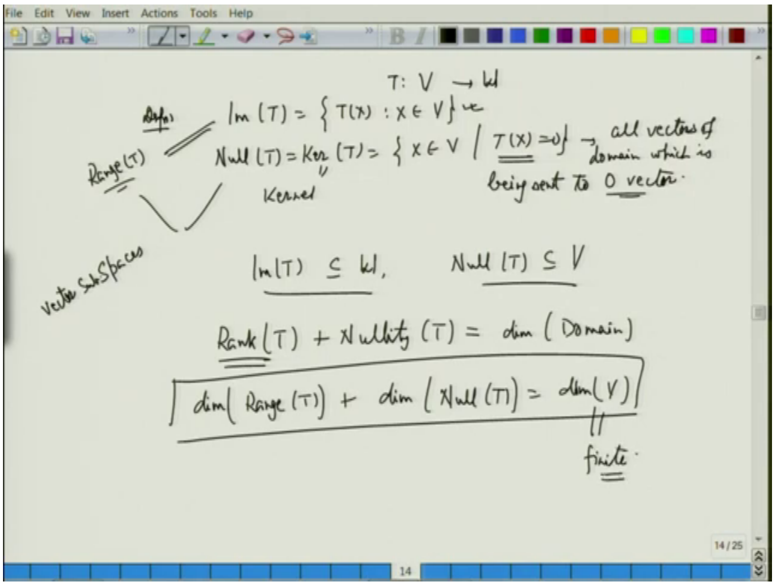

In the next class would like to at I will just state it, so that you come prepare for it. So, there these two definitions that I want you to recollect definition; one definition was is image of T. So, T is a map from say any vector space V to any space W, image of T is T of x; x belonging to V.

Then, there is null space of T and if you remember in terms of matrices, I have also wrote it as Ker of T, we said it as kernel of T, this is all x belong to V such that T of x was 0. So, this is all vectors of domain, which is being sent to 0 vector, alright; everything was being sent to 0 alright and this is the image or the range of T.

So, go back to your matrices and or you can just prove it yourself that range of T and null space of T, they are a spaces; so, vector subspaces, fine. Image of T is a subspace of W, null space of T is a sub space of V. So, we can talk of their dimensions, we can look at the diagonality theorem that we had there, rank of T plus nullity of T is equal to dimension of domain.

So, here rank of T basically means, dimension of range of T; nullity means, dimension of null space of T is equal to dimension of V. So, we will be able to talk about it, once these are finite dimension; otherwise, we cannot talk of things. So, in the next class, we will look at this and try to proceed, fine.

Thank you.