Linear Algebra
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Lecture – 30
Definition and Examples of Linear Transformations

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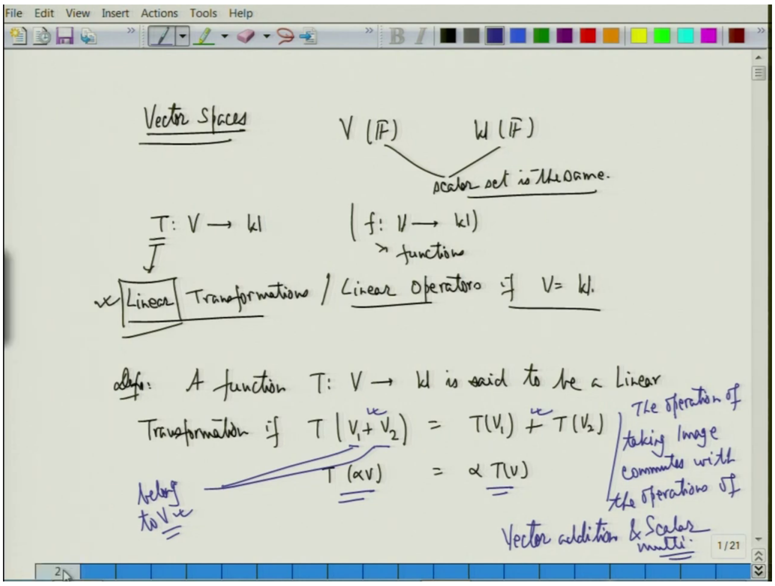

Alright, so let us start the class today. Till now we have lead what we call vector spaces alright and mostly we concentrated on examples coming from R n and towards the end we looked at fundamental spaces related with matrices; row space, column space, null space and left null space. Now, we would like to look at functions. So, I have vector space V over F and I have W another vector space over F itself. So, important thing is that a scalar set is the same.

A scalar set is the same alright, I want to define a function T from V to W. So, generally we write f from V to W. Functions were always represented with f alright, but mostly we write here T, because we use the word what are called linear transformations or what are called linear operators, if V is same as W alright, this is the language that we use linear transformation and linear operators.

There are function that you can define from V to W, but then you may not make much sense of it unless, your function is having this property of linearity, alright. So, first let me give you a definition and then look at some examples, alright. So, definition; a function T from V to W is said to be a linear transformation if T of v 1 plus v 2 is equal to T of v 1 plus T of v 2 and T of alpha V is equal to alpha times T of V.

So, what I am trying to say here is that alright, I am trying to stress here that this v 1 and v 2 they are elements of they belong to; so, they belong to V fine, the vector space V, I picked up two elements from V, I am adding it alright the image that I get can also be obtained accordingly that is one thing.

Similarly, there is something like a scalar multiplication, so we have this. What we are say in mathematics is that, the operation of taking image; the operation of taking image so, whenever you have function you take the image of the function. So, taking image alright, commutes with the operations of vector addition and a scalar multiplication, is that ok?

So, what do you are saying is that when I am going from V to W, in some sense my operations are preserved. The operations do not change much, alright. They have some nice property. So, let me make it diagrammatical thing here.

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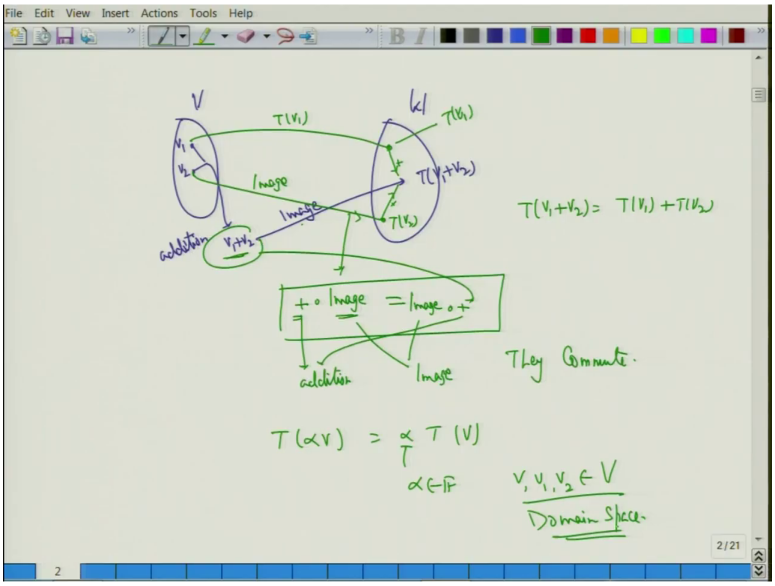

So, I have a vector space V, I have vector space W fine, I have 2 vector C or which is v 1 and v 2. What I can do is I can sum the 2, so if I sum the 2 I will get v 1 plus v 2. So, I have done the addition here, alright. So, this operation is looking at the addition and from here I can go to T of v 1 plus v 2, so this is my image that I am looking at, fine. What I can do is I can also take the image from here to here. So, I am looking at T of v 1, which comes here. So, this is my T of v 1, fine.

Similarly, v 2 will have some image T of v 2, fine. So, this is again the image that I am looking at fine. Now, what this part; so, our condition was that T of v 1 plus v 2 should be equal to T of v 1 plus T of v 2. So, what it says is that if I want to add these two vectors, the sum must go back here itself, there is no other choice. So, when I want to add here this will be the same vector, is that ok?

So, what you are saying is look at this, this is called the image alright, and then you are adding alright; so, you are looking first the image and then the addition. This should be same as first take the addition alright, and then take the image, fine. So, you are taking the image afterwards here, first you are taking the addition and then the image. So, what you are saying is, the two operations of addition alright and the operations of image they commute, is that ok?

A similar thing is true, for a scalar multiplication also and that I am not drawing here, but this is what it says that T of alpha V is same as alpha times T of V. So, alpha is a scalar and v, v 1, v 2 they belong to the V, which is the domain space is that ok? So, we will understand the implication of this that, how does this play a role to us. So, first we will look at some examples where T is not a linear transformation, we will have some problems with it, alright. So, let us look at examples now, alright.

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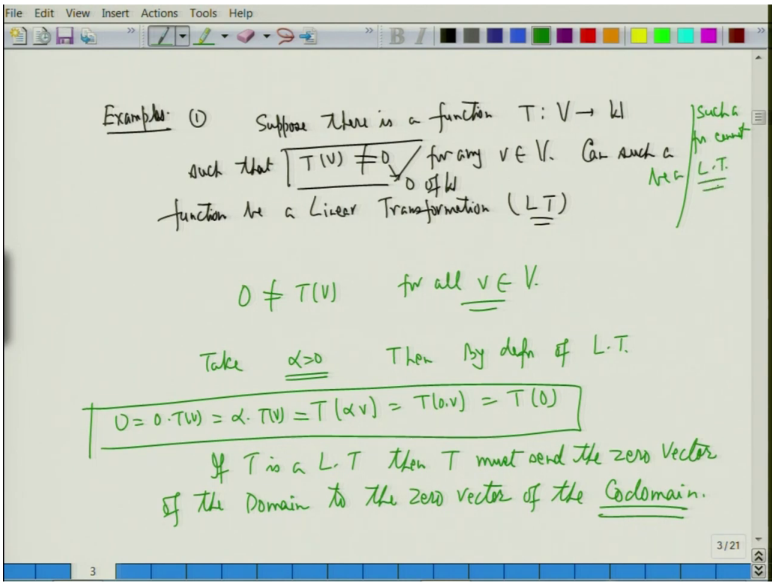

The first example is suppose, there is a function T from V to W such that, T of v is not equal to 0, 0 of W alright 0 for any v belonging to V, alright. Can such a function exist? Can such a function be a linear transformation? I can always have a function but, so, suppose there is a function this such that this happens. Can such a function; such a function be a linear transformation? Important from next, I will be writing L T for Linear Transformation the short form, alright.

So, look at here what you are saying is that T of v is never 0, alright. You are saying T of v is never 0, fine. So, 0 is not equal to T of v you are saying for all v belonging to V, this is what you are saying. So, take alpha is equal to 0, then what we know is then by definition of linear transformation T of alpha v which is same as T of 0 times v, which you know that T is 0 alright fine and by definitions in this linear transformation they should also be equal to alpha times T of v, but alpha is 0 here, alpha taking at 0.

So, it is 0 times T to v, so this has to be 0, alright. So, it says that whatever you do if T is a linear transformation, then T must send the zero vector of the domain to the zero vector of the co-domain, is that ok? So, that is what it is important, that 0 has to go to 0. It means that there is always a point which is being set to 0 therefore, such a T cannot exist. Such a function cannot be a linear transformation, is that ok?

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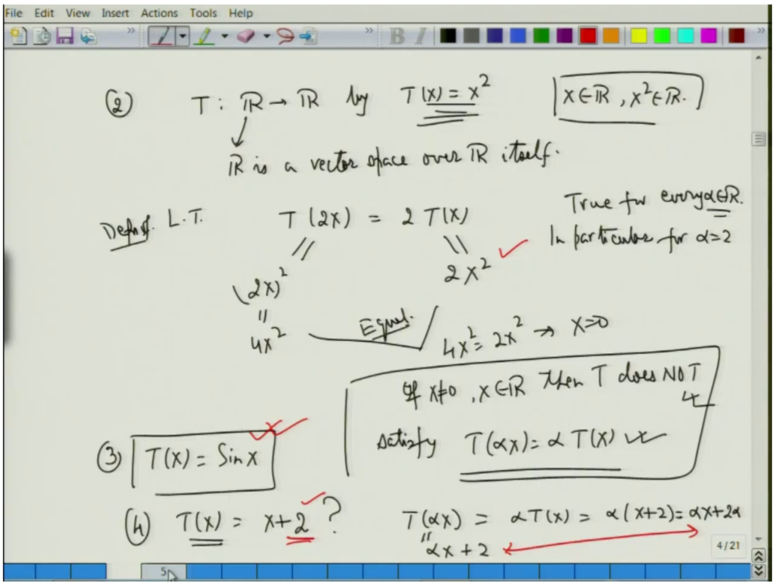

So, that is one example. Let us look at some more examples. 2, define T from R to R by T of x is equal to x square, alright. I know that x is a real number, x square is also is real number fine and therefore, Tx is equal to x square is a meaningful function. There is no problem as such, but we know that R is a vector space over R itself, alright.

So, therefore, by definition of linear transform; definition of linear transform T of 2 times x should be equal to 2 times T of x fine, because it is true for every alpha; true for every alpha belonging to R. So, in particular I am looking for alpha is equal to 2, I am trying to compute it.

So, this side is 2 times x square whereas, this by definition, look at the definition it is T of x is x square. So, it is 2 x whole square, which is equal to 4 x square. So, what you are saying is that these two are equal, alright. So, this is equal; so, 4 x square is equal to 2 x square implies x is 0, alright. So, you are saying that this is meaningful only at 0, at nowhere else it is meaningful, alright.

So, if x is not 0 and x belongs to R, then T does not satisfy, T of alpha x is equal to alpha times T of x is that ok? So, it does not satisfy that is more important, it does not satisfy, fine. So, I cannot have such a function. Similarly, I would like you to see that you cannot have T of x as sin x fine, why? If T is sin x again there is a problem with alpha x that will come into play. Can I have T of x is equal to x plus 2? Fine, so think about it again.

So, T of alpha x is supposed to be alpha times T of x. So, it will give me alpha times x plus 2 which is nothing but, alpha x plus 2 alpha, but at the same time this will give me, look at the definition, it will just give me alpha x plus 2. So, again I have a issue here, what we are saying is that alpha x plus 2 is equal to alpha x plus 2 alpha, alright. So, again I cannot have a constant here, which is coming into play. So, I cannot have a square, I cannot have a constant, I cannot have sin x, cos x and so on.

So, question is that, what type of functions can I have and is it worth reading, because they are most of the examples that we are studied in our class 11th and 12th was x square, polynomials, sin, cos, tan, and so on, all of them they are somehow failing here, fine. It turns out that there are lots and lots of examples, which are very-very important and all of them when I am looking at a vector space V as a finite dimensional vector space they lead it to what are called matrices itself, alright.

So, we have seen lot of examples here, where it is not a vector space. Now, we would look where it is not a linear transformation. Now, I would like to look at how should we define this, is that ok? So, that it makes sense and some examples there off.

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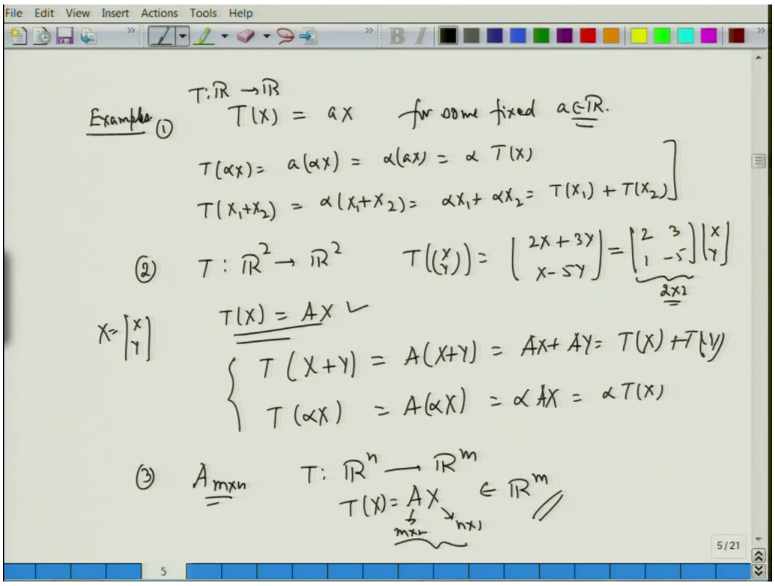

So, let me look at some examples now again back, which are linear transformations solve it examples now, which are positive answers. Examples; I can define T of x. So, T from R to R itself, define T of x is equal to a times x for some fixed a belonging to R alright, then you can see that T of alpha x is equal to a times alpha x, which is same as alpha times ax which is alpha times T of x fine and T of x 1 plus x 2 is equal to alpha times x 1 plus x 2, which is same as alpha x 1 plus alpha x 2, which is same as T of x 1 plus T of x 2.

So, both the conditions are satisfied, so it is a linear transformation, fine. 2, I can define T from R 2 to R 2 by T of X Y is equal to 2 X plus 3 Y and say X minus 5 Y, I would like you to see that this is nothing but 2 3 1 minus 5 times XY, fine.

So, basically what we are saying is that T of X capital X, where capital X is X comma Y is nothing but A times X fine, and you can see here that T of capital X plus capital Y which is supposed to be A times X plus Y is same as AX plus AY, which is same as T of capital X plus T of capital Y fine. I also have T of alpha X is equal to A times alpha X which is alpha times A of X which is alpha times T of X.

So, again both the conditions are satisfied. So, you can see that this A had nothing to do a special that it was a 2 cross 2 in this case. So, nothing as special about it, you can do it for a general also. So, I take any A which is m cross n fine, I want to talk of A times X. So, A is m cross n. So, this X has to be n cross 1 and therefore, I need to define T from R n to R m and I define T of X is equal to AX, fine.

So, once I have done that this matrix multiplication tells me that this is an element of R m alright and as above we can verify that this is indeed a linear transformation, fine. So, matrix as we have got, what more example that you can think of?

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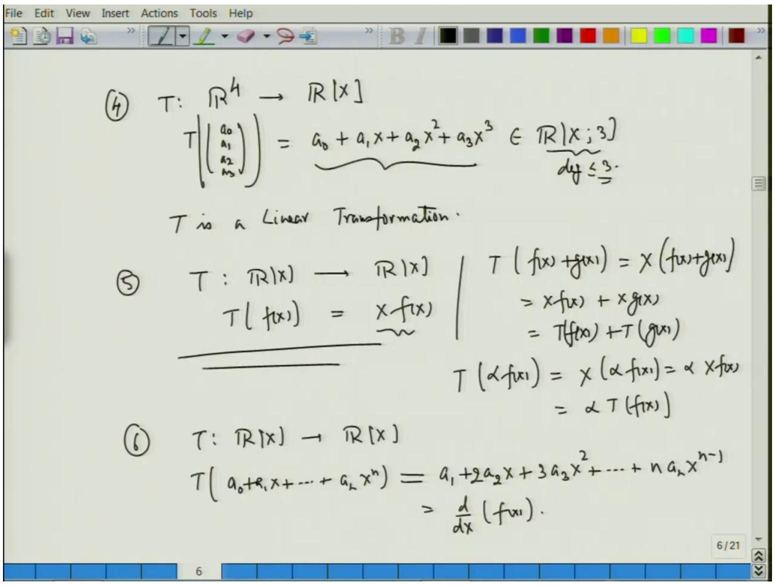

So, let me look at some more examples, I do not know it as example 3 or 4. So, 4th examples for me; I define T from R n or say R 4 to R of all polynomials alright, set of all polynomials with real coefficients and polynomials in x. I define T of, so element R 4 are nothing, but some a 0 a 1 a 2 a 3 I can think of this as four components. I define T of this as a 0 plus a 1 x plus a 2 x square plus a 3 x cube.

So, this in general is an element of R of x and 3 all polynomials of degree less than degree less than equal to 3, because that is the way I am defining here, fine. So, I can define it whatever way you want you can define it like this. So, you can see that T is a linear transformation, just verify it for yourself, 5th I define T from Rx to Rx by T of x or T of fx, T of f of fx is a polynomial by x times f of x is this a linear transform that is the question, fine.

So, what we need to show look at T of fx plus gx two polynomials. So, what we are saying is that we are multiplying the element. So, it is x times fx plus gx, so which by definition is nothing, but x times fx plus x times gx. So, this is equal to T of fx plus T of gx fine, what about the scalar multiplication alpha times f of x T of this. So, this by definition is x times alpha of f of x, which is same as alpha times x of f of fx, because alpha is a constant, it is a scalar number. So, it comes of a same as alpha times T of x, fine.

So, we can say that this is also a linear transform. I am just multiplying by x. What about differentiation and integration? So, 6; I defined T from Rx to Rx by T of say I have a polynomial a 0 plus a 1 x plus a n x to the power n. I define this function to be is equal to a 1 plus 2 a 2 x plus 3 a 3 x square plus so on plus n an x to the power n minus 1. So, what I am saying I am looking at d dx of f of x fine, I am looking at this.

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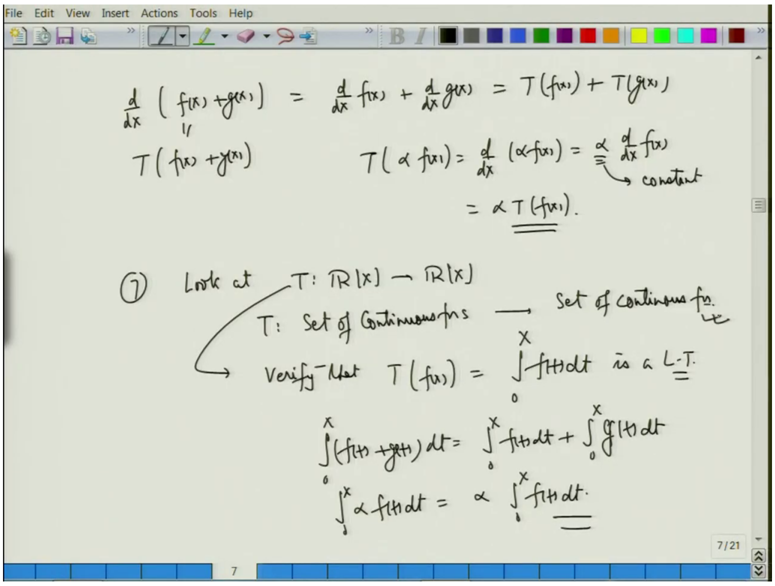

So, remember that if I am differentiating. So, differentiation of so, recall that if I am looking at differentiation of two functions f x and g x. Fine, if I am differentiating two functions, then sum of two functions when I differentiate, it is sum of their differentiations. So, this is nothing, but f x plus d dx of g x. So, this will give me T of f x plus T of g x and this is nothing, but T of f x plus g x.

So, differentiation operator in some sense is functioning nicely. Let us look at the other part. T of alpha of f of x is d dx of alpha f x, fine. So, this is same as alpha times d dx of f x, because alpha is a constant, a scalar quantity and therefore, this same as alpha times T of f of x, is that ok?

So, again this is also an example, one more example before I leave this. So 7, at least one more example. Let us look at integration. So, I want to look at integration. So, I want to look at T from Rx to Rx, you can also look at T from set of continuous functions to set of continuous functions alright, because integration is meaningful only for continuous functions and what are called slightly mode you can go demand integration and so on, try to understand where you can do integration.

So, you can define integration also here. So, verify that T of f of x defined by 0 to x f t dt is a linear transformation, alright. So, if you want to check it; so, what we know is that 0 to x ft plus gt dt is 0 to x ft dt plus 0 to x gt dt; therefore, that will happen and 0 to x of alpha f t dt is alpha times 0 to x, f t dt is that ok? So, you can see that I have given you enough examples to motivate you that they are important, fine.

The most things that we had learnt in class 12th was calculus. In calculus differentiation and integrations were the most important things. So, we see that all of them have been taken care of, alright. Whether you are looking at set of continuous function, set of differentiable functions are also continuous functions.

So, they can be integrated, differentiation and so on, everything make sense. So, we can talk of functions from one vector space to another vector space, where the vector spaces are themselves say 5 times 4 times 20 time differentiable continuous and so on whatever way you want alright, fine.

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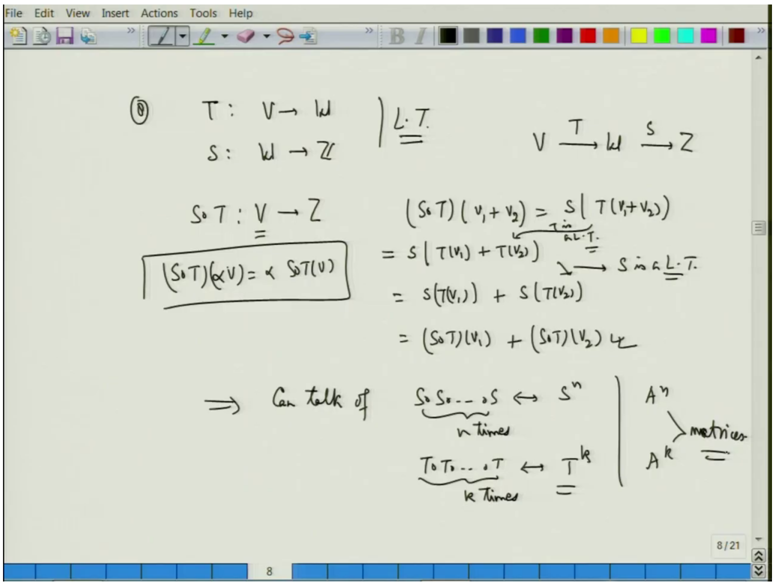

I would also like you to one more the last one I think 8th define; so, I have got suppose I have got T a linear transformation from V to W. I have another linear transformation S from W to Z. So, they are linear transformations fine, question. So, if I look at here V is here, W is here, Z is here, T is coming from here, S is coming here. So, I can talk of composition of function SoT from V to Z fine. I can talk of this; is this a linear transformation?

So, we know that matrix multiplication is a meaningful object. So, you hope that this should also be composition of functions is also a function. So, let us see what happens to this. So, SoT is I have to look at V here. So, it is v 1 plus v 2. So, this by definition is S of T of v 1 plus v 2 that is the composition of function that is the way you write.

So, this by definition is equal to S of T of v 1 plus T of v 2 fine, I can write from here to here, because T is a linear transformation. So, coming from here to here is T is a linear transformation alright, because of that I can come from here to here. Now, from here I can do, I can write it as S of T of v 1 plus S of T of v 2 why I can write this, because again from here to here I am looking at, S is a linear transformation, alright.

So, at one step, it is T is a linear transformation is giving me the one part, the next part it is the S is a linear transformation which is giving me the other part which is same as SoT of v 1 plus SoT of v 2 fine, is that ok? So, you can see that SoT will behave nicely. So, similarly you can show that SoT of alpha v is alpha times SoT of v. So, try that out yourself and show that it is indeed true, fine.

So, now we have looked at quite a few examples for matrix A also. So, from here I can say that this will imply that I can talk of; I can talk of S composed with; S composed with S n times and we generally write it as S to the power n or similarly, if T is the linear transformation T composed with; T composed with T k times, we will write it as T to the power k, fine.

So, they have behaves they are analogs of A to the power n A to the power k where these are matrices fine, so those analogs are there. Let us not worry about that part fine. So, now, let us look at some theorems in this respect what are the theorems here.

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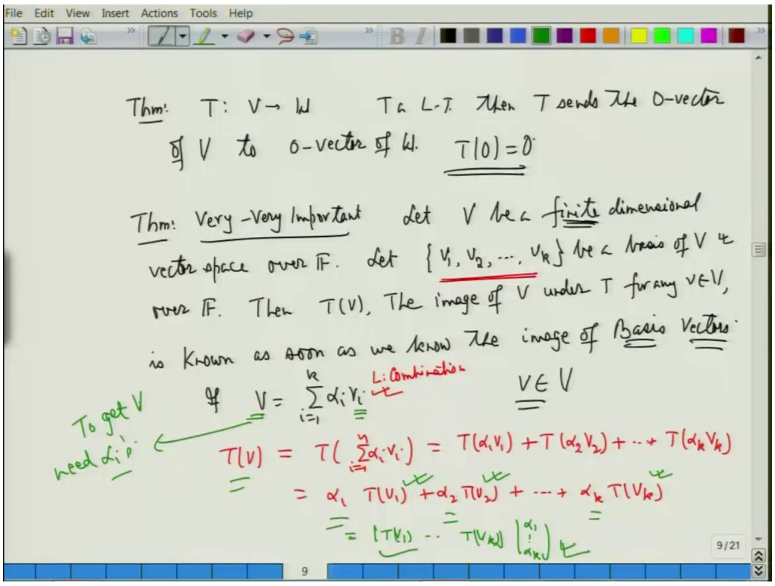

So, lot of examples have been there known examples are also there, some results, small results, one result that we had already proven earlier was theorem that if T is a linear transformation from V to W, T a linear transformation then T sends the 0 vector of V to 0 vector of W or what we are saying is that T of 0 is 0 that we have already seen, fine.

Next, thing is very-very important; vey-very important. The previous result was also important, but this is very-very important in the sense that how do I define T, alright. So, suppose; so, let V be a finite dimensional, this is very important finite dimensional vector space over F.

Let v 1 v 2 vk be a basis of V over F, then T of v the image of v under T for any v belong to V is known as soon as we know the image of basis vectors. Alright, this is what I am writing in terms of the language, but actually in mathematically what we write is that if V is equal to summation alpha i v i, i going from 1 to k.

So, what we are saying is that I have v, v is an element of capital V. So, v belongs to capital V, v 1 to v k is a basis this is a basis alright, yeah. So, this is a basis, what does it imply? Since, this is a basis it implies that I can write V as linear combination. So, I am writing this as a linear combination is that ok.

So, therefore, what I am saying is that if I look at T of v by definition it is equal to T of i is equal to 1 to k alpha i V i, but now linearity will tell me that this is same as T of alpha 1 v 1 plus T of alpha 2 v 2 plus T of alpha k v k which is same as alpha 1 T of v 1 plus alpha 2 T of v 2 plus alpha k T of v k, alright.

So, note here, what I am trying to say here is that Vi’s are known that is given to me this is a basis to get the V, I need to; to get V need alpha i's so the same alpha i's are here fine. So, I just fix my alpha i's, look at the images here T v 1 T v 2 to T v k and I know what is T of v is, alright.

So, what we are saying here is that it is nothing, but T of v 1 till T of v k times this vector alpha 1 to alpha k, is that ok. So, that finishes this theorem you can see that I have written the vector here and the scalars that come into play, alright. So, that is all for now.

Thank you.