Linear Algebra

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Lecture – 03

So, in the previous class we learned what are called matrix addition and a scalar multiplication. Now, we would like to look at what is called matrix multiplication which is very very important thing and the crux of matrix theory. So, as I said it is the crux. So, it means that it has to play an important role and we need to give sufficient time to understand matrix multiplication. So, let us me try that out.

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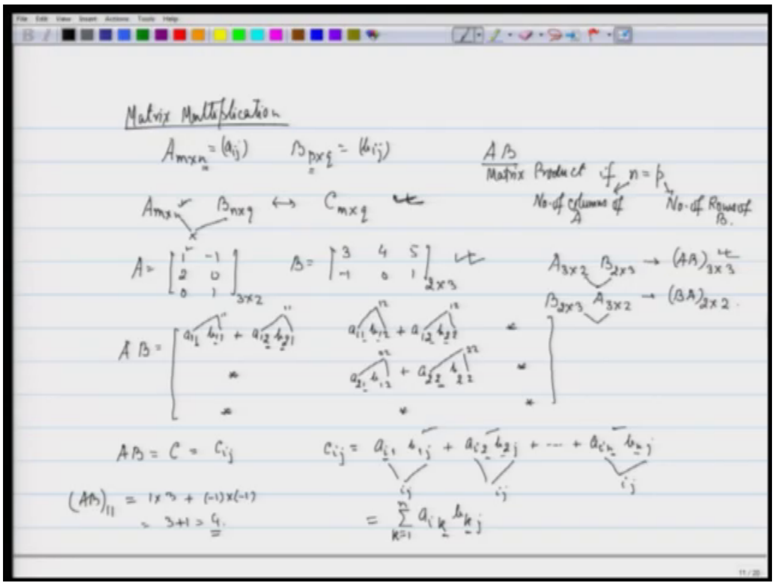

Let us try to understand matrix multiplication. So, we have learned this matrix multiplication in our school, but would like to proceed further and understand it in a better way.

So, given a matrix A m cross n and a matrix B which is say p cross q of different sizes a has entry a ij b has entry b ij, then we can talk of A times B matrix multiplication or matrix product whatever you want to say if n is same as p. So, n means number of columns of A should be equal to number of rows of B.

Now, why it has been defined this way cannot be explained now, but we look at examples and see that why it plays such an important role for us alright, because not just multiplying some things, but more than that. So, as I said we have a matrix product only when m should be equal to p.

So, I can multiply. So, I have matrix m cross n, then if I want to multiply then B has to be other size n cross q and the output of this matrix is a matrix C which has m rows and q columns. This n and n cancels out. So, you are left out with m rows and q columns.

So, let us look at examples to understand it. So, I have take A is equal to 1, 2, 3 well let me take the example that I have taken earlier I think. So, that there will be clarity. So, 1 minus 1 2 0 0 1 and B is equal to 3 4 5 minus 1 0 1.

So, this matrix A is of size 3 cross 2, B is of size 2 cross 3. So, if I want to multiply. So, A is 3 cross 2, B is 2 cross 3. So, these 2 and 2 will cancel out the product A B will be of the size 3 cross 3. Similarly, here it turns out in this example that B is 2 cross 3 and A is 3 cross 2.

So, again these 2 will cancel out I will get the product B A which is of size 2 cross 2. So, in general if I want to look at A times B is defined in this example. So, in the previous example that is A is m cross n b is n cross q then a times b is defined and it is a matrix of size m cross q.

But, when I want to define B times A, then B has q columns, but A does not have q columns. So, that is not defined. So, matrix product there are issues, but in this example everything is nice. So, let us take up example try to understand them.

So, what is the matrix product A B here. So, A B that we learn here is we have to multiply the first entry a 11 with b 11 plus a 12 with b 21 alright. So, note here this one and this one they cancel out in some sense and I have this here the first and the first here. Similarly, 2 and 2 will cancel out here I have got 1 here and 1 here.

The next entry is going to be it is a 3 cross 3 matrix. So, 1 2 entry. So, for 1 2 entry I need to look at a 11 into b 12 plus again a 12 into b 22. So, again observe here that 2 and 2 gives me 1 2 entry here these 2 cancels out this 1 and this 2 gives me the entry 1 2 and this 1 cancels out. So, here I have left out with the 1 1 entry.

Similarly, if I want to look at the 2 2 entry here. So, I have to have 2 here and something here. Again this something has to cancel out here and I should get 2 here. So, this 2 and these 2 will give me 22 entry. I also have a 2 2.

Again these 2 has to cancel out and this 2 should give me these 2 to get me 2 2 fine. So, this is the way I am supposed to multiply. So, I have a one entry here another entry here. So, these are the entry that I am going to get.

So, in general when I write A B is a matrix C alright the entry of C is c ij, then c ij has to have this property that a i has to be there then B has to come with a j. So, that these i and this j they give me ij entry and then the first entry first column first row plus again a i 2 b 2 j again i and j gives me ij here this 1 and this 1 they cancel out.

Similarly, here 2 and 2 cancels out and so on since the matrix of the size n. So, I will go till n a i n b n j this n and n cancels out I am left out with i j entry alright. So, that is the way to understand that the middle part they cancel out and whatever is left out is outside. So, we generally write it as summation a i k b k j k is going from 1 to n then this is the summation sign that we have here this k and k cancels out.

So, I left out with i and j. So, k goes from 1 to n. So, k is 1 means 11 here 22 here and n n here fine. So, let us go back to our matrix that we gave here and let us compute the entry AB of 11 AB of 11 is 1 times 3 plus minus 1 times minus 1 which is 3 plus 1 which is 4 alright. So, let us proceed and compute all of the entries now.

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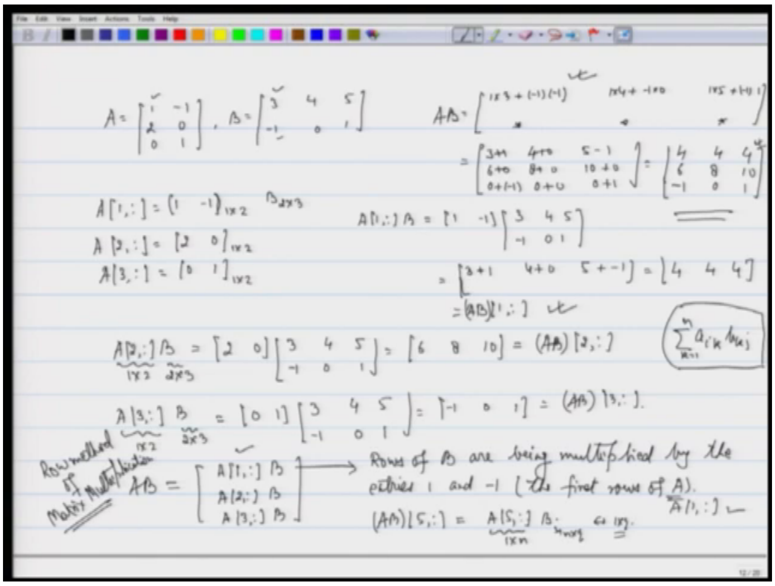

So, again let me write it A is 1 minus 1 2 0 0 1 B is 3 4 5 minus 1 0 1. So, the matrix product AB is this 1 with 3 that is this 3 plus minus 1 times minus 1 then 1 times 4 plus minus 1 times 0 then 1 times 5 plus minus 1 times 1. Similarly, the entries here. I would like you to check that this is nothing but 3 plus 1 is 4 4 plus 0 is 4 5 minus 1 or 4 plus 0 3 plus 1 4 plus 0 5 minus 1.

This is 2 into 3 is 6, then so 2 into 3 is 6 plus 0, then 2 into 4 8 plus 0 2 into 5 is 10 plus 0 0 into 3 is 0 plus minus 1. Then I have 0 plus 0 and I have 0 plus 1 which is same as 4 4 4 6 8 10 and minus 1 0 1 alright. Let me just clean it I think this is too congested.

So, this was 0 plus minus 1 0 plus 0 and 0 plus 1 fine. So, this is the matrix product AB. This we all know how to compute it the way we have computed it. This is what we learnt on out of school. Now, I would like to understand this matrix multiplication in a different way. So, what is the different way. So, let us try to see. So, I have my matrix as the first row of the matrix as 1 minus 1 or let me write this itself. The second row is 2 0. The third row of this of A is 0 1.

Now, if you look at this is a matrix of size 1 row 2 column, 1 row 2 column, 1 row 2 column. B is a matrix which is 2 cross 3 it means that I can talk of this times B. I can multiply these 2. So, multiplying these 2 basically means I am looking at 1 minus 1 times this matrix B which is 3 4 5 minus 1 0 1. This is same as look at here 3 plus 1 4 plus 0 5 plus minus 1 which is same as 4 4 4 which was nothing, but the first row of AB. Just match it with this the first row of AB fine.

Similarly, let us look at the second row of A multiply by B because this is 1 cross 2 B is 2 cross 3. So, it this matrix multiplication makes sense. So, it is 2 0 times 3 4 5 minus 1 0 1. So, it is nothing, but I am multiplying 0 to the second row here 2. So, it is 6 8 and 10. Check that this is also equal to the second row of AB.

Similarly, I can talk of this is a 1 cross 2 matrix again this is 2 cross 3. So, I can again multiply 0 1 times 3 4 5 minus 1 0 1 which is nothing but minus 1 0 1 which is third row of AB. So, if I look at AB what we are saying is AB has this property which is which looks like this times B; this times B this times B.

So, here rows of B are being multiplied by the entries 1 and minus 1 the first rows of A alright and therefore, we had A of 1 i. So, I would like you to understand here one thing that I am multiplying AB. When you multiply AB we learnt AB multiplication using this idea that is a ij a ik b kj and summation over k that was the idea. Our idea was looking at a ij a ik b kj k equal to 1 to n. That was the idea that we had, but that is as per as calculation is concerned we need to learn it.

But, there is another way of learning things here which says that I can compute the first row of AB just by multiplying the first row of A with B alright. If I want to compute the fifth row of AB if I want to compute the fifth row of AB, I just have to look at the fifth row of A and multiply it by B it makes sense because A has A is 1 cross. So, if A was m cross m, then the fifth row is nothing, but 1 cross n and B is n cross q.

So, this makes sense and this gives me a matrix of size 1 cross q. So, it gives me the fifth row of A times B fine. So, I hope you have understood let us see again once I am trying to do that, but when I want to compute AB I can compute AB A times B entry wise the A 11 entry times B 11 plus A 12 into B 21 plus so on. So, I can compute the 11 entry of AB 12 entry of AB 13 entry of B B and so on.

Another way to compute is just compute the first row of AB, the second row of AB, the third row of AB and so on fine. So, this is called the row method because we are looking at the first row of AB, the second row of AB the third row of AB. So, this was our row method of matrix row method of matrix multiplication alright. Now, let us learn what is called the column method of matrix multiplication. So, we will be looking at columns to multiply things alright.

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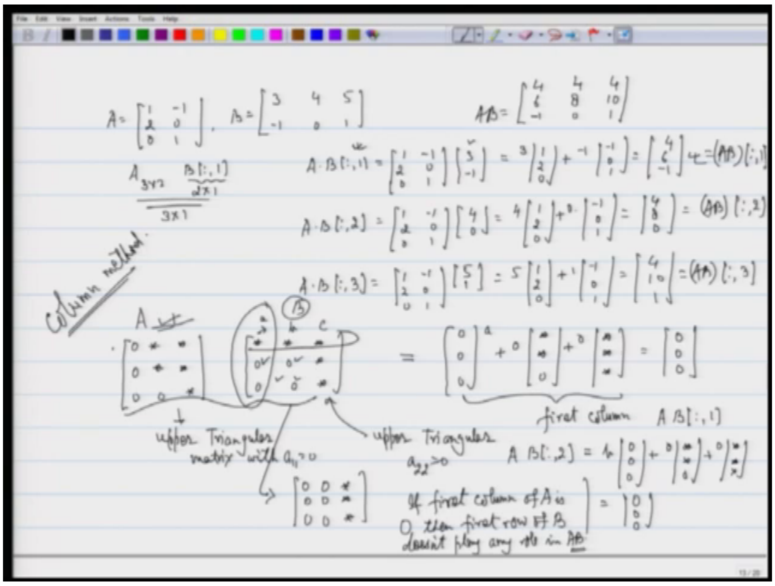

So, again the same example I have A as 1 minus 1 2 0 0 1, B is 3 4 5 minus 1 0 1. We already written what the rows and columns are. Again look at A. A is 2 cross 3. Look at the first column of B. The first column of B has size 2 cross 1. So sorry I wrote A wrongly it is A is 3 cross 2 and therefore, this matrix multiplication makes sense and it gives me a matrix of size 1.

So, let us multiply it out. So, I am looking at A times B of first row it is a first column. So, A is 1 minus 1 2 0 0 1. I want to multiply this with the first column. The first column is 3 minus 1. So, this is nothing, but look at it 3 times the first column 1 2 0 plus minus 1 times minus 1 0 1 which is nothing, but 3 plus 1 is 4 6 plus 0 is 6 and 0 plus minus 1 is minus 1.

So, I would like you to check that this was the same thing in the as in the previous example. So, let me write the previous what was AB. So, that you can match it. So, AB was 4 4 4 6 8 10 minus 1 0 1 alright. Again let us write the other way A times B of the second column.

Again I have A 1 minus 1 2 0 0 1. Now, the second column of B is 4 comma 0. So, it is nothing but 4 times 1 2 0 plus 0 times minus 1 0 1 which is nothing, but 4 8 0. So, match it that this is nothing, but look at AB the second column. Here, it was the first column alright.

Similarly, if I look at the third column is 1 2 0 minus 1 0 1 this times 5 comma 1 which is 5 times 1 2 0 plus 1 times minus 1 0 1 which is nothing, but 5 minus 1 is 4 10 plus 0 is 10 0 plus 1 is 1 which is nothing, but the third column of AB alright.

So, we have learned this also that I can in place of looking at component wise multiplication of AB we looked at computing the rows. We have also now able to compute the columns just multiplying the matrices and the entries alright. So, let us take an example another example to understand it better why do we need such a thing.

Suppose, I have a matrix A alright I have some matrix A, B and C. So, the matrix A we know is 0 here 0 here 0 here. There is some non-zero entry here fine. There is a 0 here alright. So, let me make it also non-zero first alright 0 here.

So, you can see that this is an upper triangular matrix with A 1 1 0. I take another matrix which is again upper triangular, but I have something here which is slightly different. So, this is also upper triangular, but here A 22 entry is 0 alright. I want to multiply these 2 matrices and see what do I get fined.

So, can I multiply matrix entry wise? You can multiply this the first row with the first column look at 0 times a star star times 0 and so on. You can also multiply it as 0 0 0 times a star. This star is your this one. So, say A here B here and C here if I write I am multiplying by A here, then it will be 0 times a star a star 0 plus 0 times a star a star a star.

This will give me the first column which will be nothing, but 0 0 0 alright. So, here I am writing A here alright. Similarly, so here I looked at this times the whole matrix. So, I looked at if this matrix is A this was B, then I have looked at A times the first column of B. If I compute A times the second column of B, it will be B times 0 0 0 plus 0 times a star a star 0 plus 0 times a star a star a star which will again give me 0 0 0.

So, if you multiply these 2 matrix what you get here is 0 0 0 0 0 0 and something here fine. So, this is the way sometimes it helps to understand matrix multiplication. What it tells me that if I look at the matrix A, its first column was 0. So, in my B the first row does not play a role.

So, what I am saying is that if first column of A is 0, then first row of B does not play any role in AB alright. It is only the second entries second row third row of B that plays the role fine because whatever you put it gets multiplied by 0 and you get only 0 as such fine.

So, this is the 2 ways we learned what is call the row method, this is the column method. We will come to some more examples, but now we will look at another example another multiplication alright.

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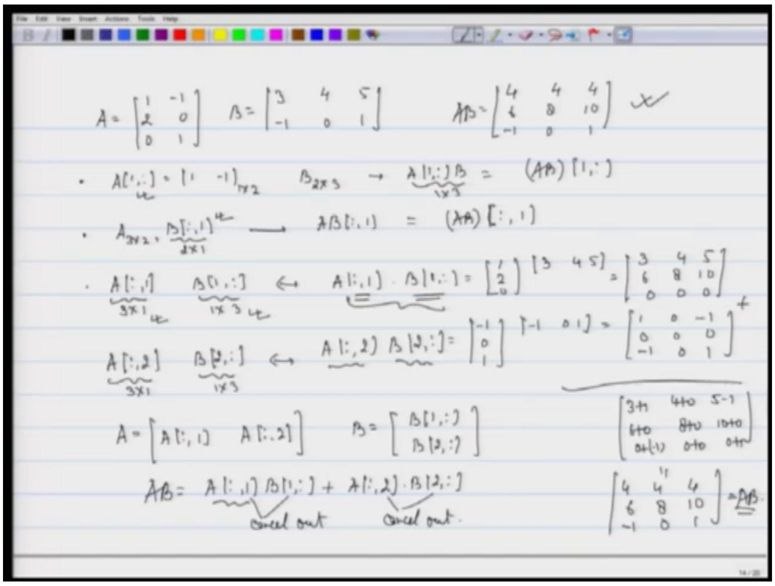

What is called the. So, let us see. So, let me write that first. So, again let me write A as whatever A was for me A was 1 minus 1 2 0 0 1, B was 3 4 5 minus 1 0 1 and the matrix AB was 4 4 4 6 8 10 minus 1 0 1 alright. So, now I would like to look at. So, again. So, here I looked at till now I was able to talk of the first row which was 1 minus 1. This was a matrix of size 1 cross 2 B was 2 cross 3.

So, I could talk of this which is again a matrix of size 1 cross 3. I also looked at A which was 3 cross 2 and B was the column wise I looked at. So, B was 2 cross 1. It gave me a matrix of size I could talk of so AB this. So, this was AB the first row, this is AB the first column. I can also talk of let us try to understand this.

So, here I talked in terms of rows, here I term in terms of columns. Now, what about the other way round something else. So, look at columns of A. The first column of A if I look at this is

A matrix of size size 3 cross 1. If I look at B the first row of B this is of size 1 cross 3 and therefore, this matrix multiplication. So, A times this into B times. Sorry first column and this should be in the first row.

This should have in the first row alright. So, I can multiply these 2. Here multiply these 2 what do I get let us see. So, A I am looking at the first column 1 2 0 and I am looking at the first row of this which is 3 4 5. What I get is 3 4 5 2 times this is 2 into 3 is 6 8 10 multiply by 0 get 0 0 0 alright. So, I could talk of this I can also talk of. So, this was corresponding to the first column, this was corresponding to the first row of B.

Now, I would like to look at second column which is again 3 cross 1 and corresponding row of B which is 1 cross 3. So, I look at A of 2 and B of this. So, this matrix has entry minus 1 0 1, this has entry minus 1 0 1. So, this gives me 1 0 minus 1. This is 0 0 0 0 with 1 minus 1 0 1. If I multiply if I add these 2 if I see if I add these 2 what do I get is 3 plus 1 4 plus 0 5 minus 1 6 plus 0 8 plus 0 10 plus 0 0 plus minus 1 0 plus 0 0 plus 1 which is nothing, but 4 4 4 6 8 10 minus 1 0 1 which is AB.

So, check that this is another way of multiplying and you are getting to get the same answer. So, what exactly we are doing here is in this matrix multiplication is we have written A as the first column and the second column. This is the way we wrote first column second column.

We wrote B as the first row and the second row and we are saying is that AB is nothing, but A times. So, first column of A into first row of B plus second row times sorry second column times the second row. So, the important thing is here again note that this 2 and 2 cancels out this 1 and 1 is cancel out cancel out.

So, this is what is important that you are able to multiply things to all what I am trying to say is that. So, let me write it. So, there are ways of multiplying matrices.

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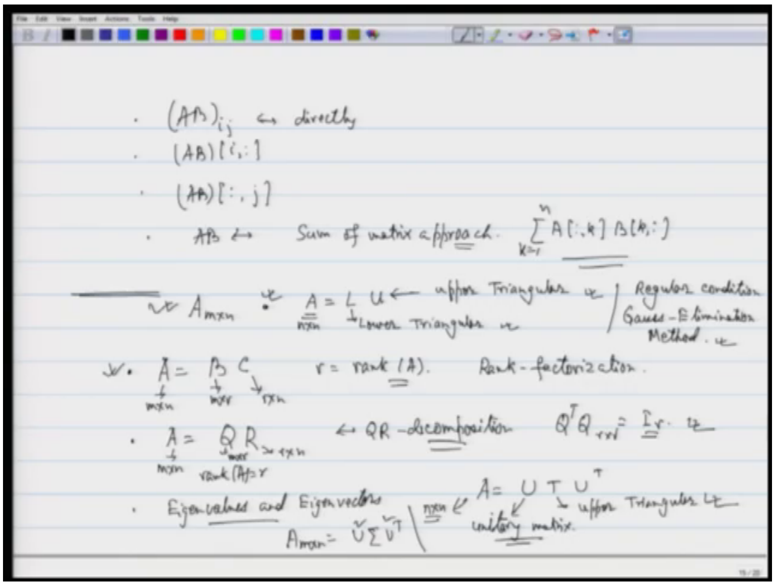

One was I can compute AB ij entry directly. I can compute AB using its rows. I can compute AB using its columns. I can also compute AB using sum of matrices sum of matrix approach.

So, here approach means I am looking at A columns of A times this and I am looking at k is equal to 1 to n. So, these are the 4 ways in which we can do matrix multiplication alright and I would like you to learn this matrix multiplication. In the next class, I will take up some more examples to make you understand this matrix multiplication.

Why I am trying to stress on matrix multiplication is because of the following aim of the whole course. The whole course that you are going to do is in different ways trying to prove the following idea.

So, we are trying to prove the following ideas that given a matrix A which is m cross n I can write A as what is called I can use A to write A as L times u where L is lower triangular and u is upper triangular. So, A is a matrix which is m cross n, then I will have some issues I will cannot talk of upper triangular lower triangular.

I will have to talk in terms of forms lower triangular matrix and upper triangular form, but if A is a square matrix then these 2 make sense. We will need conditions what are called regular conditions alright and they will lead to what is called Gauss Elimination method. So, this is nothing, but the Gauss Elimination method for us.

So, this is a first thing that we are going to understand is what is called writing the matrix A as L u decomposition L for lower u for upper triangular decomposition another thing we are going to learn is writing A as B times C. So, A is your m cross n, B is going to be m cross r C is going to be r cross n and here r is nothing, but the rank of a matrix. So, we are decomposing the matrix into ranks. This is called rank decomposition or a rank factorization of a matrix fine.

The next thing that we are going to learn is writing a matrix A as Q times R. So, if A is m cross n and as rank R. So, rank of A is r, then this Q will be of the size m cross r this will be r cross n and will turn out that. So, this is called QR decomposition decomposition.

The important thing is that this matrix Q transpose Q which is an r cross r matrix will be an identity matrix alright. So, what we call orthogonal matrices. So, we have got 3 things that is the basic idea that we have. This part will lead to what are called Gauss elimination method. As I said this is rank factorization this will also come from system of linear equation.

For the QR decomposition we will need to learn what are called inner product spaces. So, that you can understand what we are doing orthogonalization and so on Gram Schmidt process. Then there are 3 more which are related with what are called eigen values eigen vectors and eigen vectors that we can write A as U T U transpose T is upper triangular alright in certain and U is unitary matrix. I have not said what is a unitary matrix till now.

So, we will understand these things as we go along, but this eigen values and eigen vectors is going to be the last chapter. So, for that we will have to understand everything. There will be a special case of this.

So, here we need that A is n cross n A square matrix. We will look at things when we can make in place of T upper triangular we will have T as a diagonal matrix and then the final thing will be writing any matrix A m cross n as some U D V transpose where U and V are unitary and D is diagonal alright.

So, in place of D most of the books they write sigma here to understand this is singular value decomposition of the matrix. So, initially we look at the first 2 examples that is A equal to L q L u and the rank factorization. Afterwards after that we will learn what are called vector spaces and then things like that. So, that is all for today, but keep track that we need to understand matrix multiplication, so that you can do all of them.

Thank you.