Linear Algebra
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Lecture – 29
Fundamental Theorem of Linear Algebra

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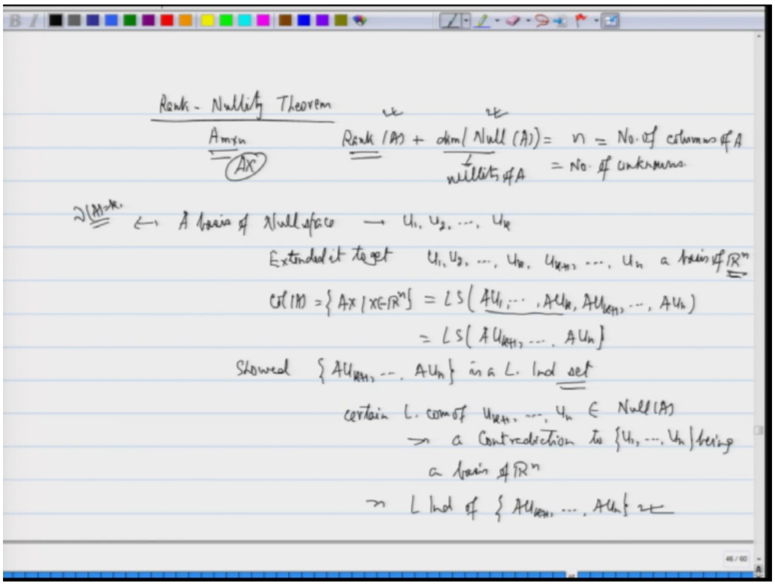

In the last class we have learned what is called the Rank Nullity theorem. So, let us recall what it was A was an m cross n matrix, then it said that rank of A plus dimension of null space of A was equal to n alright, n is number of columns of A or which was number of unknowns or variables alright. So, this was called nullity of A and this by definition itself is a rank fine.

So, this what the idea was and to prove it the idea was that you start with a basis of null space, of null space. We took it as a u 1 u 2 u k dimension was or the nullity of A, we took it as k fine extended it to get u 1 u 2 u k comma u k plus 1 to u n a basis of R n I am looking at AX. So, I am looking at this so, R n is coming into play.

And to get the column a space A which was nothing, but AX X belonging to R n, we saw that this is same as linear span of A of u 1 till A of u k A of u k plus 1 till A of u n. But, these vectors are 0 because, they belong to the null space u 1 to u k belongs to the null space. So, this was nothing, but A of u k plus 1 till A of u n.

So, we showed A of u k plus 1 to A of u n is a linearly independent set that is important. And to do that we consider the system of linear equations, then we showed that there is a certain vector. So, there was certain linear combination certain linear combination of u k plus 1 to u n, which belongs to the null space of A, and that implied a contradiction to u 1 to u n this being a basis of R n.

And this implies the linear combination the coefficient must be 0 implies. So, this implied linear independence of A of u k plus 1 to A of u n. We will again come back to it when you go to linear transformation we will use a similar trick to prove a rank nullity theorem there, but I would like you to understand it nicely fine. This is the one which relates rank with the nullity.

And hence it says that go back to the system of equations, their we have got certain nice results coming into plane that if the rank is k, then there are n minus are linearly independent elements in the null space of A fine. Let us proceed further with the fundamental spaces. So, we had looked at this example.

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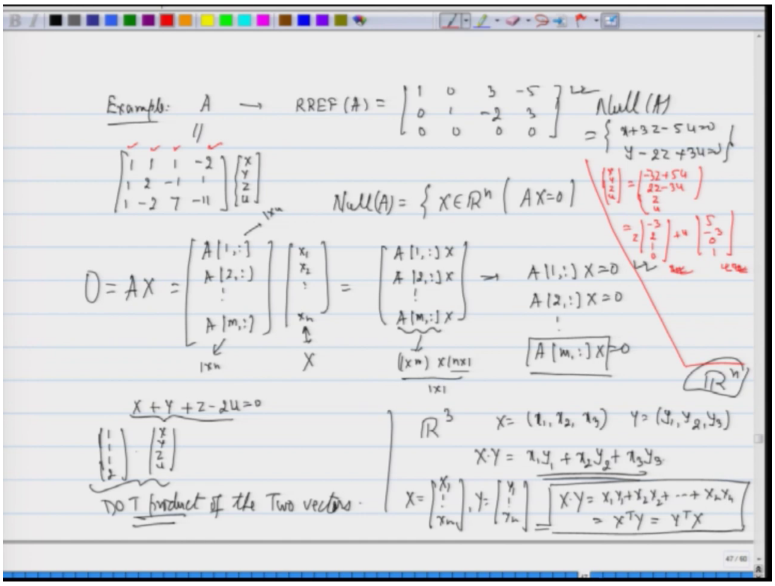

So, let us write that example again alright. So, hire the matrix A from there we had gone to RREF of A and RREF of A was 1 0 3 minus 5 0 1 minus 2 3, this was the RREF of was and A was equal to 1 1 1 1 2 minus 2 2 minus 1 7 minus 2 1 minus 11 alright.

So, I would like you to see that that when I am solving the system. So, look at this null space of A should null space of A is all AX such that all x belonging to R n such that AX is 0. So, what we exactly we are doing let us understand that? So, I have this first row, second row, the mth row of A, this is my A and my x is x 1 x 2 x n or basically x itself fine.

This a matrix of size m cross, what is the size of this matrix 1 cross n each of them is 1 cross n. So, I can write it as A of this times x this x fine. So, let us look at this one for example. So, this was 1 cross the there is only one row, there are n columns.

And I now, again multiplying by n cross 1 because x is n cross 1. So, I am getting it is 1 cross 1. So, it is a scalar quantity and we are saying that this is corresponding to AX which is supposed to be 0. So, we are saying that this implies this is 0 this is 0 and this is 0 fine. So, what are these vectors? So, let us try to understand. What are these vectors? That I am talking of fine.

So, in general if I am got here in this example if I have look at here, if I write here x y z u here, then you are looking at the first row as x plus y plus z minus 2 u is equal to 0 alright. Or and sometimes I am looking at look at this vector 1 1 1 2 x y z u, I have got these two vectors and we are looking at what is called the dot product of the so, we are looking at the dot product the two vectors.

So, recall that in R 3 if your vector X which is x 1, x 2, x 3 Y was y 1, y 2, y 3. Then the dot product of X and Y was x 1 y 1 plus x 2 y 2 plus x 3 y 3 alright that was the dot product. So, in place of 3 I have got four components here in general, I have got n components I am looking at R n I can still defined my dot product as component wise.

So, if I have got X as x n Y as y 1 to y n I can define my dot product X dot Y as x 1 y 1 plus x 2 y 2 plus x n y n, which is also equal to if you can see here in terms of the matrix notation, it is also equal to X transpose Y or Y transpose X whatever way you want to write.

So, when I am looking at X transpose Y or Y transpose X are the back of my mind I have a dot product with me. And there was a notion of dot product would said that they also give us. What are called angle between two vectors?

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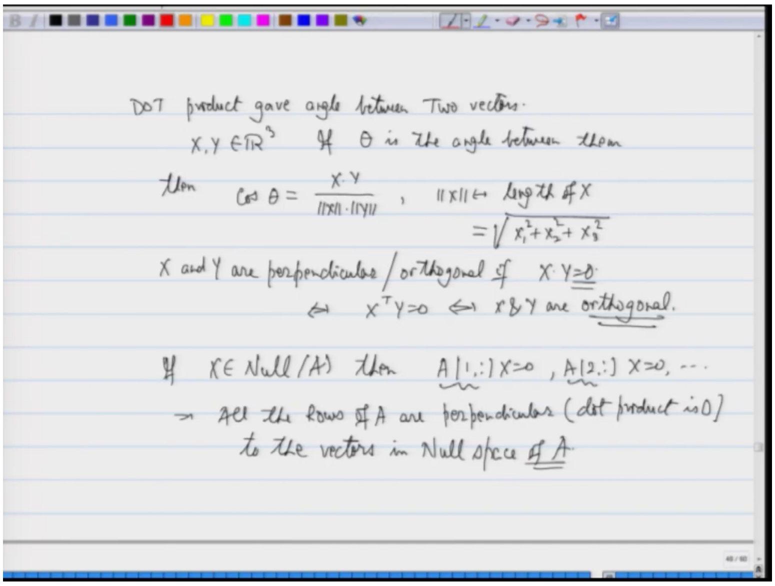

So, dot product gave angle between two vectors. So, what do I mean by this? So, given that X and Y are two vectors in R 3, if theta is the angle between them then cos theta was equal to X dot Y divided by length of X into length of Y alright.

So, this I am writing it as length of X and this is equal to a square root of x 1 square plus x 2 square plus x 3 square is that fine. And we said that X and Y are perpendicular or orthogonal. If X dot Y was 0 fine. So, here in this notion of matrix multiplication, it turns out that X transpose Y is 0, if and only if x and y are orthogonal fine.

So, I can talk a perpendicularity of things alright. So, what we see from the last part is that what we had seen was that if I look at? So, if X belongs to null space of A, then first row times X was 0, second row times X was 0 and so on. It means that these rows implies that all the rows of A are perpendicular why because that dot product dot product is 0.

All the rows of a are perpendicular to the vectors in null space of A alright. So, let us go back and see the whether this is true. So, let us look at here fine. So, the null space of A is going to consist of. Sorry, what are the null space of A here? So, null space of A was in this example it was all x y z w such that x plus 3 z minus 5 w 5 u is 0 and y minus 2 z plus 3 u is 0.

So, we can see here that 1 plus 3 z minus 5 u is 0 y is 1 here so, 1 times y minus 2 times z plus 3 times u is 0. Just multiply with this also you can see that 1 is here plus 3 is 4 alright. So, z and then 5 times u is 10 null space of u just a minute so, is column space column a space is there sorry fine. So, check that their 0 here as such. So, just look at this here alright we are saying that null space is here, take a solution of this so, take x naught to z naught y y naught belonging to this.

So, therefore, I think I am not littering in terms of this matrix form but anyway. So, for you so, that you have a clarity here. So, from here if I want to write here x y z u is equal to x is minus 3 z plus 5 u y is 2 z minus 3 u z and u. So, this is same as minus 3 2 1 0 with respect to z and then with respect to u 5 minus 3 u, u is 0 1.

So, check that now, everything is nice that minus 3 plus 2 plus 1 is 0 that is this part, minus 3 multiplied to the first component 2 multiply to the second component, 1 multiplied to the third component and 0 multiply to this is 0. Similarly look at here 5 multiplied to the first one minus 3 multiplied to this is 2, 2 minus 2 is 0 fine. So, I would like you to see that each of them is 0. So, the null space of A is generated by these two vectors.

So, these two are the basis vectors fine. So, I am going to use this idea of perpendicularity in the next thing. So, this is called the fundamental theorem of linear algebra.

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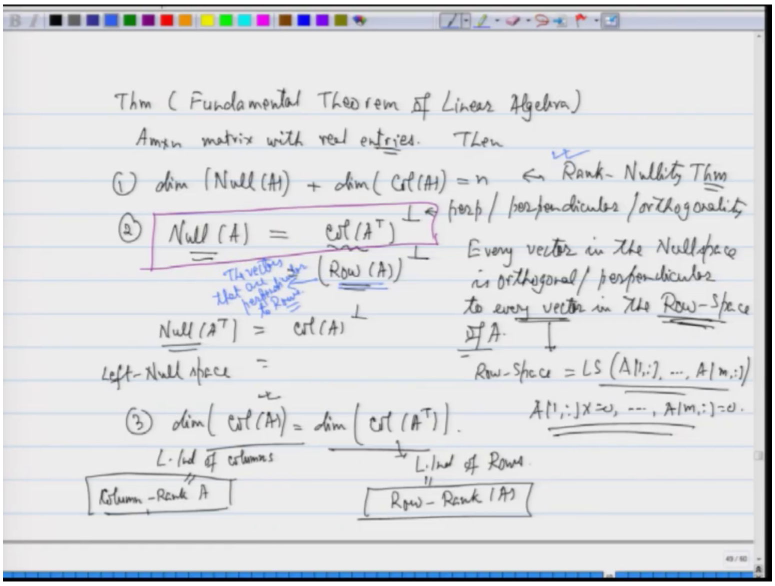

So, let me write it down. So, theorem this is called funda mental theorem of. So, I will not be able to prove all of them because, of certain reasons because, we have not had understood. What is orthogonality? But I will just state it so, that you have a picture of this.

So, A is given to be an m cross n matrix m cross n matrix with real entries fine. Then what happens? Then 1 dimension of null space of A plus dimension of column space of A is n. This was the rank nullity theorem two null space of A is equal to look at the column space of A transpose alright. Look at this thing what is called perp? For, perpendicular or orthogonality.

We just saw if you remember that what is column a space of A transpose is nothing, but row space of A. So, we are saying that every vector in the null space. So, we are saying that every vector in the null space is orthogonal or perpendicular to every vector in the row space of A alright.

Why because, why I am saying this row space of A every vector because, row a space if I want to look at is generated by to the linear span of the different rows, sorry it is A here fine. So, the row space is generated by linear span of the rows, we already saw that a times this of X was 0 all of them was 0 for us is not it, this what we already saw. So, this says that null space of A is perpendicular is orthogonal to row a space of A.

So, again look at it nicely fine then the another version of this is this I am look at A and A transpose. And also like to look at over null space of A transpose and the corresponding here. So, null space of A was reduce column space of A transpose.

So, it should be column space of A, A transpose gets A because A transpose, transpose is A and then, have to look at perpendicular here fine which is same as. So, null space of A I am going to null space of A transpose column A space of A trans to column A space of A perpendicular is that ok.

That is what I am doing here? So, this is called this was left null space alright. And the third one is dimension of column A space of A is equal to dimension of column A space of A transpose alright fine. So, this part tells you let us try to understand dimension of column space of A, is talking about number of pivots alright.

Number of elements in the columns is talking about linear independence, linear independence of columns. This is talking about linear independence of rows. What we know is that? This gives us the row rank of A fine, this will similarly give by definition this is nothing, but the column rank of A which is same as column rank of A.

So, that saying that column rank of A is same as the row rank of A is that ok. And how do we prove it? We prove it using these one and two you get the result directly alright. So, just let me write the third part, second part also we will come. So, let me write the third part first.

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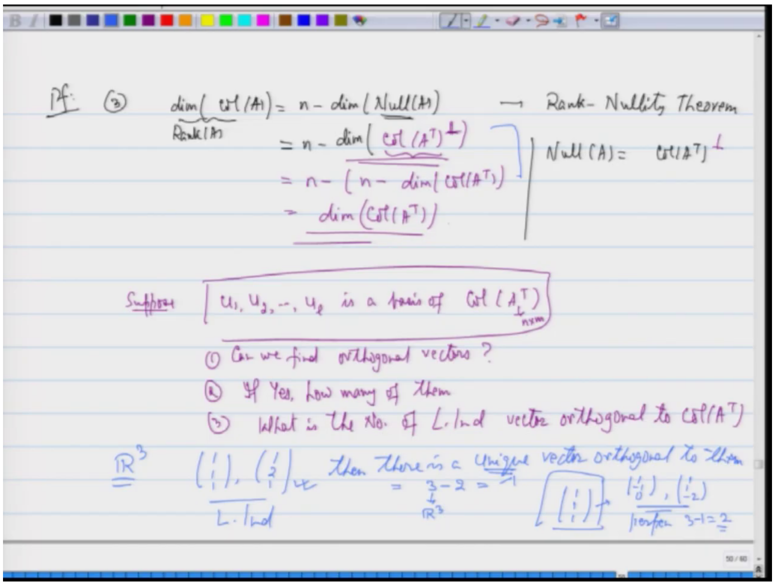

So, third part if I want to look at a proof first is already proven earlier. Second part we will you have already seen, some part in the sense that we have already shown that every row of A is perpendicular to the elements of null space. And therefore, the linear combination also be perpendicular fine because, if x is perpendicular to y z is perpendicular to y, then x plus z is also perpendicular to y alright.

So, then use those ideas to prove yourself, I am not want to waste my time there try it out yourself. So, I am just trying to prove the last part that is that row rank is same as column rank. So, now, let us look at dimension of column space of A is nothing, but the rank nullity theorem it is equal to n minus dimension is this.

So, dimension of null space of A alright, dimension of null space of A plus dimension of column space, which was the rank of A column rank is equal to this by the rank nullity theorem.

So, this because of the rank nullity theorem, this I can also write it as n minus dimension of null space of A, we saw that what was null space of A. The second theorem we had that null space of A is equal to look at the column space of A transpose alright, go back null space of A was equal to column space of A transpose perp alright.

So, null space of A is column space of A transpose perp so, A transpose perp fine. So, it is same as column space of A transpose perp. Perp means perpendicular. So, if X is perpendicular to Y things like that. So, you would like to know what is the dimension of this? So, what should the dimension of this? So, can I say that so, let us look at this part should understand it better. So, suppose I give you suppose u 1 u 2 u l is basis of column space of A transpose.

Suppose I give you this fine. So, A transpose I am looking at is n cross m matrix A transpose fine. So, each column is of size. What is the size? n things like that. So, now, if this is a basis can I find so, can we find orthogonal vectors that is the question first question 2. If yes how many of them? Then 3. What is the number of linearly independent vectors orthogonal to column space of A transpose. Because, you are looking at this part alright.

So, I would like to see that this is again a function of n itself because, each of the columns are of size column space of A transpose, if I look at A transpose all the columns are of size n therefore, it is a vector of n. So, it will be n minus n minus itself because column will get transferred into column itself is there ok. So, it will be n times dimension of column space of A transpose n n cancels out I get this as dimension of column space of A transpose alright.

So, I have not been able to give you the idea of this part that how do I go from here to here, I would like to try it yourself. For example, if you already know for example, if I if I mean R 3 I give two vectors 1 1 1 and 1 2 1 this 2 are linearly independent. Then there is a unique vector orthogonal to them. So, we are saying that this unique means there only 1 so, it is 3 minus 2 3 is the dimension of R 3 and to the vector to that we are given here.

So, this is what we get 1? If I give you only vector 1 1 1 then, they are two vectors which are linearly independent which are perpendicular 1 minus 1 0. And the another vector is 1 1 minus 2 both of them are perpendicular to this vector. So, again this number is 3 minus 1 which is 2. So, this number comes from that idea itself, I cannot prove it at this stage because I am not you talked about inner product dot product and so on but in R 3 you already know.

So, if you go from R 3 point of view we can see that everything make sense and you can get it fine. So, this is what the fundamental theorem of linear algebra is? So, what is fundamental theorem linear algebra? There is a notion of the what you call the rank nullity theorem, that you already understand. The second part says that you can relate null space with the row space in some sense. So, you are not looking at row space, you are looking at the perpendicular.

The vectors which are perpendicular to the row space look at here. The vectors that are perpendicular to rows alright fine this is important. And the third one was nothing, but relating the two things is that ok. So, let us write this in terms of a diagram. So, that there is more clarity on it fine.

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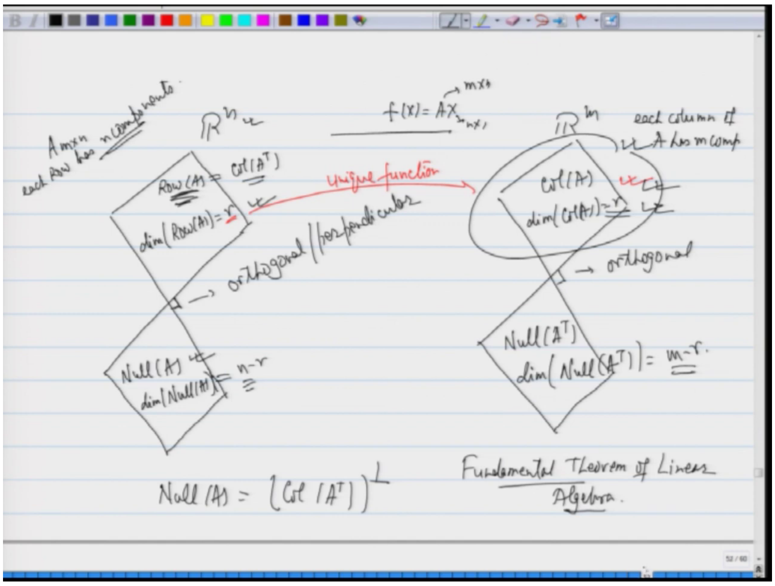

So, let me make a diagram. So, what we have I am going to write R n here, which was the domain space I got a R n here which is the column space and I have a function f such that f of x was AX fine. A was m cross n x is n cross 1. So, it I get f x which is an element here fine. So, I would like to write like this here I have something here this is perpendicular here fine.

So, let us look at what are the subspaces, I have what is called the row space here? Row space of A which are nothing, but column space of A transpose fine it is call again nicely A is m cross n. So, each row has n components fine. Each row has a n components. So, the row space is a subset of R n fine.

Now it was perpendicular to what? So, column space of A transpose was perpendicular to the null space of A. So, let me write the theorem again, statement for you state that there is clarity that null space of A was is equal to column space of A transpose whole perp fine. Similarly you have there also fine. So, row space of A was this and they are here as such, and the dimension of row space of A was r fine.

You have null space which is perpendicular here and here again dimension of null space of A is n minus r is that ok. If I look at here, here there where rows. Because, each row had how many components n components now I have columns how many components are there in the column. So, each column of A has n components. And therefore, column space will be your subset here fine and dimension of column space of A is also r fine, important again there is a perpendicular here.

So, orthogonal or perpendicular fine and we have column space here row space or column of A transpose. So, from A transpose you went a similarly from A will have to go to A transpose. So, you have to go to null space of A transpose and dimension of null space of A transpose is m minus r. So, here it is m, here it is n you have to be careful then that are two different numbers here, even though other top they are same numbers is that ok.

So, this is the picture of fundamental theorem of linear algebra. You need to understand it so, that you know which is perpendicular to what and what you can do what you cannot do alright. So, if you see here this dimension is r this dimension is also r therefore, you can get a unique function from here to here. So, whatever vector you get here the number of vectors here, it has got r component r.

So, if its dimension is r, it means that there are r vectors in the basis, there are also r vectors in the basis I should be able to get from one to the other alright. So, there is a unique map, this part I cannot have uniqueness because there are different sizes are there different issues that will come into play.

But at the top part from column space to row space and row space to column space, I can have a 1 1 map. And therefore, that also tells you that row rank and column rank should be the same fine. So, as a final part I would like you to relate this definitions that you have already done through exercise.

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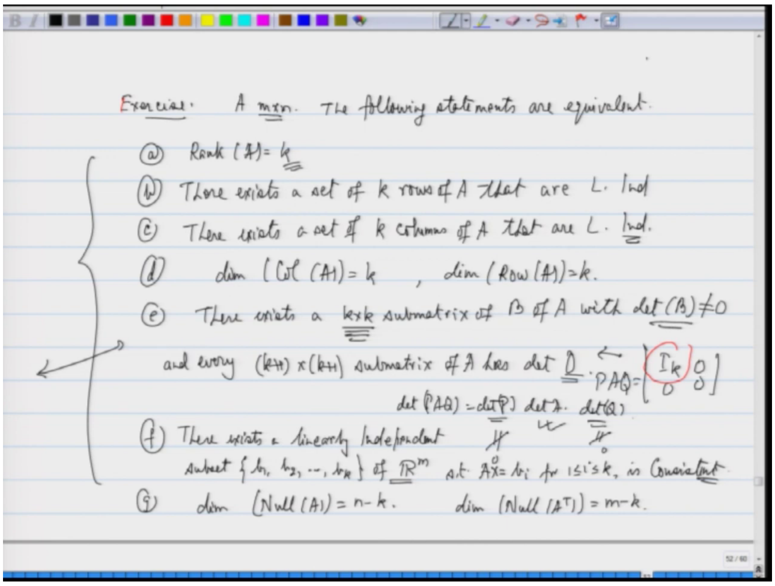

So, I have an m cross n matrix A is m cross n fine. So, the following the statements are equivalent. I will not proving them, but I would like you to understand them rank of A is k, b there exists a set of k rows of A, that are linearly independent c.

So, k rows of a linearly independent means, you are looking at column space of A transpose or row space of A, their basis is that fine. There exists a set of k columns of A that are linearly independent fine. Again the same thing you are going to from a you can prove b from b you can prove a as that angle the fundamental theorem says. Again a prove c c implies a you can do that yourself.

And they together imply you that dimension of column space of A is k dimension of row space is also k, e there exists a k cross k sub matrix of A, sub matrix B of A with determinant of B not equal to 0.

So, k cross k here it is very important and every k plus 1 cross k plus 1 sub matrix of A as determinant 0. I have not proved it, but you can look at the RREF in the RREF what happens is that you are going to get the matrix of the type, I k so, if I not the exactly the RREF, but look at PAG. So, I can write it as this.

So, recall that there was this theorem which said that given any matrix A of rank k rank is k, I can get a invertible matrix P on the left, Q on the right such that I get I k 0 0 0. And therefore, it says that if I take any k plus 1 by k plus 1 the determinant has to be 0.

And there is a k by k here which is non zero which are non zero determinant alright. So, this what this language is about relating this with this because determinant of PAQ is same as determinant of P into determinant of a into determinant of Q fine P and Q are invertible.

So, determinant of P is not 0 determinant of Q is not 0. So, they do not play a role everything is being played by this is that and accordingly, you get those parts fine. So, some of the matrix multiplication makes sense here f, there exists linearly independent subset b 1, b 2, b k of R m alright such that AX is equal to b i for 1 less than equal to i less than equal to k is consistent.

So, there are k choices in R m so there is a k dimension of subspace for which we have a solution, this what it was about saying that I have a solution here alright. So, there are is a k dimensional space here, for which I have a solution here is that fine e f g dimension of null space of A is n minus k fine. Similarly you can have another one which is dimension of null space of A transpose is m minus k alright.

So, all of them that equivalent condition you can prove on so, most all most all of them can be proven using the fundamental theorem. Because of this diagram that we have got here everything follows from here, if you have understood the definitions.

The only problem is looking at this part which is about determinant they follow from here and some properties of determinant that I have not told you. So, I would like you to understand it, this is the most important part that you need to understand is there ok. So, try that out yourself that is all for today.

Thank you.