Linear Algebra

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Lecture – 28
Rank – Nullity Theorem

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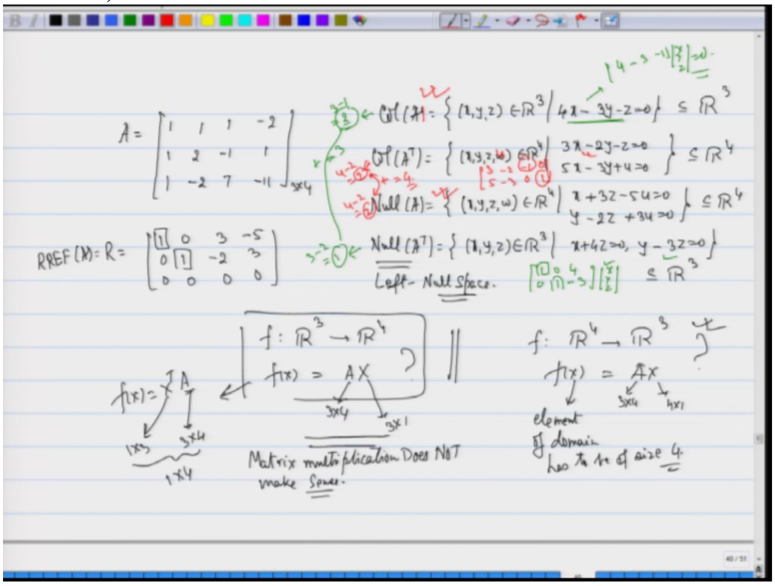

Alright. So, let us go back to the previous example in the last class. I have just written them down. I had the matrix A I had computed the RREF of A and then got column space of A, row space of A which was column space of A transpose, null space of A and null space of A transpose, fine which was same as the left null space. This was same as left null space, fine.

So, I would like you to understand them somehow look at R 4, what are the subsets of R 4 from here? Alright. So, what are the subsets of here in R 4? So, to understand what they are so, again this is matrix is of 3 cross 4.

So, I have got f from R 3 to R 4, I am defining f of x is equal to AX. Does it make sense? Or is this the function f from R 4 to R 3 and saying f of x is equal to AX which one is the right one, how do I find out? So, the idea to find out is look at the matrix and look at the matrix multiplication.

So, the matrix is given to be A is 3 cross 4. So, I need to have X which is 4 cross 1. So, this x which is an element of the domain; element of domain has to be of size 4 fine. But if I look at here A is 3 cross 4 is fixed. This X is 3 cross 1. So, this matrix multiplication does not make sense; multiplication does not make sense, fine. So, what I wrote here was wrong. I cannot write this, but this is A meaningful object for us, alright. But, you can also think of this defining it differently I can define f of x as X transpose A.

Now, X is 3 cross so, I can think of this as 1 cross 3, this is 3 cross 4. So, I get A matrix which is 1 cross 4. So, when I want to do calculation using rows I can define things this way, is that ok? So, depending on whether you want to a study things with respect to column vectors or row vectors you can define it differently. There is no issue with that and because of that all these notions have come column space, row space, null space and the left null space; because of those all these ideas have come and they are related also.

So, I would like you to understand them nicely, fine. So, this evaluate column space of A it is A subset of R 3, what about the null space? We are looking at all x, y, z. So, this is A subset of R 4, this is again we are looking at x, y, z, w 4 of them. So, this is subset of R 4 and this is A subset of R 3, fine. Look at it nicely and understand it.

Now, let us look at the dimensions here. Now there is only one condition here. So, the dimension of this space will be 3 minus 1 which will be 2, fine? Look at this part. This is also A subset of R 3. There are two linearly independent conditions or there are 2 rows over the matrix that I am looking at. So, I am looking at the null space means I am looking at this one.

So, if I want to look at here I am looking at x, y, z; 1 0 4; y is 0 1 minus 3 I am looking at this set, this is the solution space here of null space. There are two pivots here. So, the dimension of this space is 3 minus 2 which is 1, fine? You can see that this plus this adds up to 3, understand it. This is also you are looking at 4 minus 3 minus 1 of x y z is equal to 0 as such, fine.

Similarly, if I look at these two, here this corresponds to the matrix looking at 3 minus 2 minus 1; w is 0, 5 minus 3 I wrote u here in place of w. So, let me write it u here alright 0 and 1. So, I am looking at this system here A is this and x y z and u, fine. So, again I have got 2 pivots in some sense here you can see here. So, therefore, the dimension of this is 4 minus 2 which is 2. Here also there are 4, so, it is 4 minus 2 which is 2. So, this 2 and this 2 add you get 4.

So, the question is that is there A relationship between these ideas? That is one thing, fine. The other thing is that is there some relationship between column space of A and null space of A. So, the idea is to understand the relationship between these spaces one after the other. So, first we will try to understand the relationship between column space of A and null space of A, we will try to understand that and that is called the rank nullity theorem.

So, we will try to understand the rank nullity theorem and then the next stage we will try to understand something more what are called orthogonality of all these spaces. So, I will have to go to what is called A dot product and then do it. So, first let us go to what is A column space and null space what is the relationship between them.

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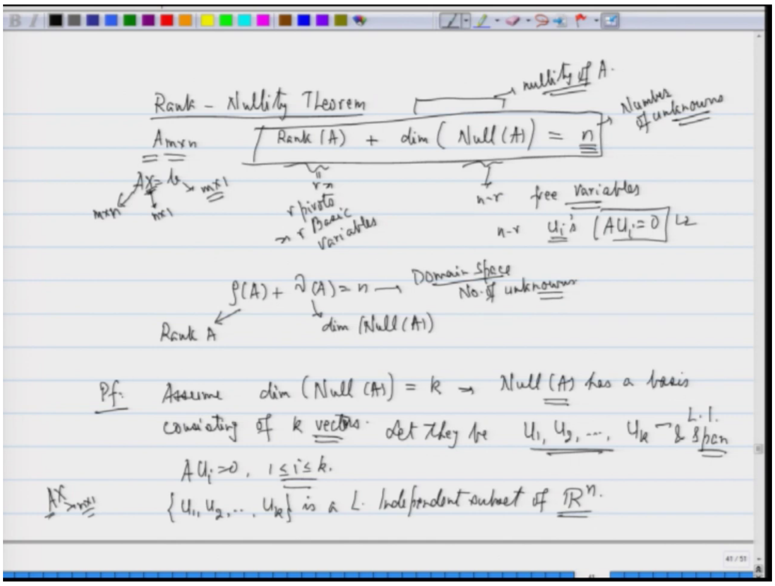

So, let us look what is called the rank nullity theorem. So, A is m cross n, we are starting with this. We know what is the rank of A alright rank of A plus dimension of null space of A is equal to n. What is n? The idea of this was we start with solving AX is equal to b; A was m cross n, X was n cross 1 and this was m cross 1 that is another way of understanding it where which subspace is where.

So, this n is number of unknowns A number of variables that you are looking at, fine. Recall when we had looked at the system of linear equations the main theorem there. If the rank was R there were n minus R free variables. So, rank R implies R pivots implied R basic variables and therefore, I got n minus R free variables fine and therefore, I got n minus R u i's. Recall that there are u i's such that A times u I was 0 right. So, you recall that part.

So, I am just trying to prove that there we got with an example; we could not prove it so well. So, I am trying to prove things now, is that ok? So, let us do that proof of this theorem now. So, we are going to prove this theorem the rank of A plus the dimension of null space of A this dimension this were dimensional null space is called nullity of A alright. And, generally if we write it as rho of A plus nu of A is n.

So, this is for rank of A, this is for dimension of null space of A, is that ok? And n is whatever the domain space or number of unknowns, fine. So, let us use whatever theory we have build up to understand it. So, proof. So, assume dimension of null space of A is some k, fine. So, this implies alright null space of A has a basis consisting of k vectors alright. So, important null space of A has A basis consisting of k vectors.

Let them be; let they be u 1, u 2, u k alright. So, I am saying that they are A basis it means they are linearly independent and a span, is that ok? Both the things are true. So, what we are saying is A of u i is 0 4 1 less than equal to I less than equal to k, fine. Now, what we had learnt was that so, now, if I look at this set u 1, u 2, u k. This is a linearly independent subset of R n. That is very important, u i's are elements of R n and I multiplying A alright. u i's are supposed to come here because I am looking at AX. So, this is n cross 1.

So, what we know is that u 1 to u k they are subset of R n, they are linearly independent set. Since it is A linearly independent set so, what does that imply? It implies by the theorem that I can extend it to get a basis.

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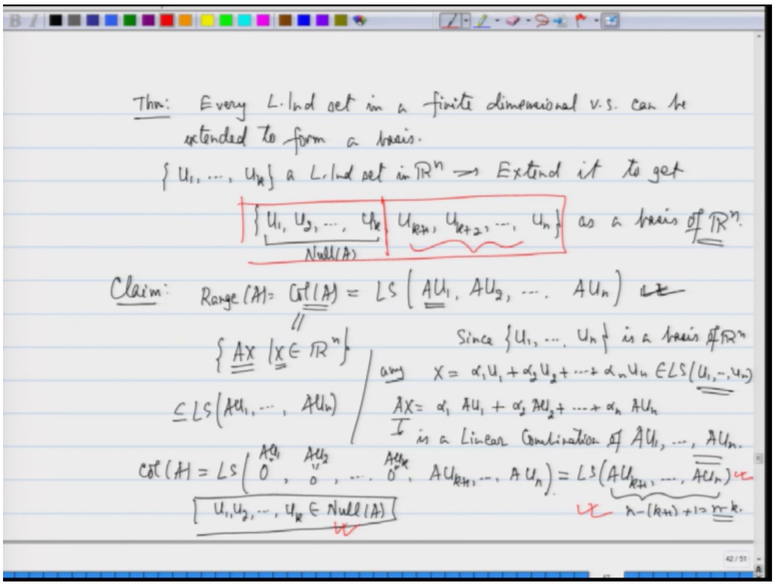

So, there was this theorem who said the theorem said that every linearly independent set in a finite dimensional vector space vector space can be extended to form a basis fine. So, here I have got u 1 to u k a linearly independent set in R n implies extend it to get u 1, u 2, u k and then u k plus 1, u k plus 2 till u n as a basis of R n, is that ok?

So, I can get it as a basis of R n that is more important this part is for null space of A and overall, is that ok? So, I have got A basis for this I want to claim. So, claim so, look at things that what is range of A? Range of A was nothing, but column space of A, how do I get A column space of A? Column space of A I am claiming that this is nothing, but linear span of the vectors A of u 1, A of u 2, A of u n I am saying this. Why I am saying this? Because if I want to look at column space of A by definition column space of A is AX X belonging to R n, is that ok?

Now, what we know is that u 1 to u n is a basis. So, since u 1 to u n this is a basis of R n; X is nothing, but any X. So, any X will look like some alpha 1 u 1 plus alpha 2 u 2 plus alpha n u n. So, any X belongs to the linear span of u 1 to u n, is not it? Fine. So, again recall what is the basis? Basis is linearly independent set and it a spans. So, it is able to a spans. So, if I take any element X belonging to R n I am able to get X using some linear combination.

So, there are these scalars alpha 1 to alpha n, I do not know what they are, but they are there, fine. So, now, if I multiply with A so, A of x is nothing, but alpha 1 times A of u 1 plus alpha 2 times A of u 2 plus alpha n times A of u n. So, basically what I am saying is that AX is A linear combination of A of u 1 till A of u n, is that ok? So, each of them is being represented as such, fine.

So, any element AX if I look at here then this is A subset of alright this is A subset of linear span of A of u 1 till A of u n, is that ok? And, the linear span of u 1 to u n is the whole of R n. So, I am looking at only this part there is nothing wrong everything is ok. So, this part is equal to this because any AX is in the linear span of A of u 1 to A of u n fine. And vice versa you will prove it yourself because linear independence will imply that they are invertible and things like that I can get A standard basis u 1 to u n using these and vice versa fine. So, I have got this.

Now, let us look at this what is column space of A. So, I can rewrite it column space of A is linear span of what is A of u 1? A of u 1 is 0, A of u 2 is also 0 and 0 till A of u k. Because u 1 to u k u 1, u 2, u k there were elements of null space of A, there are basis of this. So, A time those u i's 0 are 0 the rest I do not know what they are. So, they will remain as it is, fine. So, they will remain as it is.

So, therefore, what I see is that column space is nothing, but is generated by linear span of A of u k plus 1 till A of u n, how many vectors are in the set? n minus k plus 1 plus 1 which is n minus k, alright. So, I would like you to now understand again what is n? n is the number of elements in R n, a basis of R n. So, those were the basis elements for me. So, this is what it is, fine; out of that k of them are coming to null space the rest of them they are coming to the column space, is that ok?

So, somehow I have to prove I can see that column space of A is generated by these vectors because the linear span of these. I will have to prove that dimension of or these vectors are linearly independent. So, to imply that dimension is there.

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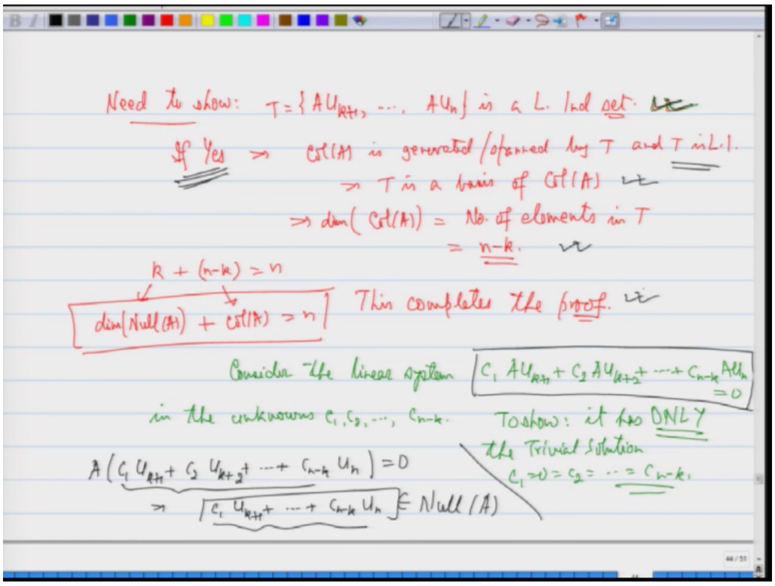

So, need to prove need to show the vector show this set A of u k plus 1 till A of u n is A linearly independent set, fine. If I can show that if yes, this will imply that column space of A is generated or spanned by spanned by this spanned by T spanned by T. And T is linearly independent implies T is A basis of column space of A and this will imply the dimension of column space of A is equal to number of elements in T which we just saw it was n minus k, fine.

And, this will imply that k plus n minus k is n k for dimension of null space of A and this for column space of A. So, dimension of null space of A and column space of A gives you this and this completes the proof, fine. So, I just have to show that this set is linearly independent. So, let us try to prove that this set is indeed linearly independent. So, we have proved that something is linearly independent, what am I supposed to look at? I am supposed to form A system of linear equation in certain unknowns.

So, consider the system consider the linear system c 1 times A of u k plus 1 plus c 2 times A of u k plus 2 plus. So, on c of n minus k A of u n is equal to 0 in the unknowns c 1, c 2, c n minus k to show it has only the trivial solution c 1 is equal to 0 is equal to c 2 is equal to c n minus k, I have to show this only the trivial solution alright. So, let us try to do that.

So, what we see here is that if I look at this part fine it is. So, I am trying to solve A system which is c 1 A u k plus 1 c 2 and so on. Now, A is A matrix so, I can take out A on the left. So, I can rewrite it as A times c 1 u k plus 1 plus c 2 u k plus 2 plus c n minus k u of n is equal to 0, fine. I have just take out A on the left matrix multiplication. So, I have got this.

Now, this part tells me that this vector imply this vector c 1 u k plus 1 plus so on till c n minus k u n this belongs to null space of A, is that ok? This is important. So, it says that this part this vector whatever that vector is this vector is A linear combination of u k plus 1 to u n, but it is an element of null space.

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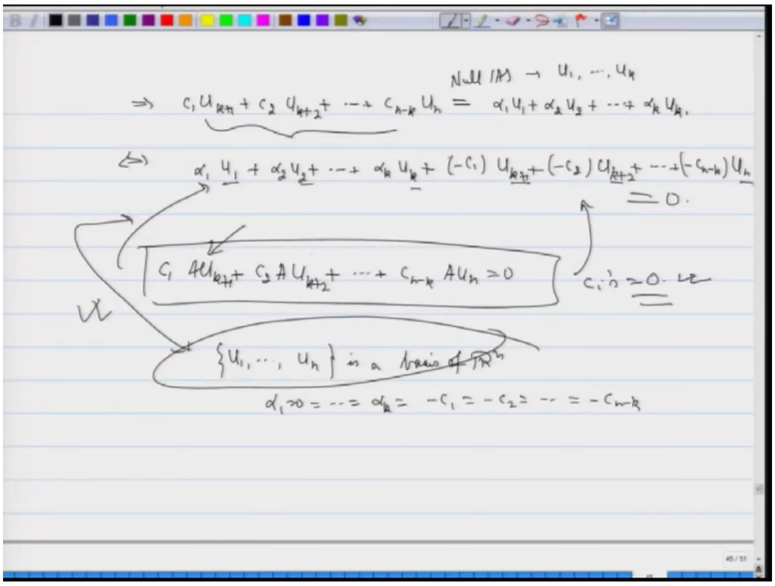

Now, since it is A base it is an element of null space alright so, it will imply that I can write this vector c 1 u k plus 1 plus c 2 u k plus 2 plus c n minus k u n, this is an element of null space of A. Null space of A has A basis. What is the basis? It is basis is u 1 to u k. So, I can express this vector as linear combination here. So, I can write this vector whatever that is as some alpha 1 u 1 plus alpha 2 u 2 plus alpha k u k, is that ok? Understand?

The left hand side is an element of fine, somewhere I do not know where it is, but at the same time we have shown that A times that is 0. So, therefore, this vector belong to the null space. Since this belongs to the null space I can express it in terms of the basis of the null space so, I can write like this. But, this is equivalent to looking at just look at it alpha 1 u 1 plus alpha 2 u 2 plus alpha k u k plus minus 1 times c 1 or minus c 1 times u k plus 1 plus minus c 2 times u k plus 2 plus minus c n minus k times u n is 0, fine.

So, the system of equation that I had which I had started with which was related with the idea of c 1 A u k plus 1 plus c 2 A u k plus 2 plus c n minus k A u n 0. This system of equation that I had started with has led me to a new system of equation which has all the n variables. So, it has got the vectors u 1, u 2, u k, u k plus 1 u k plus 2 u s. So, I have got n vectors and I am looking at a system of equation in all those n vectors.

Now, what are those vectors? These vectors are nothing, but a basis of R n alright. So, u 1 to u n this is a basis of R n and therefore, this as a unique solution which is given by alpha 1 is 0 is equal to alpha k is equal to minus c 1 minus c 2 minus c n minus k and therefore, I get that each of these c i's has to be 0, fine. So, understand here that from this system we went to this system which was about a basis of R n, alright. This was the basis of R n which was there and for a basis of R n we need that each of these linear combination each of these coefficients must be 0.

So, the alpha is are at 0 c i's are 0 therefore, the c i's are 0 here. Therefore, these vectors that I am looking at they are linearly independent and this is what we wanted to prove that this set has to be linearly independent we have done that. And, since you have done that therefore, the answer is yes once the answer is yes, so, columns of A are generated by T and T is linearly independent we have already shown it now. So, that T is A basis and therefore, this is this and we have the result with us, is that ok? Fine.

Thank you.