Linear Algebra
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Lecture - 27
Fundamental Spaces Associated with a Matrix

Alright, so let us go back to one of the previous results that I had done in the class, I would like you to remind it to understand it further. What we had done was that we had a collection of vectors u 1, u 2, u k.

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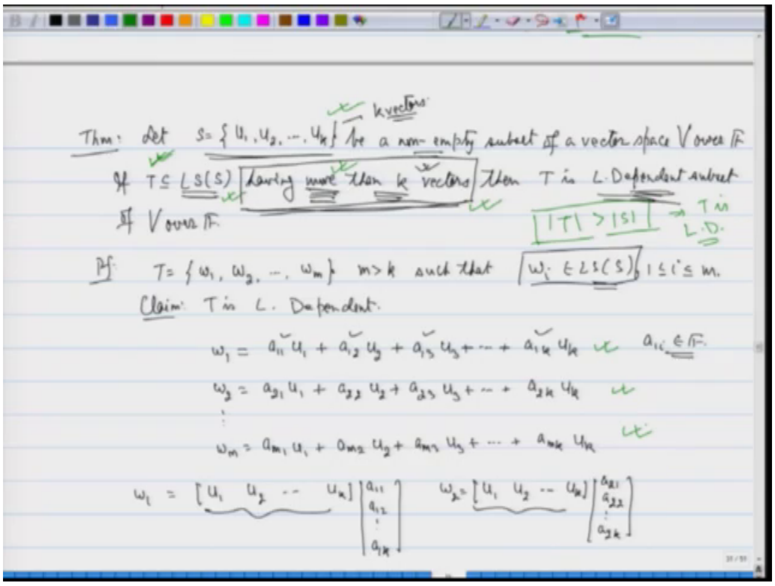

And, from there from these vectors u 1, u 2, u k we looked at a linear span LS of S and then from that linear span we picked up certain vectors T, fine. The only important thing that we said was that the number of elements in T is more than the number of elements in S. So, what we said was that number of elements in T fine and that implied that T is linearly dependent fine, that was the idea. And, what did we do it? We wrote the combinations as this different combinations.

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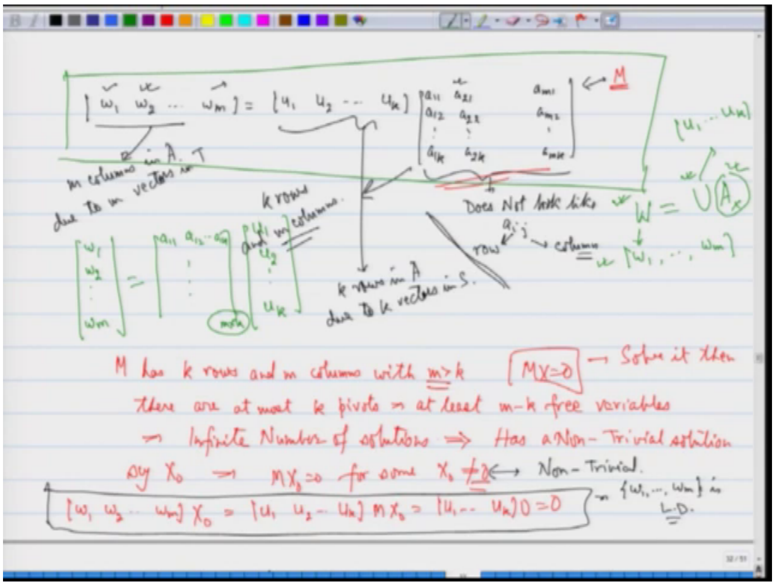

And, from there we wrote the w's in whatever way we could and the next stage what was there is that we had this matrix for us some sort of matrix representation, where w 1 to w m writing it as; so, I have matrix of W which is consisting of w 1 to w m. And, I am writing W as U times the matrix A and U was again a vector of u 1 to u k, fine and A was your whatever the size of the matrix, that I have I do not worry about it. But, there is some size that is coming into play.

So, we wrote a collection of vectors as vectors into a matrix, fine. In here I looked at vectors as column vectors, I could have also done here it will it as w 1 w 2 w m here and then written here u 1 u 2 u k. And, now you can see that the first one was a 11 u 11 a 12 u 2 a 1k u k alright, that was the first vector that was w 1.

If I want to get w 1, now I will have to write the matrix as w 1 is a 11 u 1 a 12 u 2 and a 1k u k, fine. So, I will now get a matrix which is of the type m; so, it is m cross k, fine. So, now, let me put it everything in the next page and have a better clarity of things, alright. So, what I am trying to say here is that I have a collection of vectors.

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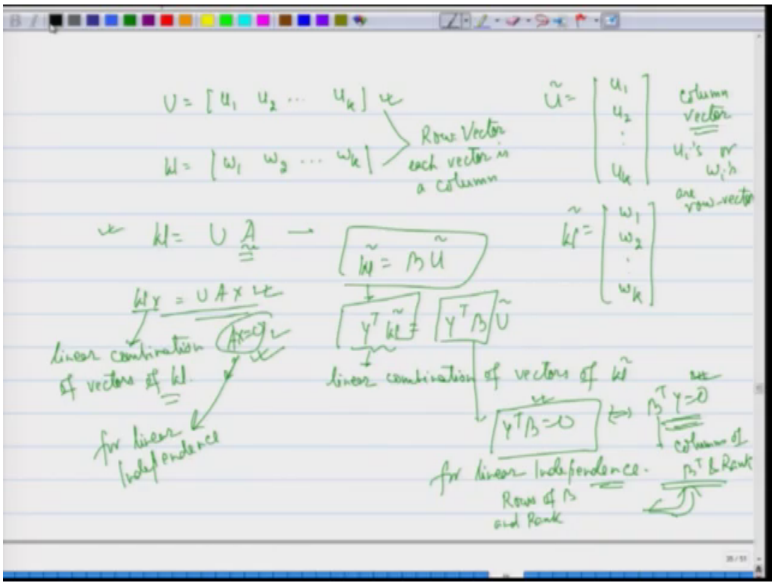

So, I have collected vector say U which is I am writing it in terms of columns or rows whatever it is u k, I could also write in writing it like this I can also write it as say s say u tilde as u 1 u 2 u k. So, depending on what way we want to write, we can write things here, fine. Similarly, I had W so, I had I think I wrote it as w 1 w 2 w k.

Similarly, I can have W tilde which is w 1 w 2 w k, alright. So, depending on how our vectors are what how we want to represent, we can say that U or W looks like U times a matrix. So, this is in terms of the rows here, if you look at here these are the rows. So, it is a row vector, row vector; row vector or each vector is a column; each vector is a column, fine. Here, these vectors are column vector.

But, we are writing it each of these u i's, u i's or w i's are row vectors. So, depending on where I am I can look at things, fine. So, this is one way of writing it, if I want to write using this idea then W tilde will be equal to A times or depending on whatever A is, this A and this A could be different, so it was basically transpose if you see nicely B times u tilde, alright.

So, there are different ways of looking at things, here I wanted to look at W times X transpose was U times A X transpose or if I want to write X as a vector, then I have this nice way. Otherwise, if I want to do it here this is the column vector; so, I will have to write here Y transpose W tilde as Y transpose BU tilde. Then it will make sense, then I can say that I am looking at linear combination of vectors of W tilde, fine.

Here, I am looking at also linear combination of vectors of W. So, the vectors are same, here the in W tilde the vectors are w 1 to w k. In W also there are w 1 to w k fine, but it is the way of writing which makes all the difference. If you want to write like this, then you have to look at the system AX is equal to 0 for linear independence dependence. So, this is for linear independence, alright.

If I want to look at from this point of view; so, I want to look at this. So, that give me the time to look at Y transpose B is 0, and then again I am looking at again this is for linear independence. So, depending on what representation I have, I need to look at AX equal to 0 or I have to look at Y transpose B equal to 0 which is equivalent to saying that B transpose Y is 0 alright, fine.

So, where you have to multiply what you have to do and how do you use your theorems of infinite solutions depends on you, you have to be clear about it, fine. So, be careful when you make a statements, you cannot just do whatever way you like, fine. So, here if I want to look at again here it is if I want to look at infinite number of solutions for this case; I need to look at columns of B transpose and rank fine, which is same as looking at which is equivalent to looking at rows of B and rank.

But, rank remains the same for us therefore, it is here it is rows which played the role, here it was columns which played the role, fine. So, you have to be careful when you say what are free variables, what are basic variables and so on. It will depend everything on these ideas where you are looking at, fine. So, sometimes you have to look at AX is equal to 0, sometimes you have to look at B transpose Y equal to 0 to make sense of things, alright.

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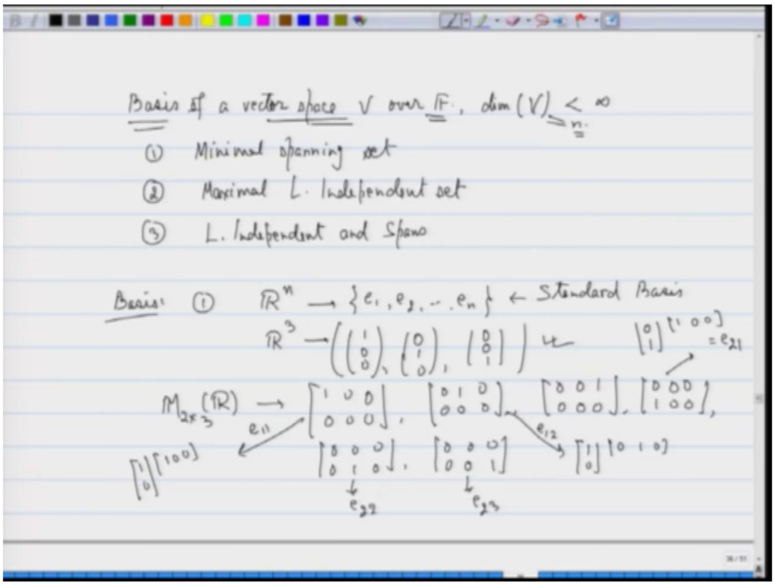

So, now with this idea let us start further, also recall that we had done what was called a basis of a vector, so alright. We had a notion of basis of a vector space V over F, alright. Mostly in our example as I told you from very beginning, it will be real numbers; once in a while it will be complex numbers. But, most of the theory is true even for fields which are finite number of elements like z 5, alright. So, you just have to take things and have clarity of things, fine.

So, what are the basis? The basis was alright either it was a minimal spanning set alright, also I am assuming here that my vector space has dimension finite, dimension of vector space V is finite; so, suppose that it is n fine, it is finite. So, it has minimal spanning set, this is what a basis is or I can say it is the maximal linearly independent set or the third was a set which was linearly independent and a spans.

So, if I want to show that something is a basis any three can do and we have seen the basis for us, so let me remind you once more. So, basis examples 1 first example, if I look at R n the basis consists of e 1 the standard basis e 1 to n, this is the a standard basis. So, what I am saying is that if I look at R 3 this corresponds to 1 0 0 0 1 0 and 0 0 1. If I am looking at say all 2 cross 2 matrices fine, then I have basis as 1 0 0 0 or let me write it 2 cross 3 0 0 0 1 0 0 0 0 0 0 1 0 0 0, fine.

So, I would like you to find out, how do you get these matrices that is more important alright, using these ideas, fine. So, try that out for example, if I want to write this, this is 1 0, so this is 2 cross 2. So, on the left it is 2 and on the right it is this. So, I multiply these two matrix, I will get this part fine. What about this? This is nothing, but again 1 0, but we have multiplied to 0 1 0, fine. For example, if I want to look at this matrix, look at 0 and 1 and multiply it with 1 0 0, alright.

So, they are basically coming from these ideas itself, the sizes are different; so, you have to be careful about it. So, this matrix for example, we write it as e 11, this as e 12, fine. This is e 21 that is second row and first entry; this is e 23 and e 22, fine. So, different ways of writing it, but you will need to understand all of them fine.

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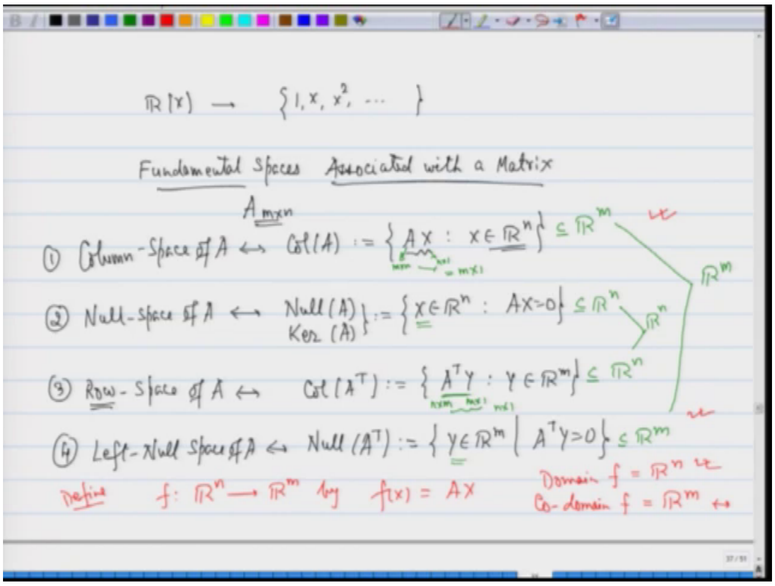

Another example is set of all polynomials with real coefficients. So, its basis is going to look like all polynomials of different degrees. So, there is no end to it, it just goes on and on, fine. Now, we now alright one more example, I think I should look at one more example, so consider this no, alright. So, now, let me look at what are called fundamental spaces associated with a matrix, alright. So, this will give me lot of examples and for them we can look at the basis fine.

These are fundamental spaces. So, I have a matrix A which is m cross n, I define certain things what is called column space of A. So, let me write I think full form here and then the column space, so it is column space of A, the notation for this is col of A and this is defined to be equal to column I am looking at all the columns.

So, I am looking at A times X, X belonging to R n; is that ok? Fine, this is the first space. Second space is we have already seen this also, null space of A. We write it like this, many books they will write it as this. So, whatever way you want to write you write it, this is all X belonging to R n such that, A X is a 0 vector, so I am solving the homogeneous system.

Third is called the row space of A, even though it is called row space; I do not write row here, because in most of the books they look at that way. So, we look it as column space of A transpose because, the rows of A are nothing, but columns of A transpose. So, therefore, we write it as column space of A transpose. So, therefore, this is equal to A transpose Y, Y belonging to R m, fine. And, the fourth is what is called a left null space of A.

So, it is nothing but, null space of a transpose which is nothing but, all Y belonging to R m such that A transpose Y is 0, alright. So, please try to understand what they are. So, these are the different definitions what is called column space. So, if I look at column space I am varying here X is varying over R n. But what is the size of these vectors? If I look at the size of these vector A is m cross n, X is n cross 1. So, I get here this is size is m cross 1.

So, this is a subset of R m, fine. This is looking at all X belonging to R n. So, this a subset of R n. What about here? A transpose, if I look at this is of size m cross 1, Y is m cross 1, so this is n cross 1. So, this is a subset of R n and this again you can see here that this is a subset of R m. So, what I am looking at is, I have got 2 subspaces which are coming from R m and 2 subspaces coming from R n; is that ok?

So, we need to understand them what is the relationship between them and how do I come across all those things, is that ok? Fine. So, let me try to define a function now for you. So, I define a function f from R n to R m, I am defining a function. So, define f by f of X is equal to A times X fine. So, domain of this function if I look at the domain of this function, domain of f is R n, fine. What is the co-domain? Co-domain of f is R m, fine.

So, therefore, if you look at these fundamental spaces alright; so, the first two first and the last one, 1st and 4th they are coming from the co-domain and the 2nd and 3rd they are coming from the domain, fine. So, basically they have come from functions. So, if you remember in our school days we used to learn given a function, what is the domain of the function, co-domain was fine. But, more than co-domain it was looking at fine, what was looking at?

The range of the function; so, there was a notion of what is called the range of a function, fine. So, I would like you to see that this is nothing but, the range of f, why range of f? Look at this, range of f by definition is f of X, X belonging to domain fine, which is same as AX, X belonging to R n, fine. This is what it was. Then we looked at; so, that we are looking at given a function we looked at domain, co-domain, range, range is same as the column space of A in this case.

And, we also did many things like f is 11, f is on 2 and things like that, fine. So, how does we get ideas about 11 and on 2? So, it turns out that I am looking at functions which are of this type which are called linear functions which we will come to afterwards or linear transformations. Because, of that it will tell us that I just need to understand the null space to say whether the function is 11 or on 2 or not, fine.

So, the idea is we need to understand these spaces. So, that you can go back and understand whether f is 11, whether f is on 2, what is the range of f, whether the where is the function defined and so on, is that ok? So, let us go and build up these ideas for us.

You can check that column space, I am saying it is a space it is a vector subspace of R m, null space is again a vector subspace, but is a subspace of R n, column space of A transpose is a subspace of R n, fine. And, you can see that they are somehow related that you can see and we will prove them first. So, before doing that let us look at some examples to have a better clarity alright example, alright.

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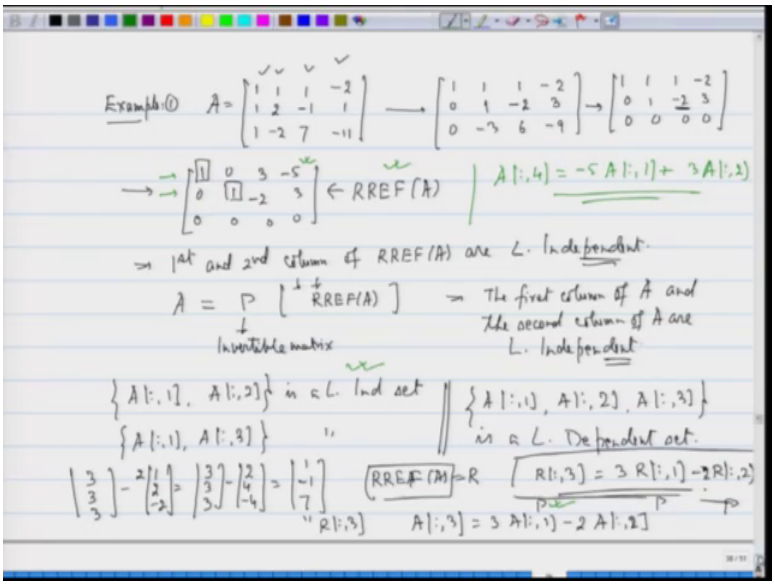

So, I have A is given to be equal to 1 1 1 1 2 minus 2 1 minus 1 7 and minus 2 1 minus 11, fine. So, what we do is that we want to look at what is the row space, column space and so on. So, let us apply the RREF, try to get the RREF of this fine; I am doing it slowly.

So, 1 minus 1 is 0, 2 minus 1 is 1, minus 1 minus 1 is minus 2, 1 plus 2 is 3, 1 minus 1 is 0, minus 2 minus 1 is minus 3, 7 minus 1 is 6 and minus 11; so, 1 minus 1 yeah, so minus 11 plus 2 is minus 9, alright. This is what I have, just like it just verify it 1 1 1. So, I will verify it once I get the final thing.

So, let me see 1 1 1 minus 2, 0 1 minus 2 3, fine. Now, this the last row is 3 times minus 3 times this 1; so, minus 3 6 minus 3. So, I get 0 0 0 0 fine, I can rewrite this as again further I have got 0 0 0 0 0 1 2 minus 3, I can use this to make 0, so 1 0 this minus this; so, 1 plus 2 is 3 I hope minus 2 minus 3 is minus 5, alright.

So, this is my RREF of A fine, I have got the RREF. Now, from here I want to conclude all my ideas, alright. So, now, what it says is let us look at this matrix carefully. The pivots were these were the pivots, fine. Since, these were the pivots; the 1st and 2nd; so, the implies that 1st and 2nd column of RREF of A are linearly independent, fine. So, if I look at A, A is nothing, but some P which is invertible matrix, invertible matrix times the RREF of A, fine.

Now, we are saying that, the first two columns are linearly independent fine. So, let us go back. So, go back to this ideas of writing vectors in terms of whatever way you want. Once you have written it, then you can say that this will imply that the first column, first column of A and the second column of A are linearly independent, fine.

So, just have a look at it, you can see that they are linearly independent. So, what I am writing is, this is a linearly independent set alright, the first two columns. You can also see that I can get the third column alright in terms of the first two columns. How do I get it? Think about it, for yourself, is that ok? So, how do I get the third one? You can think of this as system of equation and you are trying to solve it here.

So, it is 3 times the first one and 2 times the second one, something like that minus 3 minus 2 and so on; think about it how do you get it, fine. You can also say that here you can see that first and third are also independent because, I cannot get this column from here, fine. So, you can see that first and third are also independent fine. Similarly, first and fourth is linearly independent, second and third is linearly independent and so on.

At the same time you can see that if I want to look at this first column, second column and the third column then, this is a linearly dependent set. Because, you can see that in the RREF look at the RREF, in RREF RREF of A, if I look at this matrix; suppose I write this as R. Then, what I can see here is that the third column of this is equal to 3 times the first column plus 2 times the second column, alright.

I hope I have done the calculations correctly; so, no there is a mistake somewhere I think because minus here. So, there is a minus here, I should have done minus here, that was a mistake. So, there should be a minus here, 2 times this, fine. So, now, you could; so, just verify this is this is true, I can multiply P on the left throughout.

So, I can multiply by P here, P here, P here. Once I have done that, I get that A of this is equal to 3 times, A of this minus 2 times, A of this fine, just verify it. So, 3 times the first one it is 3 3 3 fine minus, why I am not getting it? Now, I think I getting it. So, 2 times this one; so, 2 times 1 2 minus 2 which is same as 3 3 3 minus 2 4 minus 4, which is 1 minus 1 and 7, 1 minus 1 7 which is your R 3rd 1, fine.

So, you can see that I have computed the RREF; generally RREF is done to solve a system of linear equations. But, in place of solving a linear system, it has also given you something more that is I am able to say what is happening to the columns here. You can similarly check here that in this example that the fourth row of A alright, fourth row of A or fourth column of A sorry is equal to minus 5 times; just look at here minus 5 times the first one and plus minus 3 times the second one.

See again there is a mistake here. So, this should have been 3 here. So, plus 3 times A of this alright; so, I wrote it wrongly from there. So, therefore, we are not getting it. So, you can see that everything is nice here, you can get things. So, the idea is whatever is true for RREF, you have that for you, apply P to it; the matrix which gave you RREF and then go back to A, is that ok? That is one thing that we got. So, we have said that these are linearly independent, the first two columns.

Further let us look at these two rows; these two rows in themselves are linearly independent, fine. Since, these are linearly independent; so, you can get again that, I think I should write it down the next line; otherwise, I will have a problem. So, for me the matrix is there, but RREF I want the RREF of A.

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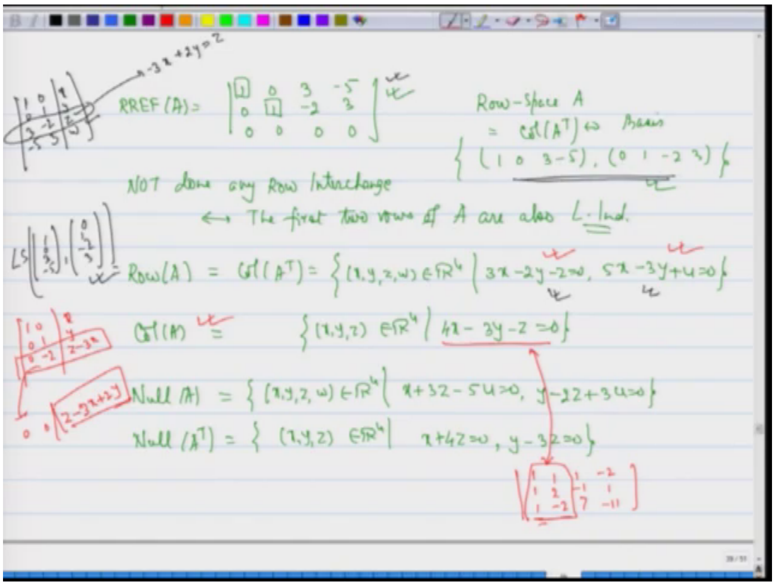

So, RREF of A is 1 0 3 minus 5 0 1 minus 2 3 0 0 0 0, whatever A that let us not worry about it, fine. So, this is my RREF of A, these were the pivots. So, we are saying that row space of A which was same as column space of A transpose, its basis is going to be 1 0 3 minus 5, this vector and the another vector is 0 1 minus 2 3, these are the two, alright.

In this example, it turns out that I have not done any row interchange, not done any row interchange. And therefore, the first two rows; the first two rows of A are also linearly independent, fine. So, if I had done interchange accordingly I will have to go back and then see; which were the rows which gave me these as the pivots and things like that, alright. So, whatever rows I have accordingly I will have to go back to A and then get my answers that is the basic idea.

So, what we have done here is that I have been able to get what is a row space of A. So, row space of A is nothing but, column space of A transpose which is generated by these two, think about it fine; how I can write it out? So, please check that this turns out to be x, y, z, w this belonging to R 4 such that, 3 x minus 2 y minus z is 0, 5 x minus 3 y plus u is 0. Column space of A, we already computed also there also, what was that? Column space of A was x, y, z belonging to R 4 such that 4 x minus 3 y minus z is 0.

I would like you to see that, null space of A was x, y, z, w belonging to R 4 such that x plus 3 y sorry x plus 3 z minus 5 u is equal to 0, y minus 2 z plus 3 u is equal to 0. And, null space of A transpose is x, y, z belonging to R 4 such that x plus 4 z is 0, and y minus 3 z is 0. This is what I am writing here, but let us try to understand from where did I get it, how did I get it, fine. So, here I am saying that these are in the; so, if I want to look at the row space of A, alright.

So, row space of A is generated by these two vectors, fine. So, it is alpha times this plus beta times this. So, it is supposed to be equal to linear span of the vectors 1 0 3 minus 5 and 0 1 minus 2 3, fine. So, you are supposed to check whether they are or not. So, if you remember what we had done was to check things, we need to look at the system which was 1 0 3 5 0 1 minus 2 3, look at x y z w and get the conditions.

So, you can see here that there are 4 equations here, only alright there are 4 equations for us. And therefore, we will need here two conditions to come into play, you can look at that, so just look at this part, it says here that x y will remain as it is. I can cancel out z, I can cancel out w. How do I cancel out z? Look at z, if I want to cancel this out this vector.

So, this vector is nothing but, 3 times x just remove the first one 3, this row minus this row will give me this part 0 and this plus this will give me 0, alright. So, I just need to add this. So, minus 3 x plus 2 y, so just look at this part, this alright should be equal to z; so, that they cancel out. So, what I am saying is I have got 1 0 x 0 1 y. So, I just multiply by 3 and subtract out it 0 minus 2 z minus 3 x.

Again, I want to make this 0 again; so, I will have to do with the second one; so, from 0 0 z minus 3 x plus 2 y, alright. So, for this system to have a solution, I will need that this is the condition and which is this condition. Similarly, the next one will give you this condition, is that ok? Fine, column space of again you can get it, that is or you can just look at whatever you have, try to get them yourself.

What I would like you to stress on is trying to look at examples and verifying things on your own, fine. So, just go back and see whether they make sense to you or not. So, for example, if I look at my matrix was 1 1 1 1 2 minus 2 1 minus 1 7 minus 2 1 minus 11. So, you can see that I am looking at column space of A, look at these two vectors, 1 1 1 1 1 1 will give you 0, 1 2 minus 2 if you just plug it here, you will get that 4 minus 6 plus 2 will give you 0, alright. So, these two were the basis elements as I said they satisfy this, fine. You can also see that something more is important, that we will come to in the next class.

Thank you for now.