Linear Algebra
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Lecture - 26
Basis of a Finite Dimensional Vector Space

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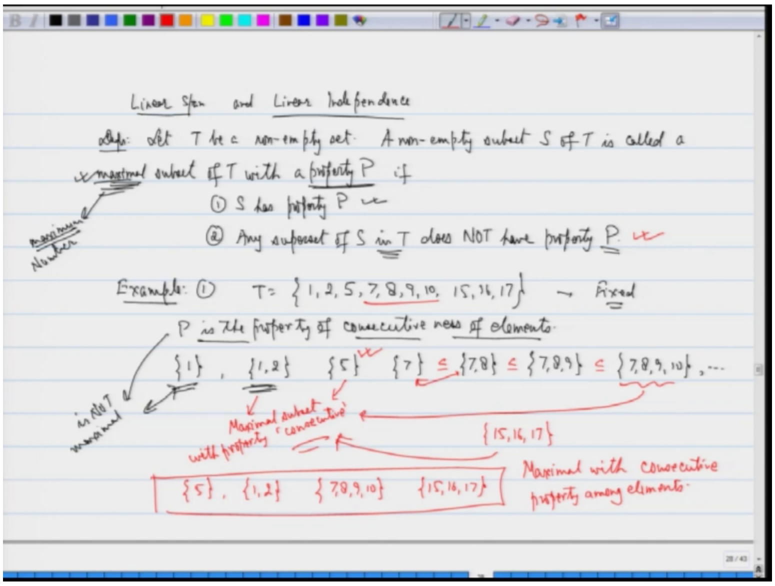

So, we learned what are called linear span and linear independence and they were related by a theorem which said that if I want to expand my linearly independent set, then I have to pick new vectors from elements which are not in the linear span. Is that ok?

So, we will try to get depend to it and come out with an idea which will say that I the maximum linearly maximum said that you can think of which linearly independent will give us some ideas about certain size of the vector space and similarly the linear span will also get related alright.

So, let us get to certain definitions which are important, fine. So, definition what are called maximal linearly independent set and minimal linearly independent set, linear spanning set also definition. So, let T be a non empty set non empty set. A non empty subset S of T is called a maximal set a called a maximal subset of T with a property P.

Maximal subset of property P if 1; S has property P. 2, any superset of S any superset of S in T that is in what in T does not have property P is that important. So, I have given a set T that is fixed once the T is fixed I define some property, alright.

So, I define a property P. So, fix a T which is non empty set define a property P. Now, pick up elements from there. In such a way that the set that you are making has the property P, but at the same time when I want to say it is maximal then I should not be able to add any other elements from T to S. So, that it becomes it still has property P, alright.

So, let us look at an example to understand it better example 1. So, I pick this T as 1, 2, 5, 7, 8, 9, 10, 15, 16, 17. I suppose I pick this set T alright. So, T is fixed alright. Now I want to a study the property P, P is a property that I am looking at P is the property of consecutiveness; property of consecutiveness of elements, is that ok? Fine.

P is the property of consecutiveness of elements from T I am looking at T, is that ok. So, if I look at the set T, I can just take this I can take 1, 2 they are consecutive 1, 2, 5 is not consecutive, 5 is in itself consecutive. Then 7 is constitutive because, there is only 1 element in it 7, 8, then 7, 8, 9, then I have 7, 8, 9, 10 and so on, these are the consecutive element that I am looking at consecutive sets fine. So, what I need is I want to talk of maximal subset. There is a difference between maximal and maximum.

So, maximum leads to what is the notion of number of elements number; larger, smaller and height and so on, whatever you want to talk off. Maximum is comparing things, is that fine ok. Comparing with respect to those numbers and those things. Property here I am talking in terms of maximal maximal means it has that property that is all I am not comparing 1 with the other.

So, for example, so, you say that S has property P. So, if I look at 1 has property P because there is only one element. So, it is consecutive among it is own elements, but it says that any superset of S in T does not have property P.

So, this 1, 2 is a super set of 1 and it has that property fine. So, therefore, S is not this is not maximal with the property this P with this property, is that ok. If I want to look at 1, 2 now. If we look at 1 2, then I cannot pick any element from T adjoin to it. So, that consecutively is still maintained.

So, therefore, 1 2 become some maximal; maximal subset with property, what is a property? Constitutive, fine. Similarly 5 is there is no problem with 5 because I cannot adjoin any element from T.

So, that this is lost fine. Any superset of S in T does not have property P. So, I cannot do that if I want to make it consecutively consecutive then I need to add 6 to it, 6 is not there in T therefore, this is again a singleton 5 is also a maximal subset with property consecutiveness fine. 7 is not because with 7 I can adjoin here there is a problem 7 8 I can join 7 8 9.

So, second part is lost again I can join this. So, again that is lost, but this is nothing but again this maximal alright. In 7 8 9 10 I cannot pick anything from here. Similarly, I would like to see that this set 15, 16 and 17 this is again maximal with respect to this fine.

So, what we see here is that the following sets 5 in itself 1, 2 7, 8, 9, 10, 15 16 and 17. These are 4 sets which are of different sizes, but all of them they are maximal with the property of consecutiveness. They are maximal with consecutive property consecutive property among elements, alright.

Five in itself with a single vector or single element. I cannot join anything from T, so that it becomes consecutive 1 2, I cannot join 3 there is no 3 here. So, again 1 2 is as it is. Similarly, 7, 8, 9, 10 I pick from here I get that I cannot either join from the left or from the right hence it is the maximal set. Similarly, 15 16 17 is a maximal set fine.

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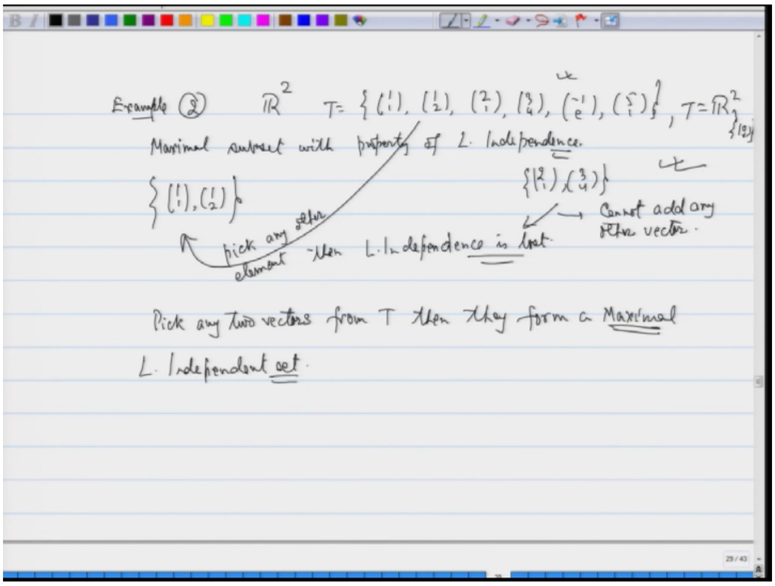

Another example 2. Suppose I am looking at R 2 alright; I can pick some vector say 1 1, 1 2, 2 1, 3 4 minus 1 e, 5 1 suppose I pick this as my T fine. Then what are the maximal? So, I want to look at maximal subset to with property of linear independence fine.

So, I can see that I can pick 1 1, 1 2 this is a set which is linearly independent and from here I cannot put anything extra here to a still get linear independence because as soon as I pick anything else. So, pick anything from here, pick any other element, then linear independence is lost , is that ok.

Similarly, if I take say 2 1 and 3 4. If I want to add anything extra from here then again I will be lost alright. So, cannot add any other vector, fine. So, you pick any two from here if you see pick any two. So, pick any two vector from T, then they form a maximal linearly independent set is that ok.

Now, this I am looking at any T, I can expand it further to say that if I just take T as R 2 itself minus the zero vector fine and I pick 2 vectors from R 2 which are linearly independent, then they will form a maximal linearly independent set fine.

So, what we are trying to say is that I am trying to evaluate the idea of maximal with the largest size in some sense alright of a linearly independent set fine. So, given any vector space I want to pick the maximum size the largest size of a maximal lineally independent set, is that ok.

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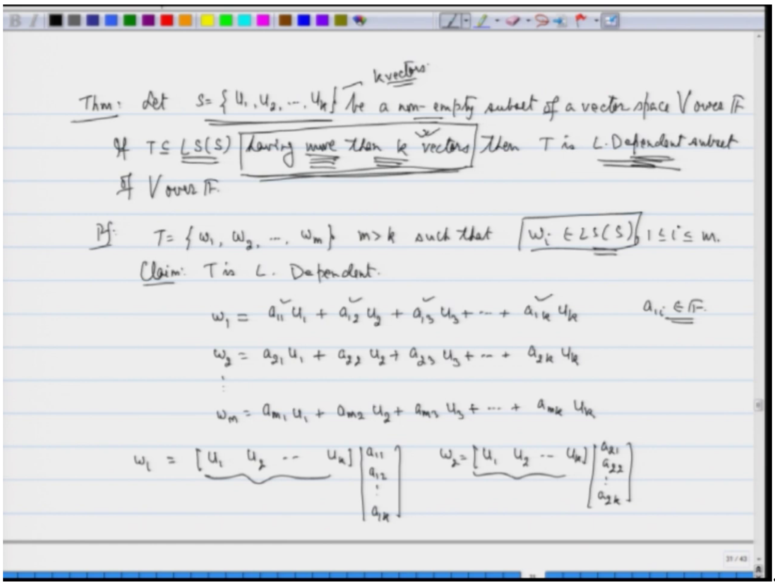

So, let us go into that part. So, let me prove this theorem, statement of the theorem let s is equal to u 1, u 2 u k be a non empty subset of a vector space V over F fine. If T is a subset of linear span of S; is linear span of S having more than k vector, this is more than k vectors alright k vectors. Then T is linearly dependent subset of V over F, alright.

So, let us try to understand what I am trying to say here I am not saying that S is linearly independent. I am just saying that S is some set which is non empty alright, is that ok? So, I have some set s which is non empty.

So, they are some vectors in it, they are k elements in this. So, they are k vectors here fine. Now, I am looking at the linear span of all these vectors. So, it is a subspace. So, I have a huge collection of vectors now. From that huge collection of vectors I pick any subset T in which the number of elements is more than k fine.

So, suppose. So, let us try to understand this. So, suppose I am picking a vector say w 1. So, let me pick T is equal to w 1, w 2, w m and m greater than k such that is w i belongs to linear span of S fine; 1 less than equal to i less than equal to m. We are claiming claim T is linearly dependent fine, this is what it says.

If I pick any subset of LS of S, linear span of S and their subset has more than k vectors then that subset is linearly dependent fine. So, I will have problem when I want to increase the size in subset that is what I am trying to say.

So, let me prove it. So, it says that w i's are in the linear span. So, when I said w i in the linear span what does it mean? It means that w 1 is some linear combination of elements of S; S has elements u 1 to u k.

So, I can write w 1 as a 1 1 u 1 plus a 1 2 u 2 plus a 1 3 u 3 plus so on till a 1 k u k, I can write like this fine. I can write w 2 as; so, these are some scalars that I am getting I do not know what they are, but this a i. So, a 1 i it belongs to F and I am writing w 1 as linear combination of u 1 to u k. I can do that because each w i is in the linear span fine.

Certainly for w 2 I can find another set of a scalars or maybe the same set of scalars I do not know, because I am not saying that S is linearly independent, I have got something. a 2 1 u 1 plus a 2 2 u 2 plus a 2 3 u 3 plus a 2 k u k, I can write like this. Similarly, I can go to a w m and write it as a m 1 u 1 plus a m 2 u 2 plus a m 3 u 3 plus a m k u k I can write like this fine.

So, be careful have a look at it nicely what we are trying to say is that; each of the w i's are elements of linear span since that elements of the linear span therefore, we can find a scalars there was a definition of linear span and linear combination when I say that something is in the linear span it means that I have a solution.

So, I have some scalars such that I can write w i is in terms of u i, so I can do it fine. So, I can write this as if I want to see it nicely let me write it in a different way. So, that you understand it. So, I can write this as w 1 as u 1 u 2 u k times a 1 1, a 1 2, a 1 k fine; w 2 as again u 1, u 2, u k and again a 2 1, a 2 2, a 2 k fine.

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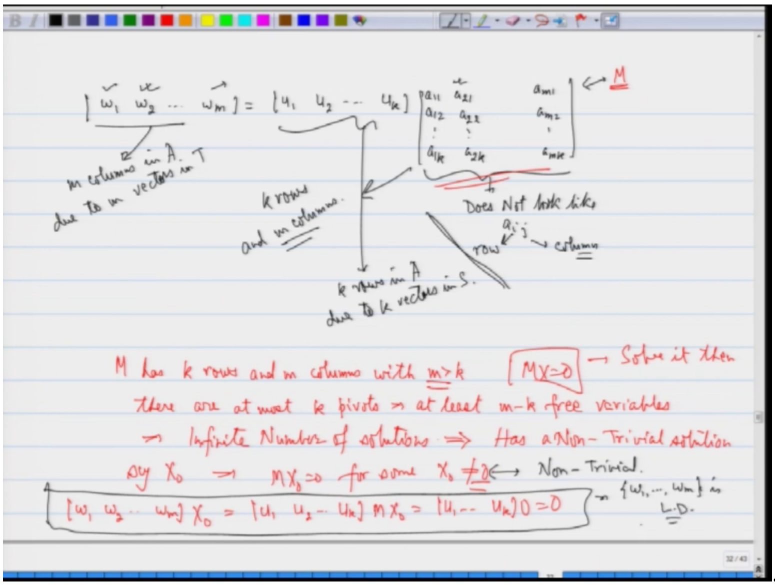

So, you can see that each of them can be written nicely like this. So, in general, what I can do is I can write this vector w 1 w 2 w m as this part is fixed u 1 to u k u 1 u 2 u k is fixed.

And then I have got for w 1, it is; a 1 1, a 1 2 till a 1 k. For w 2 I have the second set here which is a 2 1 a 2 2 a 2 k. And for w m I will have a m 1 a m 2 a m k alright. So, I have been able to write w i’s in terms of u i's with this new matrix coming into play for me is that.

So, I have got this matrix for me which is w 1 to wm is u 1 to u k times this. Now, this matrix look at the size of this matrix. The size of this matrix is how many rows are there? 1 to till k. So, this does not look like; does not look like a ij with i for the row; with i for the row and j for the column somehow it has got reversed alright fine.

So, even though it has got reversed what we see is that there are how many rows? First row, second row till k row. So, there are k rows in this matrix, k rows and m columns fine. Column number 1, column number 2, column number k m and they are coming from the columns are coming from here; m columns in A due to m vectors in T. And k rows because of k rows in A due to k vectors in S, is that ok.

So, this is the way we are doing it fine. So, now, let me write this matrix has something here. Now, what we see that this matrix have the property that it has got how many rows? It has got k rows and m columns. So, this matrix so, let me write this matrix as say M. So, M has k rows and m columns with m greater than k.

So, if I want to solve the system MX is equal to 0. Solve it then there are at most k pivots, this implies at least m minus k free variables and since I am look at the homogeneous system therefore, I have got infinite number solution. This implies infinite number of solutions fine. Since I have got infinite number of solutions; so, therefore, I get that this will imply that has a non trivial solution say X naught.

So, what I saying is that this will imply that MX naught is 0 for some x naught not equal to 0, is that ok. So, therefore, now if I multiplied this out with. So, this is my matrix M. So, if I want to look at. So, let us look at w 1, w 2, w m times X naught fine.

So, look at this times X naught, then this is nothing, but u 1 u 2 u k times M times X naught which is u 1 to u k times 0 which is 0 fine. So, what we see here is that I have got a vector X naught consisting of a scalars alright. I got X naught nontrivial such that this vector this collection w 1 w 2 w m times X naught is 0, alright.

So, it means that I have got a nontrivial solution and hence this said w 1 to w m is linearly dependent. This implies the set w 1 to w m is linearly dependent and this is what we want it. If look at the statement it says that if T is any subset of linear span having more than k vectors then T is linearly dependent alright.

So, since it has more than k vectors therefore, I could get free variables that is more important alright. I had k from k I had more than k. So, I had free variables and therefore, I got linear dependence that is important for us is that ok.

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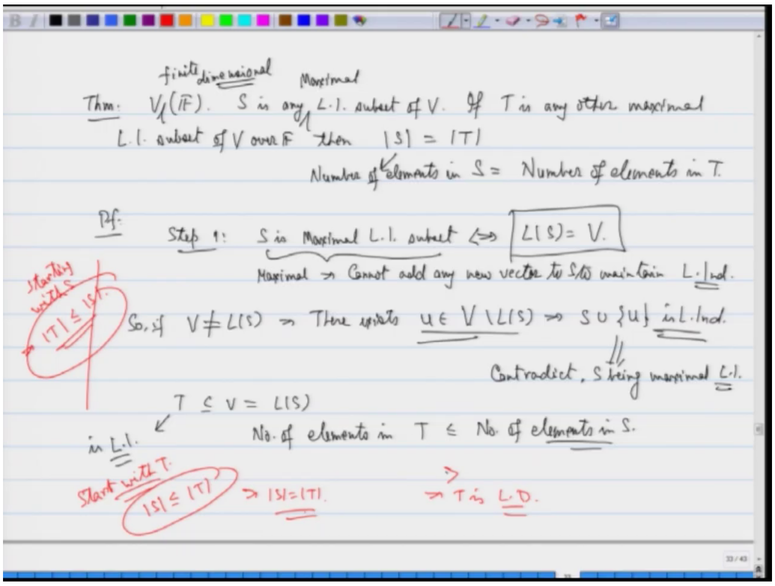

So, now let us this use this idea to come to the next theorem which is theorem this theorem. So, V is a vector space over F fine, S is any linearly independent subset of V, S is any maximal lineally independent maximal linearly independent subset of V. If T is any other maximal linearly independent subset of V over F, then number of elements in S is equal to number of elements in T. Number of elements in S is equal to number of elements in T.

I am not saying that S and T have same set same elements I am not saying that, I am just saying that the number of elements are same. So, if you go back to the slide one of the previous slides, this is what we had here that in R 2 I had picked up a lot of linearly independent subsets each had only 2 elements they could be different, but each of them had only 2 elements is that ok.

So, this is what we are saying here that V is so, I had to say that V is a infinite dimensional I will be assuming always at V is finite dimensional otherwise I cannot talk of number of elements fine. So, from now on everything is finite dimensional for us unless I say it is infinite it will be only finite dimensional alright.

So, what we are saying is that; if S is any maximal linearly independent subset of T of V T is any other. So, you have S and 2 which are maximal linearly independent then the number of elements is the same alright and the proof is simple, is that ok.

So, let us try to understand this 2 steps are there; step 1 a step 1; S is maximal linearly independent subset implies linear span of S is whole of V can I say this fine. So, S is maximal linearly independent what does it mean? I cannot add any other vectors alright fine.

So, I have so, maximal implies cannot add any new vector to S to maintain linear independence alright fine. So, if I pick any V so, if. So, if V is not equal to L S will imply that there exist u belonging to V minus L S and this will imply that we call the previous result that S union u is linear independent, alright. So, you recall that.

So, if V is not equal to L S then there is at least 1 vector u which is in V, but not in L S and therefore, S union u will be linearly independent fine. If all have been given this so, linearly independent S union u will become linearly independent and this will imply that this we contradict, S being maximal linearly independent, is that ok.

So, be careful what we are saying? What we are saying is that S is maximal linearly independent means; I cannot increase the size if I cannot increase the size of the linear independent set then, that has to be equal to the linear span, is that ok.

Converse is also true that if S linearly independent and linear span of S is V then I get back maximal lineally independent, alright. So, what the previous theorem says that; now if I look at this T is a subset of V is same as linear span of S. So, from the previous theorem we understand that the number of elements in T has to be less than equal to number of elements in S. Why?

Because T is linearly independent; T is linearly independent fine. Because, if number of elements in T is greater than, if it was a strictly greater here, if it was strictly greater, then this will imply that T is linearly dependent this is what a theorem was that if I pick more elements than the spanning set then I get linear dependence alright. So, therefore, I will lose that part is that ok.

So, therefore, what we see here is that number of elements in T will be less than equal to number of elements in S. Similarly, if I start with T alright. So, here I started with a step 1 was a starting with S and implying that number of elements in T is less than equal to number of elements in S.

Similarly, if I start with S, let us start with T will get that number of elements in S will be less than equal to number of elements in T and these 2 together will imply that number of elements are same is that ok. So, what we see is that for any maximal lineally independent set in a finite dimensional vector space, the number of elements are same. So, take any V vector a space finite dimensional is that ok.

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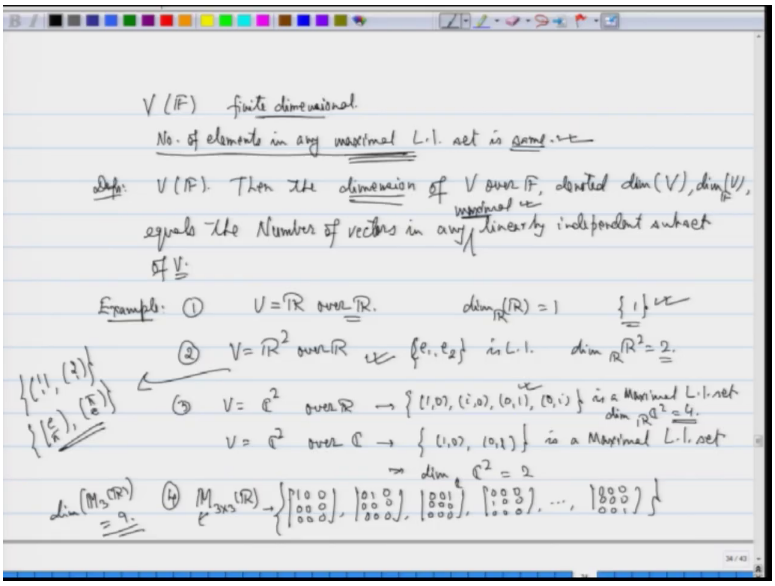

Finite dimensional number of elements in any maximal linearly independent set it is same alright. Since it is same we can use this to define something. So, definition V is a vector space over F, then the dimension of V over F denoted dimension of V.

Sometimes dimension of V over F like this sometimes this is dimension of V over F denoted this equals the number of vectors in any linearly independent subset in any maximal lineally independent; any maximal linearly independent subset of V is that ok.

Dimension is number of elements in a maximal linearly independent set alright. Because, what we are seen is that number of elements in any maximal linearly independent set is same. So, I can use that number to get my results is that ok, fine.

So, example 1; so, if I look at say vector space V as R over r itself fine, then dimension of this over R is 1 because any vector any scalar say 1 is a maximal linearly independent set and which generates. If I take V as R 2 over R, then e 1 and e 2, these 2 alright is linearly independent.

And I cannot get bigger than that. If I want to add anything extra I will have a problem and therefore, dimension of R 2 over R is 2 I cannot add any extra vector they will become linearly dependent alright. 3; I would like to see that look at this V and again C 2 over R over C.

Then here this vector 1, 0 and 0, 1 is a maximal linearly independent set. And therefore, dimension of C 2 over C is 2. Here you can see that this set 1 0, i 0, 0 1, 0 i is a maximal linearly independent set and therefore, dimension of C 2 over R is 4, alright. We will come to this again after some time, but I would like you to understand that fine.

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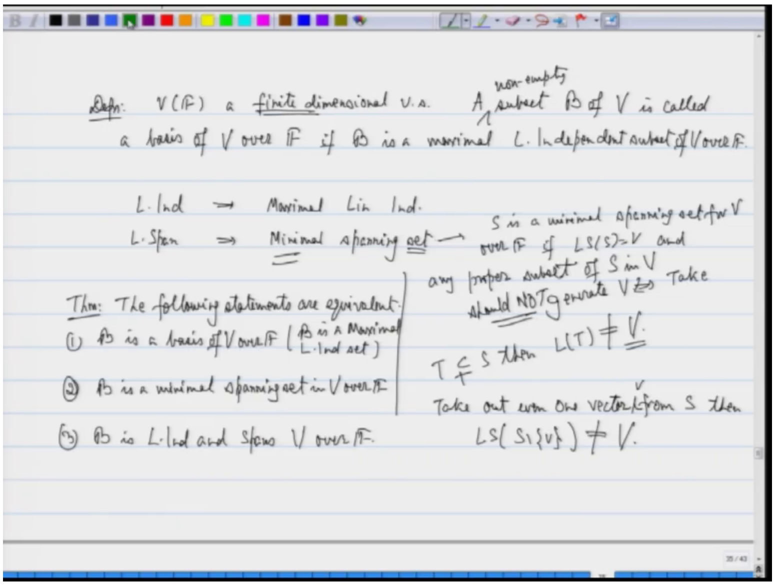

Now, what is a basis definition? So, definition of a basis let me right. So, V over F of finite dimensional vector space fine. A subset B of V so, non empty subset or non empty subset for us I can talk of empty also, but then that is for the zero vector I do not want to get into that part ok.

Non empty subset of B of V is called a basis of V over F if B is a maximal linearly independent subset of V over F, alright. So, in the previous examples, this is a basis this is another basis for R 2. I can also think of the vector 1 1 and 2 1 as a basis of R 2.

I can also think of e pi, pi e as a basis of R 2 I can take any collection whatever it is I just need them to be linearly independent fine. Similarly, these are the 4 elements which give a basis of C 2 over R; e 1 and e 2 is always a basis I also have if I look at the set of all matrices of say 3 cross 3 matrices over R then you can see that these matrices which are I do not know whether I wrote it or not, but they are the type 1 0 0, 0 0 0 this.

So, you can build up on this till finally, you get. So, this is a basis alright. So, therefore, the dimension of this over R is 9, is that ok. So, you can build up on these ideas yourself spend some time fine you can also look at somewhere else some notes and things like that and you can form it yourself, is that ok.

The next part is what is called. So, what we did was we had linear independence from there we went to maximal linear independence, alright. The other idea that we had was what is called a spanning linear span fine. From there I want to go to what is called minimal spanning set. So, what will the minimal spanning set?

So, definition will be that it should a span. So, S is a minimal a spanning set for V over F. If LS of S is whole of V and I am looking at minimal any proper subset of S in V should not generate V or which is same thing as saying that. So, take T which is a proper subset of S the linear span of T is not the whole space, is that ok.

So, from S you take out any vector even one vector alright. So, take out even one vector from student:, then linear span of S minus that single vector should not give you whole of V, is that ok? Take out even one vector V I should not get the whole space, is that ok.

So, this what minimal spanning set means; so, there is the theorem which says that. Theorem; the following statements are equivalent; the following statements are equivalent. Please note that I am still having finite dimensional with me, V is finite dimensional for me everything is finite dimensional alright 1, b is a basis of V over F.

So, basis means; recall again B is a maximal linearly independent set alright. 2; B is a minimal spanning set in V over F. 3; B is linearly independent and spans V over F alright. So, try to understand what I am trying to say.

I am trying to elate the different ideas that we had. What are the ideas? The ideas were let us go back the ideas were we started with vectors in those vectors where the notion of linear span that given set of vectors what is the largest vector subspace that I can generate. Then we also had this idea that given set of vectors are they linearly independent.

So, if I am saying that something is not linearly independent I can remove. So, recall a theorem which said that if I have linear dependence and there is a vector the first vector is linearly independent is nonzero vector, then I can remove alright. u k is a linear combination of u 1 to u k minus 1 alright fine.

So, if I have a set which is linearly dependent, I can remove vectors from them to get a smaller set which will be linearly independent fine. So, those ideas were there. So, what we are trying to do in finally, here in this part is saying that; I take a finite dimensional vector space finite dimensional means I am able to generate it using finite number of vectors.

Once I have finite number of vectors, look at that set from them I want to pick up the smallest possible a spanning set, is that ok. That is 1 way of talking of things given S it is giving with the whole space from there I want to pick the smallest size the minimal spanning set that is one question that I am looking at. What I am looking at is; what is the maximal linearly independent set that I can get from that set. S is given to me from them there I want to pick the maximal linearly independent set.

So, two questions I want to pick minimal spanning set I want to pick maximal linearly independent set two questions alright. The answer to both the question is the same, this is what it says this theorem says; that we want to go to maximal linearly independent set, then that is also equal to the minimal spanning set and not only that those two together or that set itself has the property that it is a basis means it a spans as well as is linearly independent, is that ok.

So, all the three ideas that start with any vector space which is finite dimensional you pick up a single vector which is nonzero. Keep on building keep on adding extra vectors to it till you have exhausted all the space alright and at each stage your vectors were linearly independent your sets were linearly independent alright. The collection of vectors were linearly independent.

Then finally, you will get a set which will give you the full spanning set that full spanning set will be linearly independent because that is the way you have added and that full spanning set will also be the minimal spanning set. You cannot add anything extra, is that ok. So, all these ideas are similar and I would like to end here itself this lecture. Think about it, do some assignments you will understand them.

Thank you.