Linear Algebra
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Lecture – 25
Results on Linear Independence Continued..

So, we learnt a very important theorem in the previous class which was about starting with a so, I have vector space V over F fine.

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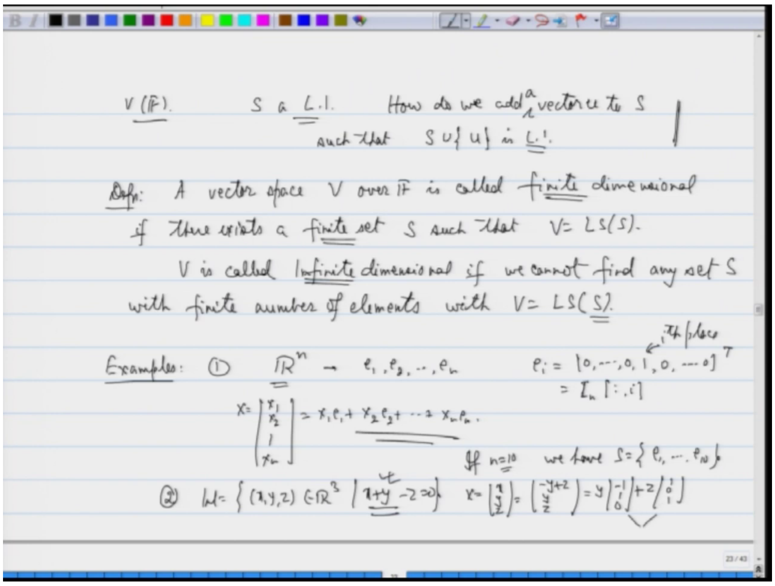

I have a S a linearly independent set. How do we add vectors to S such that vectors mean say. So, add a vector let me write add a vector u to S, such that S is union u is linearly independent alright.

So, we will take up this idea and then proceed further. So, before that let me now talk of what is called finite dimensional vector space and infinite dimensional vector space. So, a vector space V over F is called finite dimensional finite dimensional if there exist a finite set alright, there is a finite set S such that V is linear span of S is called.

So, V is called infinite dimensional infinite if we cannot find any set S with finite number of elements with V is equal to L S of S alright. So, finite number of elements which I can expand the whole thing alright. So, you already saw that when we looked at linear independence dependence here, we went to the linear span. So, they are related as such and therefore, I wanted to have this idea with me.

So, let us look at some examples now to differentiate between the two things examples. 1st example, if I look at say R n then I can generate R n using the vectors e 1, e 2 e n. And what were they? e i was the vector which had everywhere 0, 1 at the ith place and 0 again everywhere else alright or it was same as. So, let me write transpose of this because it is a column vector.

So, look at the identity thing, look at the ith column of that then this is e i for us. So, any element x here is of the type x 1 x 2 x n. So, this is nothing, but x 1 e 1 plus x 2 e 2 plus x n e n alright. So, if n is 10, if n is 10 we have S which consist of e 1 to e 10 10 vectors which expands it, fine.

Similarly, anything you can look at that for example, if I look at this example W which is x, y, z belonging to R 3, such that x plus y minus z is 0. Then this condition tells me that look at this condition it tells me that x here element here is x, y, z. It is of the type x is nothing, but minus y plus z, y remains as it is, z remains as it is.

So, it is y times minus 1 1 0 plus z times 1 0 1. So, it is linear span of these two vectors alright; any element here is the linear span of these two vectors. So, again it is finite dimensional alright.

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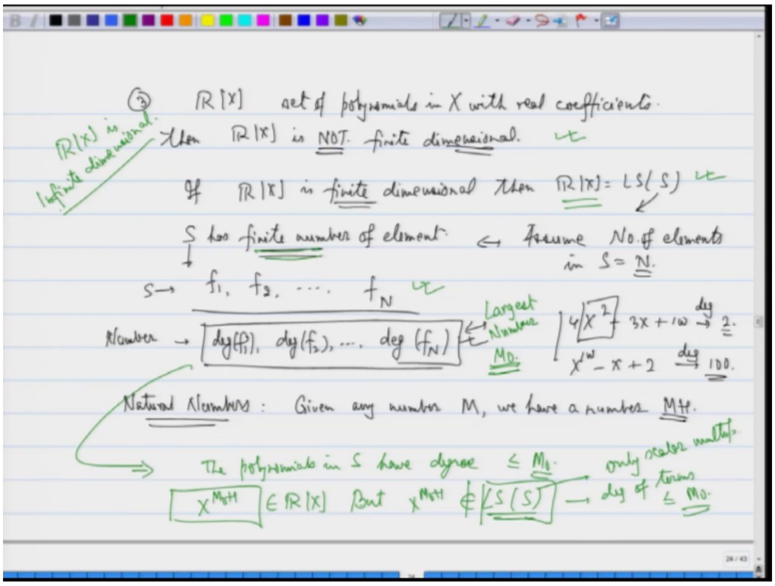

3rd example which is not finite dimensional is look at this set of polynomials x set of polynomials in X with real coefficients fine, then R x is not finite dimensional. So, the idea is simple. Suppose it is finite dimensional, then there will be some size; the degree of the polynomial will be some finite number alright.

So, if r x is finite dimensional, if it is finite dimensional then R x is by definition linear span of S and S has finite number of elements fine. So, if it is finite dimensional, it has to be the linear span of a finite set. So, S has finite number of elements. Now, if S has finite number of elements; what are the elements of S? The elements of S are nothing, but some polynomial f 1 f 2 so on till f N alright.

And, what is N? N is the number of elements in S fine. So, again understand it nicely; S has finite number of elements. So, I can assume number of elements in S is N. I look at all the polynomials f 1, f 2, f N; for each one of them I can compute. So, for each S here I am computing a number. And, what are those numbers? Those numbers are degree of f 1, degree of f 2, degree of f N alright. These are polynomials.

So, each of this polynomial has degree. What is the degree of a polynomial? The highest degree term, alright. So, if I look at for example, 4 x square plus 3 x plus 100 alright, then degree of this is 2 because highest degree is 2 here alright, fine.

Similarly, if I have got x to the power 100 minus x plus 2; its degree is 100 fine. So, depending on the highest degree, the degree of a polynomial is defined. So, what we see is that each of these numbers are going to be a number between some number to something alright. So, each of them will give me some number fine.

Can I say that there is always a number? So, we know; let us look at natural numbers natural numbers. Natural numbers have the property that whatever number you give, I can always find a bigger than that. So, if you give; so, given any number N or getting any number M, we have a number M plus 1; all that is we always have a larger number.

It never ends, natural numbers never end fine. So, if you are saying that I have certain set of polynomials, their degrees are again only finite number of them; I can look at the largest value here I can. So, these are collection of numbers, I can look at the largest number here; largest number here fine.

Suppose, the largest number in this collection alright, largest number in this collection is some M naught fine. It means what? If the largest number in this collection is M naught, this will imply that there the polynomials in S have degree less than equal to M naught fine.

So, if I look at this polynomial. So, if you look at the polynomial X to the power M naught plus 1, look at this polynomial. This belongs to R x, but X to the power M naught plus 1 does not belong to linear span of S. Because, when I am looking at linear span of S, I am only multiplying by scalars; only a scalar multiplication is allowed fine.

Since, only a scalar multiplications are allowed, the degree of the polynomial does not increase. So, the degree here degree of terms will be less than equal to M naught itself alright. So, again I started with R x, I am assuming that it is finite dimensional. Since it is finite dimensional; so, I have a finite set S which generates it, which gives me the linear span as the whole set.

I take the collection of all the elements, since it is finite I can enumerate it. It is a finite number of element; so, I can enumerate all the elements. Once I have enumerated all the elements, I look at degree of all those now; see this is a finite number of numbers. So, I can look at the largest number among them. Suppose, it is M naught then we are claiming that X to the power M naught plus 1 does not belong, is that ok?

And therefore, this assumption that every all the polynomials can be generated with elements of S is wrong alright, we get a contradiction. Therefore, it is infinite dimensional, is that ok? So, R x is infinite dimensional fine. So, this is important. Now, some implications of the previous result that I had done; so, some corollaries of that let me write it.

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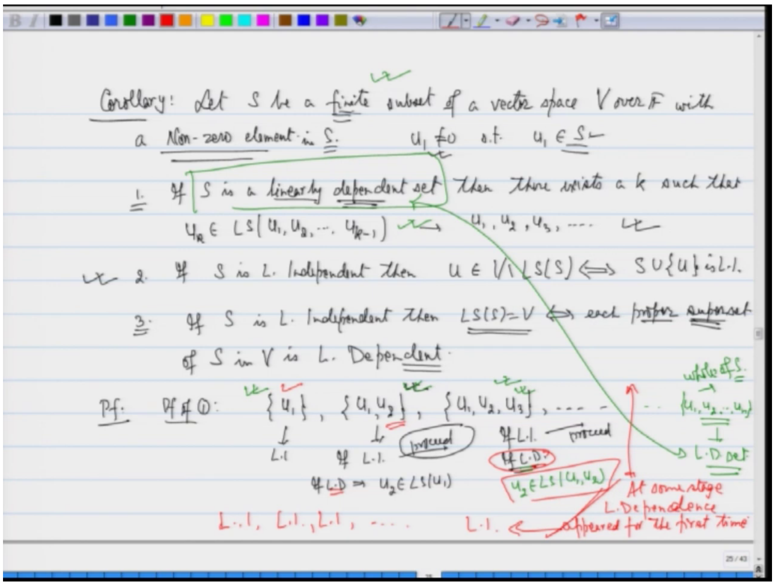

Corollary, 1st corollary be careful. So, let me write the definition for; so, let corollary. So, let S be a finite alright, S be a finite subset of a vector space V over F with a non-zero element, with a non-zero element in S alright.

So, I am starting with that S already contains a vector alright. So, there is a vector u which is non-zero such that u 1 belongs to S. So, this is already assumed fine. Then, if S is a linearly dependent set dependent set, then there exists a k such that u k belongs to linear span of u 1, u 2, u k minus 1 alright.

So, what we are saying here is that I am looking at u 1, this is a non-zero vector, then I am looking at u 2 which is in S, u 3 and so on; is that ok? So, I am looking at this collection fine.

2, statement 2 I will prove them afterwards 2; if S is linearly independent alright and then u belongs to V minus L S of student: if and only if S union u is linearly independent.

This is rewriting the previous statement itself. Recall that there we wrote that S union u is linearly independent, if u does not belong to linear span which is same thing as saying that u belongs to V minus L S of S alright. So, it is just the rewriting of that statement. 3rd, if S is linearly independent pendant then L S S of S is equal to V, if and only if each proper super set. So, here it is proper superset alright, set of S in V is linearly dependent fine.

Let us understand it nicely. So, proof. So, it is rewriting the same thing in a different languages. There we said that recall what we said that, I started with a u which is non-zero and then I want to add certain vectors to it. If I able to add vectors it means that I am picking vectors which is not in the linear span, this is what it says here fine. This is also the same thing that once I am said that everything is full, I have been able to get whole of V.

Then if I want to add anything extra, I will only get linearly dependent part alright; linear independence will be lost. And, the first part says that I have got u 1 which is linearly independent, u 2 is linearly independent when I join u 1 and u 2, u 1 u 2 become linearly independent. I again join u 3 to it, I again get linear independence. I keep on going, I will get u k the first time which will be linear dependent then it has to belong to the previous ones, is that ok? That is all it says.

So, I will just prove the first one, second, third are already implications there. So, proof of 1. So, I am looking at this collection of vectors now u 1, u 1, u 2, u 1, u 2, u 3; I am looking at this collection and so on fine. Now, if this is linearly independent that is already given to me; if this is linearly if linearly independent, then I cannot write u 2 as linear combination of u 1.

If linear dependent will imply that u 2 alpha u 2 plus beta u 1 is 0 and this will imply that u 2 is equal to minus 1 upon alpha times beta u 1 fine. I am saying alpha linearly dependent, it means what? Linearly dependent means u 2 belongs to the linear span of u 1; this is what it is fine. Recall the previous theorem, it says that u 1 was linearly independent or let me write it this way for you. So, that there is no problem.

If linearly independent proceed to the next, else if linearly dependent implies by the previous theorem that u 2 belongs to linear span of u 1 itself. And therefore, we have been able to prove this part, that k such that u k belongs to linear span of the previous ones. So, u 2 is a linear span of u 1 itself alright. If I come here, if this is linearly independent alright then I can proceed to the next step.

If it is not, if linearly dependent then by the previous theorem we know that see I came here because, of this part proceed. Proceed means this was linearly independent, fine. Now, if this is linearly independent and this becomes dependent then this u 3 by the previous theorem belongs to linear span of u 1 u 2, is that ok? So, therefore, I have been able to again prove this, is that ok?

So, at I am just looking at all the collection of elements, is that ok? S is a finite set; so, at some stage I am going to look at whole of S. So, I will get u 1, u 2, u n somewhere; this will be whole of S, is that ok? That is important.

So, I am a starting with an linearly independent set. If whenever there is a linear dependence, I can see that it is linear combination of the previous ones. What it says is that finally, I have a linear dependent set. So, this is a linearly dependent set that is given to us.

So, look at this part, it says that S is a linearly dependent set here, is that ok? It tells me as a linearly dependent set. So, finally, I have a linearly dependent set, it means what? I am starting with independence, independence, independence and suddenly I get finally, at linear dependence. It means that at some stage in the middle I would have got linear dependence.

So, at some stage alright at some stage this stage linear dependence dependence appeared for the first time. So, what do you mean appeared for the first time? Appeared for the first time basically means that till that a stage everything was linearly independent. I had linear independence, then linear independence, linear independence.

At each stage till the previous stage, I had linear independence and I got linear dependence for the first time. For example: this is linearly independent, if this is linearly dependent for the first time I get linear combination. If this is again linearly independent, I get linear dependence, if this is linear dependence then this is a previous combination and so on; is that ok?

So, I am just a starting with linear independence, linear independence and finally, there is linear dependence. So, linear dependence would have come for the first time, just a stop there. So, it says that since this is the first time before that it is linearly independent, it means from the previous theorem that this vector is a linear combination of these ones, is that ok? This is what it says. So, you have to keep track of that, that is very important for us fine.

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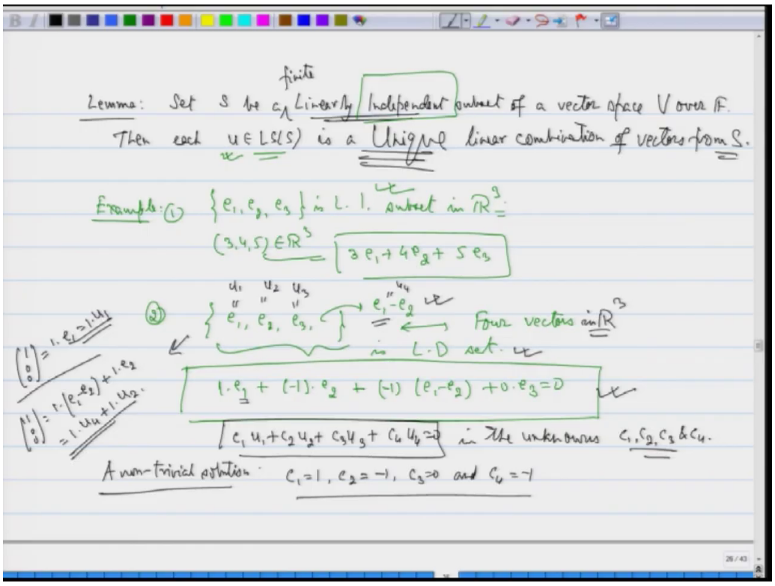

So, that finishes this corollary fine, one more result lemma. Let S be a linearly independent set subsets, I will assume finite be a finite subset of a vector space V over F fine. Then each u belonging to linear span of S is a unique linear combination of vectors from S alright.

So, this is important very very important. What we are saying is that I know that S is linearly independent alright, we are saying S is linearly independent. Once it is linearly independent, I look at the linear span of the whole set fine of S and picking any element of u. Then it says that I can write u uniquely, I cannot write it in two different ways; is that ok? That is important. For example so, what it basically says is that if I had try to understand by example alright.

So, I know that these two vectors 1 1 1 or let me write e 1, e 2, e 3 is linearly independent subset in R 3. Then if I take any vector say 3, 4, 5 belonging to R 3 then there is no way other than writing this as 3 times e 1 plus 4 times e 2 plus 5 times e 3. This is the only way I can write, there is no other way alright. And, this is happening because this is linearly independent set fine.

In place of that, if I have a linearly dependent set. So, let me take e 1, e 2, e 3 and e 4 fine, not e 4. So, let me write it as e 1 minus e 2 there are four vectors in this now, four vectors fine. Then I know that this is linearly dependent, this is linearly dependent set. Why it is linearly dependent set?.

Because, look at this here it is 1 times e 1 fine, plus minus 1 times e 2 plus minus 1 times e 1 minus e 2 plus 0 times e 3 is 0. Look at this system here alright; e 1 1 minus 1. So, this the minus cancels out. So, I am looking at the system c 1. So, let me write this as u 1 u 2 u 3, this as u 4; c 1 u 1 plus c 2 u 2 plus c 3 u 3 plus c 4 u 4 is equal to 0.

I am looking at this system in the unknowns c 1, c 2, c 3 and c 4. What we see here is that it has a non-trivial solution, trivial solution and what is the non-trivial solution? c 1 is 1, c 2 is minus 1, c 3 is 0 and, c 4 is minus 1; fine. Please verify, I may have done a mistake. But, the idea is that you have a non-trivial solution and therefore, you have more than one way of looking at things; is that ok?.

So, this set has four vectors and there are elements of R 3 it is a four vectors in R 3 fine and therefore, they are linearly dependent, is that ok? Fine. So, here if you want to write for example: if I want to write say this vector itself 1 0 0 which is e 1, I can write it in two ways. So, that is 1 times e 1 itself, this is one way of writing it. The other way of writing this will be, use this u 4; I can write 1 0 0 as fine.

Look at here 1 times e 1 minus e 2 that is u 4 plus 1 times e 2. So, I am writing this as 1 times u 1, this is 1 times u 4 plus 1 times u 2. So, I have been able to write it in two different ways, is that ok? So, the unique linear combination is lost, is that ok? Fine.

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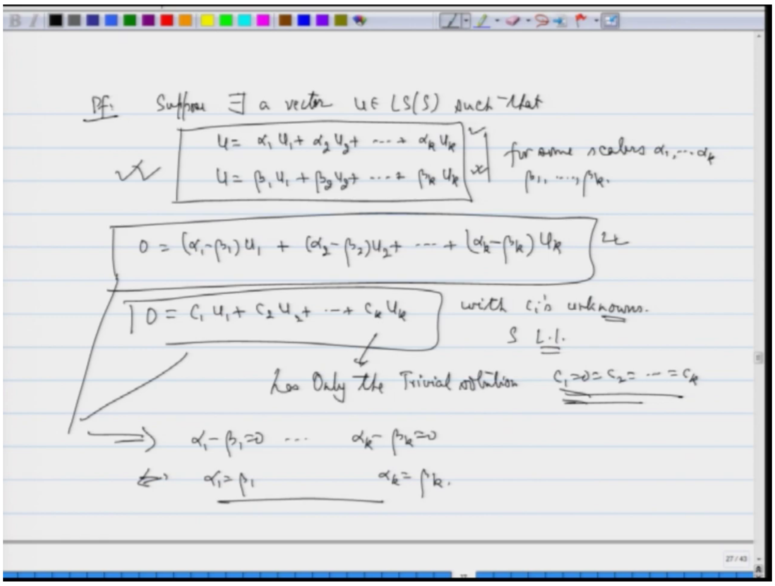

So, let me just prove it for you. So, proof suppose there exists there exist a vector u belonging to linear span of S such that u is equal to alpha 1 u 1 plus alpha 2 u 2 plus alpha k u k alright. And, u is also equal to beta 1 u 1 plus beta 2 u 2 plus beta k u k for some scalars alpha 1 to alpha k, beta 1 to beta k; is that ok?.

I am able to write u in terms of two ways alright. I have put the same u k k and k by adding extra 0s alright. So, I can always patch with extra 0s. So, that the vectors are same for me, is that ok? Now, from here what I see here is that 0 is equal to alpha 1 minus beta 1 times u 1 plus alpha 2 minus beta 2 u 2 plus alpha k minus beta k u k alright. So, what I am looking at? I am looking at a system which is 0 times c 1 u 1 plus c 2 u 2 plus c k u k, I am looking at the system with c i's unknowns alright.

Since, the c i's are unknowns, I have to look at the system. We have been given that S was linearly independent, S linearly independent. It means what? This system has only the trivial solution c 1 is equal to 0 is equal to c 2 is equal to c k alright. It has only the trivial solution. And therefore, what I can relate this idea with this part, these two ideas to imply that this is also looking at alpha 1 beta 1 alpha 2 beta 2.

So, using these two idea what I get is that, if this has to be 0 then alpha 1 minus beta 1 has to be 0 implies that alpha 1 minus beta 1 is 0 so on till alpha k minus beta k is 0. And, which is same thing as saying that alpha 1 is beta 1 and so on till alpha k is equal to beta k.

So, my expression here that I wrote is a unique expression alright. So, I end with the lecture here itself. We will look at the next idea in the next class.

Thank you.