Linear Algebra

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Lecture - 24
Basic Results on Linear Independence

So, let us recall what we have done. We had started with the definition of vector space alright. So, we had a notion of vector space, then we went to subspace.

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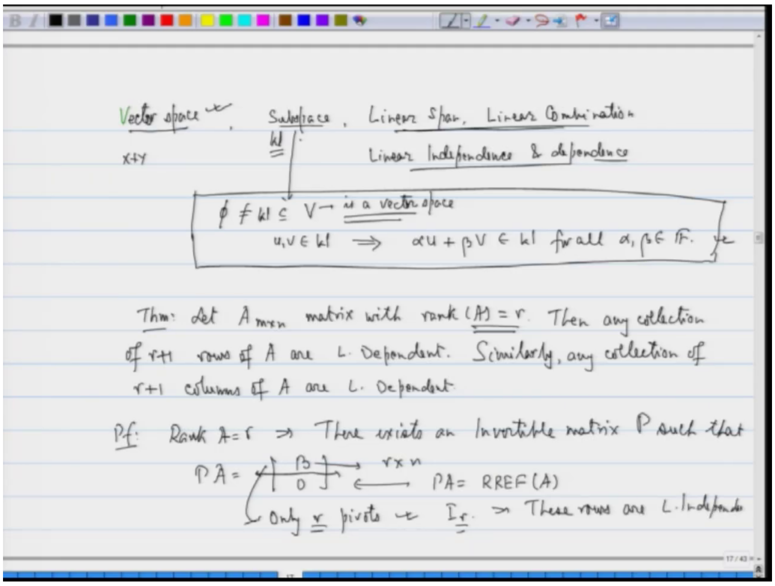

Then we went to what is called a linear span, then linear combination the ideas were related here and then there was a notion of linear independence and dependence.

So, we have learnt all of them in some sense fine . It is important to emphasize that I had not tried to prove anything as such to be vector space. I just gave you examples. I did not verify the conditions which you required for checking whether something is vector space or not.

So, if you recall. To show that something is a vector space what you needed was, you have define one vector addition you are define X plus Y, you have to check that whatever you define it is inside that set itself fine.

Then that sum has to satisfy certain things and what were they? They were associativity holds, commutativity holds, this is a 0 vector and for every X there is a negative vector fine. So, those properties were required, then there was a scalar multiplication that had to be checked alright.

So, you have define one alpha times X and that should be inside the set. To also check that one times X is X and, then distributivity holds. So, they are lot of things that needs to be proven, if you want to prove that something is a vector space. To prove that something is a vector subspace, you need less work.

What you are supposed to do is, you have to say that. So, you are saying that W is subspace alright we need to show that W is a subset of V where V is a vector space. So, I just have to point out I just have to say that W is a subset of some set fine. Also, W has to be non-empty.

And what we need to do is that take any u and v belonging to W, we should be able to show from here that they should imply that alpha u plus beta v should belong to W for all alpha beta belonging to F alright.

So, to show that something is a subspace, I just have to point out; I just have to point out that this is the subset of something and it is non-empty and then just do only one thing which is show that linear combination is there in the set. Nothing more than that, is that ok? So, that way its saves our lot of time.

Now, let us look at something about independence dependence. I had said it about the rank of a matrix. So, let me get into that with an example so that you can recapitulate this idea which are important. So, theorem. So, let A be an m cross n matrix with rank of A is equal to m alright sorry rank of A is equal to r, then any collection of r plus 1 rows of A are linearly dependent.

Similarly, any collection of r plus 1 columns of A are linearly dependent alright. So, this statement is true in general. I will just give you a glimpse of the first part, because for the second part the idea is similar, but you are not used to till now looking at column transformation, that is one thing.

The other thing is that I have not yet proven to you that row rank is same as the column rank and hence, I am not getting into that part fine. So, let me just say and then after sometime you can see yourself that you will be able to prove it yourself. So, I will not get into it, but I will try to prove the first part about the rows. So, let me look at the proof fine.

So, what we know is that the rank is r. So, rank is r implies there exist an invertible matrix P such that P times A have the type B 0 where this is r cross n alright. Or what I am saying is that I am looking at the Gauss elimination method or I can say that I am looking at the R R E F of A whatever you want to say alright.

You can think of this as P of A being the R R E F whatever what you want to look at I can think of this. Once I have looked at it, what I see is that there only r rows here only r pivots fine. Since there are r pivots, what happens to the pivots? This pivots they will give me I sub r into play and this will imply that these rows are linearly independent. Why will they will linearly independent? So, let us try to prove it.

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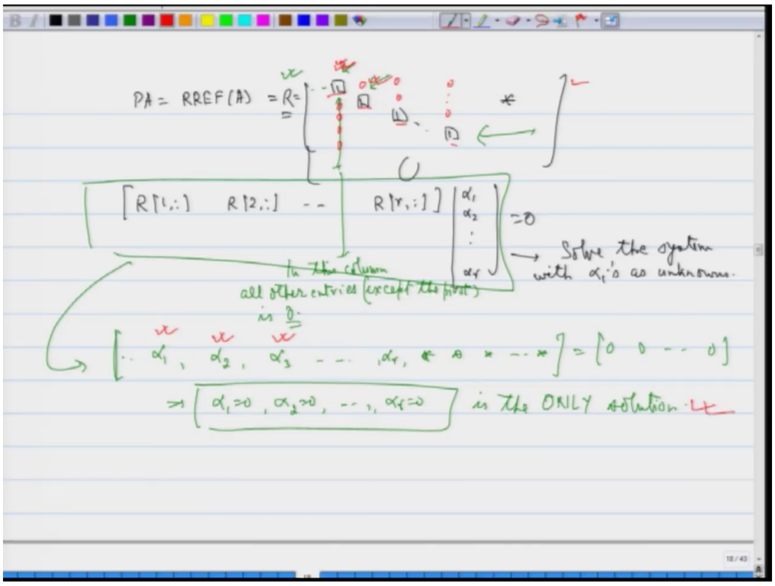

So, suppose I have the rows. So, I have got P of A which is R R E F of A it has. So, let me write this matrix as R in short.

So, R looks like something here I have a pivot here, I have a pivot somewhere here, I have a pivot somewhere here and I have this pivot R of them and then this is 0 here and there is something here fine, is that ok? So, now, I am supposed to look at these vectors alright. So, vectors are R of this is my first vector, this is my second vector and this is my R th vector fine. I am supposed to look at alpha 1 alpha 2 alpha r of 0.

I want to solve the system; solve the system with alpha i's as unknowns fine I want to solve the system. So, now, if I look at important please note this here, that I am trying to solve the system I want to show that each alpha i is 0 fine. So, if I look at this part this column alright. I know that in this column; in this column all other entries except the pivot is 0 alright.

So, if i look at here alpha 1 times this, alpha 2 times this and look at the corresponding entry here, corresponding entry corresponding to this entry whatever this entry is fine. I will get only alpha 1 here, is that ok? So, I will have something's here I do not know it has appeared somewhere. Similarly, this has come at some place I do not know which place it is. So, this will give me again alpha 1 times 0, alpha 2 times 1, alpha 3 times 0 and so on.

So, it will give me alpha 2 somewhere similarly, alpha 3 somewhere and so on till alpha r and then it will be I do not know what is here. So, I do not know what these parts are, is that ok? So, this is a this multiplication is basically giving me this vector alright. And we are saying that this vector is 0 0 0 here.

So, this part will imply that alpha 1 is 0, alpha 2 is 0, alpha r is 0. So, therefore, I get that this is the only solution is the only solution fine again. So, again note it down. I am looking at the R R E F of A, was I am looking at the R R E F of A I know that there is a ladder like thing coming into play.

Since, there is ladder like thing coming into play, this once are appearing at different stages that is the one thing. Every other entry on those pivots are 0. Since, every other entry in that pivot is 0.

So, what happens alpha 1 times the first row and look at this part alpha 1 times the first row will give me alpha 1 in this entry. All the entries here, will be since every entry here is 0. So, alpha times 0, alpha 3 times 0, alpha r will give me 0 here. So, I will be left out with alpha 1.

Similarly, if I look at here it will be alpha 1 times 0 is here already, because it is an R R E F, it will be alpha 2 times 1 alpha 3 times 0 alpha 4 times 0 so, it will give me alpha 2. Again this is a pivot so every other entry here is 0 fine. So, I will get only alpha 3 and so on. So, now, I equate the two sides I will get that alpha 1 is 0 alright. So, I will get this is the only solution.

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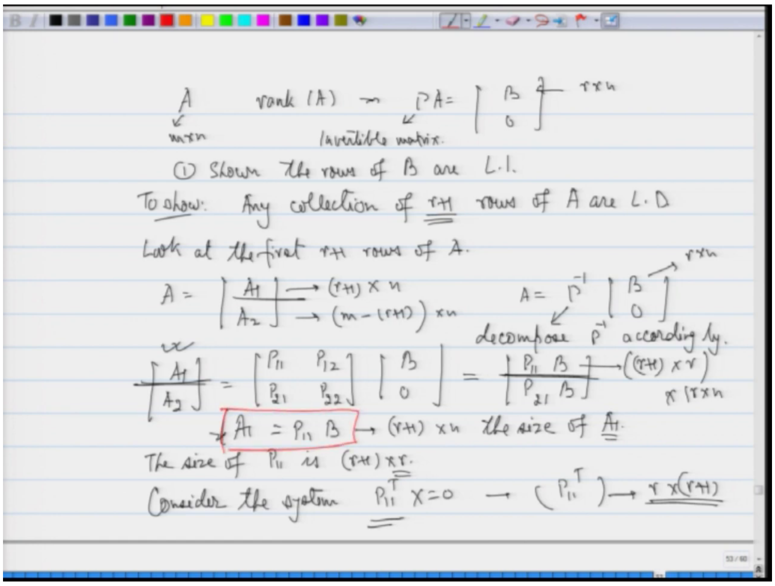

We have seen that that if I look at A which was of rank r, then we had this P times A is a matrix B with 0 and this matrix has size r cross n. So, cross n and we are looking at this rank is r and P is your invertible matrix invertible matrix fine and shown the rows of A, rows of B are linearly independent fine.

So, now, we need to show to show any collection of r plus 1 rows of A are linearly dependent any r plus 1. So, we will just look at the first r plus 1. So, look at the first r plus 1 rows of A fine.

So, I writing A as some A 1 A 2 and A 1 consist of this is of size r cross 1 plus n and this is of the size m minus r plus 1 cross n. I also have A times A is equal to P inverse of B 0. So, we decompose B; decompose P inverse accordingly means the matrix product make sense.

So, I write here A 1 A 2 as something here B 0. So, I can write it as some P 1 1, P 1 2, P 2 1, P 2 2. Now, what is the size of P 1 1? It has to be defined so that I can multiply that I can do the matrix (Refer Time: 12.45) multiplication. So, if I have to do the matrix multiplication. I can look at it is this times B and this is P 2 1 times B the rest part is 0.

So, this is going to be the size P 1 1 cross B according to this part, if I look at this part this is going to be of the size r plus 1 cross B, B the size of B is r cross n. So, therefore, r r has to cancels. So, I get r here cross r cross n. So, this will imply that P 1 1 B has size r plus 1 cross n, the size of A 1 fine.

So, the matrix product make sense. So, what we see here is the size of P 1 1; size of P 1 1 is r plus 1 cross r fine. So, now, let us look at the system alright. So, consider the system consider the system P 1 1 transpose of X is equal to 0 fine. So, now, if I look at P 1 1 transpose this is a matrix of size r cross r plus 1 since this is of size r cross r plus 1. So, this system.

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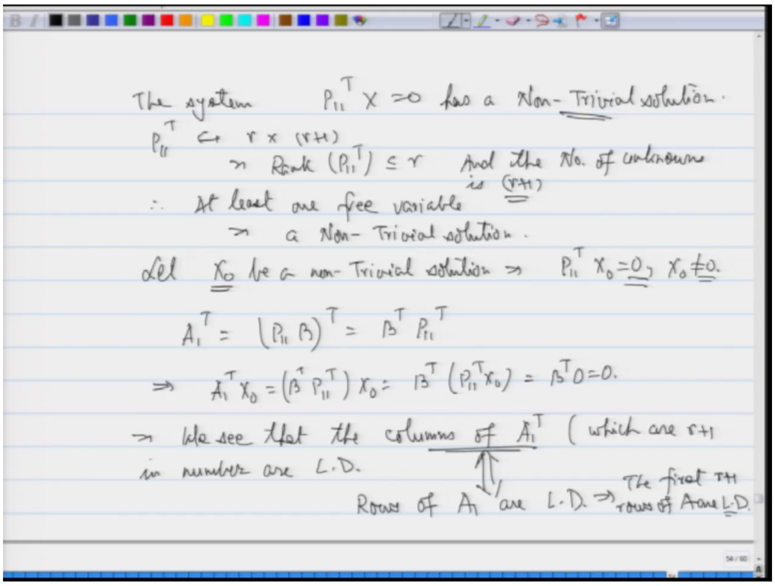

So, the system the system P 1 1 transpose of X is equal to 0 has a non trivial solution, why it has a non trivial solution? Why non trivial? Basically because P 1 1 transpose this is of size r cross r plus 1 implies rank of P 1 1 transpose is less than equal to r and the number of unknowns is r plus 1. So, therefore, at least one free variable implies a non trivial solution fine.

So, let X naught be a solution. So, let X naught be a non trivial solution this implies we are looking at X naught implies that P 1 1 transpose X naught is 0 fine. And we have X naught is not equal to 0 fine. So, let us go back to the question, so let us go back here. So, we need to look at A 1 1 fine in place of A 1 1 we are looking P 1 1 transpose basically, because A 1 1 for us is this. Sorry A 1 is this. So, this is what our A 1 is A 1, is this fine?

So, let us look at what is A 1 transpose. So, let us look at so A 1 transpose if I look at it is P 1 1 B whole transpose which is B transpose P 1 1 transpose and therefore, A 1 1 transpose X naught is B transpose P 1 1 transpose X naught which is B transpose P 1 1 transpose X naught by associativity which is B transpose 0, which is 0 alright.

So, this implies we see that the columns of A 1 transpose which are r plus 1 in number which are r plus 1 in number are linearly dependent. So, columns of A transpose are linearly dependent if and only if rows of a one are linearly dependent and this implies and this implies what we are saying is that the first r plus 1 rows of A r linearly dependent alright.

So, you have shown that if I have taking the first r plus 1 rows of A 1. So, looking at the first r plus 1 rows of A 1 here alright, then they are linearly dependent. We can do the same for any row fine a similar argument has to go through the problem will be how do I interchange rows and so on.

So, that will come through the permutation matrix, but the same idea will tell you that things will be nice basically, because of this matrix P 1 1 that to I get will have this property that it will be of size r plus 1 cross r alright. I will not get into the column part as I said I will just want you to understand it that similar thing can be done, but I have to define what is called the column rank, I have not yet done it, I have not related that two ideas, but at least you can show something's using the idea that.

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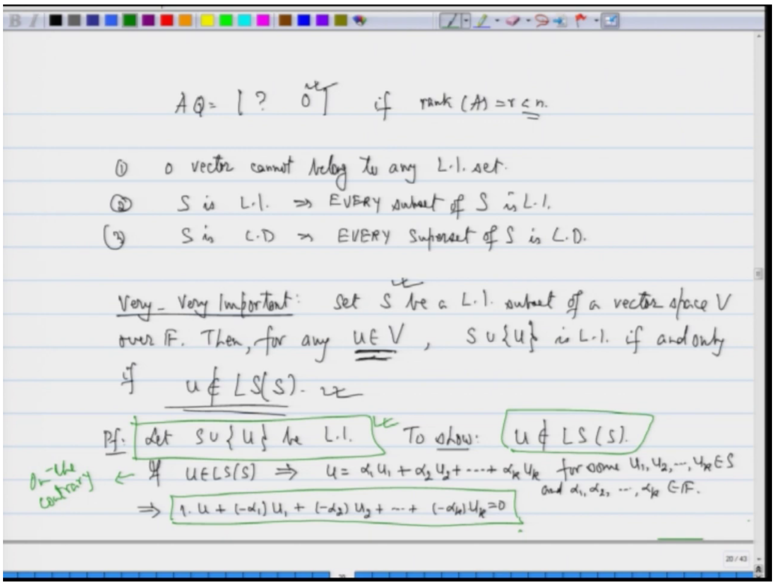

A times Q will look like something here and then 0 here if rank of A is r which is less than n alright, there will be some 0 which will come into play and you can use those ideas to prove yourself fine.

I will not get into that part, try that out yourself. Now some very important results alright very very important results. So, now, I will come back to some results on linear independence dependence.

So, the first thing that we are learned was 0 vector cannot belong to any linearly independent set, 2, S is linearly independent implies every subset S is linearly independent, three S is linearly dependent implies every superset of S is linearly dependent fine, we had all learned this. So, now, the most important theorem, very very important theorem. Let me write it. So, let S be a linearly independent subset of a vector space V over F vector space V over F fine.

Then for any V belonging to V look at. So, I already have S as linearly independent I am on look at S union V, is that ok? This V could be in the linear span of S. It could be outside, it could be element of S itself, it could be anything. So, I am looking at any V belonging to V, then for any V this is linearly independent, if and only if V does not belong to linear span of S, is that fine.

So, let me write not V, let me write u here. I think there is a problem with this notations, I think as usual. So, u here u here fine, u here, is that ok? So, what I am saying is that, let S be a linearly independent subset of a vector space V over F, then for any u, u could be either in S, u could be in linear span of S and things like that it could be anywhere.

Then we are saying that S union u is linearly independent if and only if u does not belong to linear span. So, what we are trying to do is that, I have a linearly independent set, I want to increase the size of the linear independent set I am adding extra vectors.

And trying to see whether what the new set I am getting, it is that set linearly independent or not fine the idea being very simple that I start with a single vector alright, which is the non zero vector. Then we have already seen that the single vector which is non zero is linearly independent, I want to add another linearly independent vector to, I want to add a vector.

So, that the new set having two elements linearly independent what it tells me this theorem, tells me that you have to only choose those vectors which do not belong to the linear span fine, that is very important fine. So, let me prove it for you proof. So, let S union u be linearly independent to show u does not belong to linear span of S fine.

So, if u belongs to linear span of S implies u is equal to alpha 1 u 1 plus alpha 2 u 2 plus alpha k u k for some u 1 u 2 u k belonging to s and alpha 1 alpha 2, alpha k belonging to F fine. So, this implies I am writing it as this implies. So, rewrite this as 1 times u plus minus alpha 1 times u 1 plus minus alpha 2 times u 2 plus minus alpha k times u k is 0.

So, I am rewriting the previous part as this I have just rewritten it, fine that is all I have done. Now if I look at this it tells me that the system, it tells me that look at the system here c 1 u plus c 2 u 1 plus c 3 u 2 plus c k plus 1 u k is 0, look at this system or let me write it I think.

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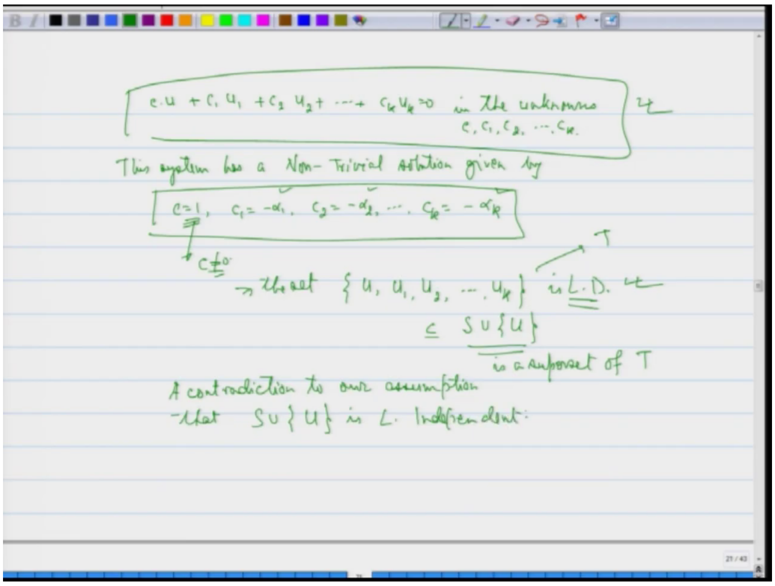

So, what we are saying is that look at the system say c times u plus c 1 u 1 plus c 2 u 2 plus c k u k is equal to 0 in the unknowns c c 1 c 2 c k look at the system in the unknowns this.

This system has a non trivial solution given by c is equal to 1 c 1 is equal to minus alpha 1 c 2 is equal to minus alpha 2 c k is equal to minus alpha k. So, I do not know whether alpha 1 alpha 2 alpha k are non zero or not, but I know that at least c is not 0 fine. So, therefore, I have a non trivial solution since the system has a non zero trivial solution implies that the set u u 1 u 2 u k is linearly dependent fine.

Now this is a subset of S union u. So, you are saying that this set is linearly dependent, this is a superset of this set T superset of T and therefore, what you are saying is that there is a linearly independent set which contains a linearly dependent set that is a contradiction.

A contradiction to our assumption that S union u is linearly independent. If I look at this our assumption was we assume that let this be linearly independent alright. So, we had assumed that our set S union u is linearly independent, S was linearly independent. We assumed that u S union u is linearly independent and from there we wanted to show that u does not belong to the linear span on the contrary we assumed.

On the contrary we have assumed here contrary we are assuming that u is in the linear span and then get a contradiction, the contradiction is to the fact that u is in the linear span, is that ok? So, therefore, we have shown that that if S union u is linearly independent then u cannot belong to the linear span of S.

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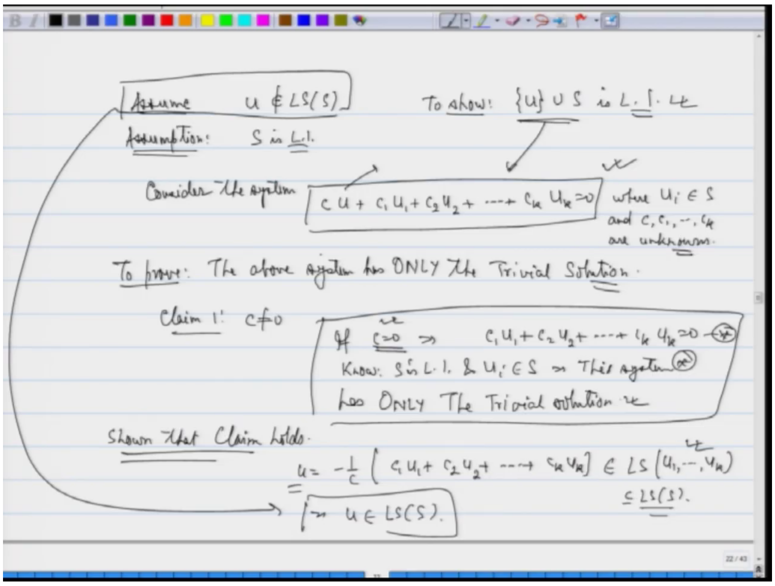

Now, let us prove the other way round. So, assume. So, assume u does not belong to linear span of S to show u union. So, this union S is linearly independent. So, be careful here you already know. So, a starting point the first assumption that we have is, first assumption.

First assumption was that S is linearly independent that is very important alright. So, we are assuming now that u does not belong to the linear span, we want to show that this set is linearly independent. So, let us consider, so if I want to show that something is linearly independent I need to consider a system of equation. So, consider the system c times u plus c 1 u 1 plus c 2 u 2 plus c k u k is equal to 0.

Where u i belongs to S and c c 1 till c k are unknowns; fine, is that ok? So, we are trying to solve a system here where u is the one that I am looking at and I have got collection of vectors u 1 u 2 u k from S any collection alright. This not only just a few of them I can take any collection, but this collection is finite collection, is that ok? And I want to solve the system.

What we are saying here is that this has the unique solution c is 0 c 1 is 0 c 2 is 0, I want to prove that, to prove the above system has only the trivial solution alright fine. So, claim 1 c is not equal to 0 fine. So, if c is equal to 0 this will imply that we are looking at the system c 1 u 1 plus c 2 u 2 plus c k u k is equal to 0. We are looking at this system alright.

So, if c is 0 I am looking at this system and I know S is linearly independent and u i is belong to S. So, implies, the only solution of this is the implies this system star has only the trivial solution fine. So, if I am assuming that c is 0 then this whole system has only the trivial solution fine, because c is 0 and this part is also 0. So, this part will have a non trivial solution.

So, now, therefore, we have shown, so shown that claim holds fine. So, again lets understand this very important idea that I have looked at a condition here; a system of equations where we have to show that each c i and c is 0. So, the first thing we are assuming that c is not 0, we want to claim that. So, if c is 0 then there is a problem here that we can see that if c is 0 then we are solving a system of equation where the vectors are coming from S.

And those vectors are already linearly independent that gives me a trivial solution which is 0 0 0. So, if c is 0 everything is 0. All the c's are 0 alright. Now if c is not 0 then what happens? I can write u as 1 upon c times c 1 u 1 plus c 2 u 2 plus c k u k alright. So, I can write u in terms of linear combination.

So, this belongs to the linear span of u 1 to u k which is a subset of linear span of S fine. So, if c is not 0 then I can write u as a linear combination of elements of u 1 to u k, and therefore, I will get that that belongs to the. So, u belongs to linear span of S and that is a contradiction to our assumption that u was not in the linear span alright.

So, that finishes this lecture here. I want you to understand that this is very very important theorem where we have said that how to go from a linearly independent set to the next linearly independent set and so on. How do we expand the size of the linearly independent set alright, that is all.

Thank you.