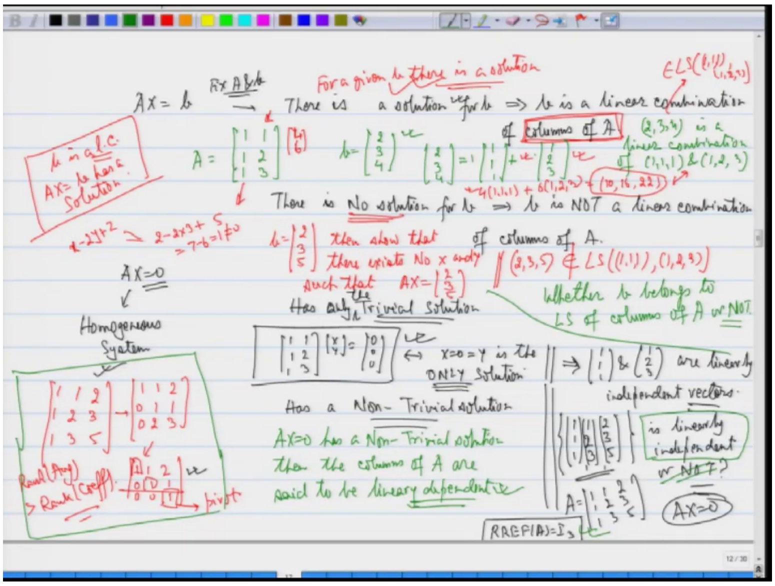
Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 23
Linear Combination, Linear Independence and Dependence

I hope you have understood the idea of vector space alright and what is the linear span? Linear span means you want to create a plane which has all those vectors and the zero-vector alright.

(Refer Slide Time: 00:31)



Now, let us go back to the system of equations what exactly we did in the system of equation. Where the system of equation as Ax is equal to b. What we had was there was a solution.

This solution may be unique, may not be unique alright, there could be lot of solutions, then or so, fix A and b then, we had this that there was a solution or there was no solution alright.

And then, we had another system what was called Ax is equal to 0, homogeneous system corresponding to that geneous system and here also we have two parts: one was what was one part? That has a trivial solution and has a non-trivial solution.

So, we had these ideas with us. Not only that if you remember, that if when A was a square matrix, then having a trivial solution was equivalent to saying that there was a solution for every choice of b alright. So, all those notions were there, and we looked at, we try to understand them.

Now, we would like to understand those ideas using vectors alright. So, this idea that there was a solution or there is a solution; is a solution for b, this goes to the idea of saying that idea says that b is a linear combination of columns of A alright. There is no solution means for b means that b is not a linear combination of columns of A is that ok.

So, let us take an example to understand it. So, here I am taking the example as say 1, 1, 1, 1, 2, 3 suppose that this is my A fine. I take b as say I take b as 2, 3, 4 then, we are saying that 2, 3, 4 is 1 times 1, 1, 1 plus 2 times sorry 1 times itself 1, 2, 4 1, 2, 3 because 1 plus 1 is 2, 2 plus 1 is 3, 3 plus 1 is 4.

So, we are saying that this 2, 3, 4 so, 2, 3, 4 is a linear combination of 1, 1, 1 and 1, 2, 3 fine. I could also take say 4 times the vector 1, 1, 1 plus 6 times 1, 2, 3 which gives me 6 plus 4 is 10, 4 plus [FL] 16 and 18 plus 4 is 22. So, again this vector is a linear combination of these two vectors is that. This is important.

We are trying to say that there is a solution for a given b there is a solution. So, for a given b, there is a solution corresponds to saying that b is a linear combination of things alright. So, b is linear combination of columns of A that is important b is a columns. 1, 1, 1 is a column 1, 2, 3 is another column.

So, you can see that they are linear combination of columns and this is what you say, what how do you get this part? You get this part using this vector 1 comma 1. Because multiplying on the right corresponds to saying that you are looking at one times this, one times this is that. This one corresponds to 4 times this and 6 times this is that.

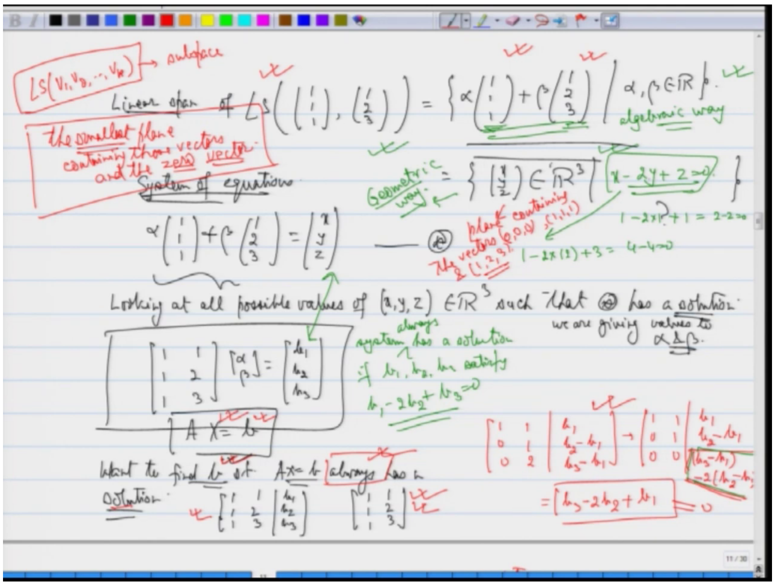
So, when I say that b is a linear combination it means that I am able to solve so, b is a linear combination in short lc linear combination means that the system Ax is equal to b has a solution. This is important. Linear combination means there is a solution is that ok?

When I say that there is no solution now, let us look at this idea there is no solution means that for example, if I look at this vector b is equal to 2, 3, 5, then you can show that; then show that there exists no x and y such that Ax is equal to 2, 3, 5 alright you can show that. How do I show that? Let us compute the RREF of this matrix augmented matrix.

So, the RREF of this augmented matrix will be 1, 1, 1, 1, 2, 3, 2, 3, 5 which gives me nothing, but 1, 1, 2, 1 minus 1 is 0, 2 minus 1 is 1, 3 minus 1 is 2, 3 minus 2 is 1, 1 minus 1 is 0, 3 minus 1 is 2 and 5 minus 2 is 3. So, this again if I proceed, I get here 1, 1, 2, 0, 1, 1, 0, 0, 1 alright.

So, you can see that there is a pivot here fine. So, these two were also pivots, but what we are able to see here is that rank of augmented matrix is greater than rank of the coefficient matrix and therefore, the system has no solution is that ok. This you could have a also understood from the previous example also.

(Refer Slide Time: 07:33)



In the previous class, what we had done was if you look at the previous class what we have done that we had this condition that where these vectors 1, 1, 1 and 1, 2, 3 the condition for the linear span was x minus 2y plus z is 0 alright. So, x plus 2y so, here x is 2, 3 and 5 so, what we see here is that 2 x minus 2y plus z is 2 minus 2 times 3 plus z is 5 which is 7 minus 6 which is 1 not equal to 0 fine.

So, therefore, 2, 3, 5 does not belong to the linear span. So, this also tells me that 2, 3, 5 does not belong to the linear span of the vectors 1, 1, 1 and 1, 2, 3 fine. Here, this tells me that this vector 10, 16, 22 or 2, 3, 4 they belong to the linear span of 1, 1, 1 and 1, 2, 3 fine.

So, whatever we had studied in the previous class about linear span, we are just trying to check now whether a certain vectors belongs to that linear span or not is that ok? So, when I am solving the system Ax is equal to b, there we are looking at intersection of planes, intersection of lines and on. Here, we are saying that whether a vector b belongs to column space or not. So, we are talking of whether b belongs to linear span of columns of A or not is that ok.

So, the things have changed. There we are looking at whether a system has a solution or not fine. The language there gets changed to whether b belongs to linear span or not. So, whether some vector is in the subspace or not fine. So, that is one important idea. I would like you to keep track of this we will be using it again and again.

The next idea that I said was looking at trivial solution and a non-trivial solution. So, let us look at example to understand what mean trivial solution. So, look at these two itself, you have the two vectors 1, 1, 1, 2 and 1, 3. I want to look at the system Ax is equal to 0. So, I want to solve 1, 1, 1, 2, 1, 3.

Does there exist x, y such that this is equal to 0, 0, 0 alright. So, when I solve it, what I see is that x is equal to 0 is equal to y is the only solution alright. So, what we had said is that this system has only the trivial has only the trivial solution is that fine that is important for us. What we are trying to say is that I have a set of vectors, again I am looking at columns, I am asking that whether the corresponding homogeneous system has only the trivial solution or has a non-trivial solution fine.

So, here we see that this has only the trivial solution and since it has only the trivial solution, we say that these two vectors 1, 1, 1 and 1, 2, 3 these are linearly independent alright. So, why you are saying they are linearly independent vectors? Because this system has only the trivial solution is that is very important for us. Be careful when you are saying that something is linearly independent, it means they solution is only the trivial solution.

Another set of examples for example, again in R 3 itself suppose I look at 1, 1, 1, 1, 1, 1, 1, 2, 3 and 2, 3, 5 fine look at these three vectors for us. So, I should have written it with some gap here. So, I have got these three vectors alright. So, I make a set out of it. So, we say that this set this is linearly independent or not that is the question fine. So, what we do when we say that this system these three vectors is linearly independent or not we solve a system Ax is equal to 0.

Now, what is the size of the matrix A? This matrix A has size 3 1, 1, 1, 1, 2, 3, 2, 3, 5 there are different ways of doing it. We will learn that if determinant of this is not 0, then we will get that this is linearly independent fine or you can look at the RREF, we have already computed the RREF of this matrix 1, 1, 2, 1, 2, 3, 1, 3, 5.

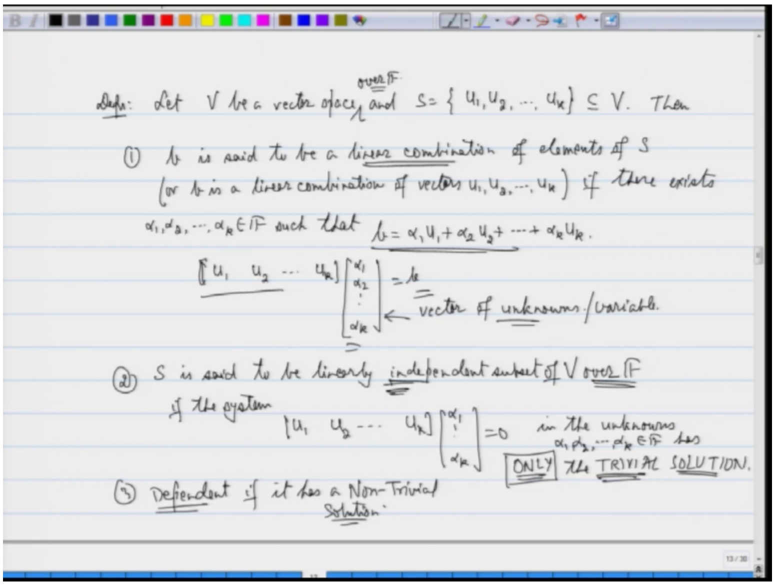
What we see is that the RREF of this matrix is identity so, RREF of A is identity and therefore, what we know is that the system Ax is equal to 0 has only the trivial solution alright. It does not have anything else, it does not have a non-trivial solution that is only the trivial solution because RREF of a is I 3 and therefore, this set of vectors is linearly independent is that.

So, it turns out that these three vectors are linearly independent. We have already done the calculations here and we will look at them again and again fine and when the system Ax is equal to 0 has a non-trivial solution; trivial solution, then the columns of A are said to be linearly dependent fine. So, there it was independence, here it is dependent is that.

So, there are four ideas that we had when we solve the system Ax is equal to b whether it had a solution or not that gave us the idea of solution, there is a solution means linear combination is there or the vector b belongs to the linear span. If there was no solution, it means that there is no linear combination, or the b does not belong to linear span fine those ideas were there.

Then, we go to the homogeneous system and look at the idea that Ax is equal to 0 if it has only the trivial solution only is stressing on it, it means that the only solution is 0, 0, 0, 0, 0 then we get independence, if it is a non-trivial solution, it means we are looking at the set of vectors or the set of columns that I am looking at that is linearly dependent fine.

(Refer Slide Time: 14:53)



So, let me write the actual definitions now. I have given you examples to make you understand. So, definition. So, let V be a vector space and S is equal to u 1, u 2, u k be a subset of V. V is a vector space; I am not writing over what over F I have to say that this is very important over V fine over F fine.

Then b is said to be a linear combination of elements of S or b is a linear combination of vectors u 1, u 2, u k if there exists; if there exists alpha 1, alpha 2, alpha k belonging to F such that b is equal to alpha 1 u 1 plus alpha 2 u 2 plus alpha k u k fine.

So, linear combination means I have this idea with me or what I am saying is look at the system u 1, u 2, u k fine. I have this alpha 1, alpha 2, alpha k and this is b I have. So, this is the vector of unknowns or I have this as variables and I am solving the system Ax is equal to b for me is that fine.

So, if able to solve this system, if there is a solution, then we say it is linear combination otherwise, it is not a linear combination. So, it is important for you to see that I am writing a matrix coming from u 1, u 2, u k. U 1, u 2, u k need not be an element of R n it could be any whether it is polynomials or anywhere that you are looking at some set of vectors, I can write them in terms of a matrix.

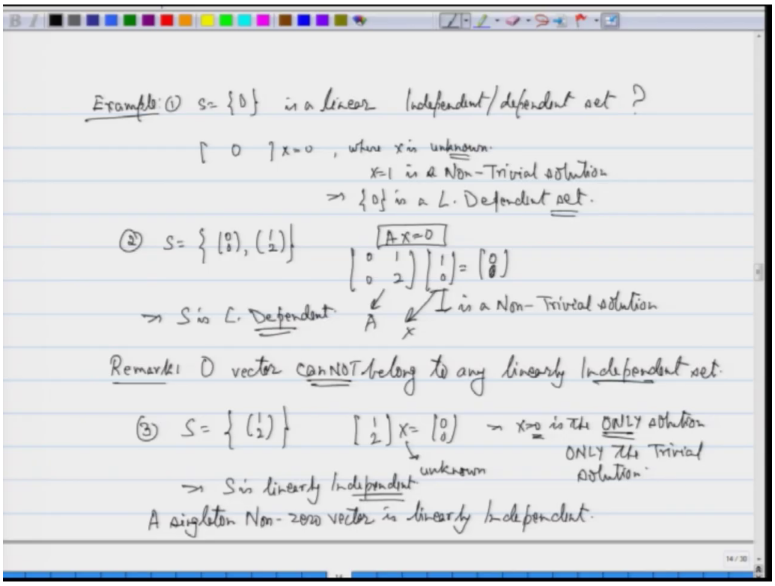
So, here my columns are those vectors is that ok. How that look like is immaterial they could be function, they could be polynomials, but I am writing it in that form fine.

2: S is said to be linearly independent subset of V over F I am a stressing on it over F if the system u 1, u 2, u k, alpha 1 to alpha k is equal to 0 in the unknowns alpha 1, alpha 2, alpha k belonging to F has only the trivial solution fine.

So, is said to be linearly independent fine, if it has only the trivial solution fine. Dependent if it has a non-trivial solution. So, dependence means non-trivial solution and independence means only the trivial solution only is very very important only the trivial solution is that.

So, you have to be careful when you do things. You have to understand them. So, let us take some examples to have a clarity on these ideas. Linear span you have already understood, linear combination I hope you have understood by examples.

(Refer Slide Time: 19:47)



Let us look at linear independence dependence so, that we proceed further alright. So, examples I am not going to linear combination only linear independence dependence alright. Suppose I have this S consists of just the 0 vector fine. What can I say about S? S equal to 0 is what? Is a linear independent oblique dependent set that is a question alright.

So, again let us go back the idea was that I have to form a system of equation; I have to form a system of equation in which the matrix A was nothing, but the vector that has given to me let us go back here that I have to form a system with A as vectors coming into it so, I have the vector A here, unknown for me is x and I am solving it here is that ok. So, I need to solve where x is unknown fine. What we see here is that x is equal to 1 is a non-trivial solution and this implies this is a linearly dependent is that ok?

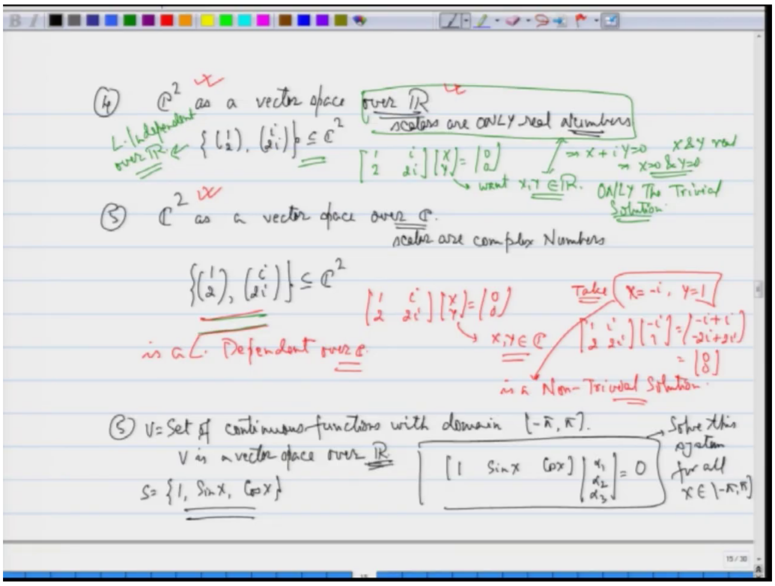
Example 2: S is equal to say let me write in terms of 0, 0 and say 1, 2 fine. What about this again from the vector 0, 0, 1, 2 you can see here the solution is 1 comma 0 is a solution to give me 0, 0. So, this is a non-trivial solution implies S is linearly dependent fine. So, you are always solving a system here Ax is equal to 0 this is what you are doing. This is your x, this is your A and this was your 0.

So, we see that again the matrix multiplication is here to help one time 0, zero times this gives me this. So, what we can see from here is that any linearly independent set as a remark, you can see here 0 vector cannot belong to any linearly independent set fine. So, this cannot belong to any linearly independent set. As soon as you have a 0 vector in a set, that set becomes linearly dependent fine.

Let us look at third example. I take S as just a single vector 1, 2. What happens? Now, I again look at 1, 2 this vector times this should be 0, 0, x is unknown and this will imply that x equal to 0 is the only solution.

So, it is the only solution means 0, 0 is the only solution. So, this is only the trivial solution only fine. This implies S is linearly independent. So, a singleton non-zero vector so, what you are saying here is that a singleton non-zero vector is linearly independent fine.

(Refer Slide Time: 24:11)



Some more examples 4th. Now would like to look at the role of a scalars being playing the role. So, let us look at C 2 as a vector space over R and 5: C 2 as a vector space over C. So, over C means a scalars are complex numbers here; a scalars are complex numbers, here a scalars are only real numbers fine.

So, let us look at two sets here same set of vectors in the two. So, I would like to look at the vector here 1 comma 2 and i comma 2i. These two are subset of C 2. Here also I have the same thing 1, 2 and i and 2i fine. C 2 is a vector space.

Here it is over real numbers, here it is over complex numbers. So, in the first case here, I want to look at so, make this matrix 1, 2, i and 2i. Let us write x, y is equal to 0, 0 want x and y belonging to R that is important because a scalars are real numbers so, therefore, I am only allow to take x, y belonging to r fine.

So, therefore, if you would like to solve here, this implies that x plus i times y should be 0, x and y real implies x is 0 and y is 0 fine. Similarly, for the second equation you can see here that x has to be 0, y has to be 0 alright. So, therefore, we only have the trivial solution. So, only the trivial solution is that ok. So, therefore, this is linearly independent dependent over R fine.

But let us look at here in this example now alright. In this example, you have 1, 2 i and 2i. I have to look at x, y equal to 0, 0, but x and y are now complex numbers fine. Since, they are complex number, we can see that take y to be equal to minus i and x to be equal to or let me take x to be minus i and y to be 1, then what I see here is that we get 1, i, 2 comma 2i times minus i, 1 which is minus i minus i, plus i, minus 2i, plus 2 i which is 0, 0 alright. So, therefore, this is a non-trivial solution alright. So, therefore, this is a linearly dependent over C fine.

So, this is how the a scalars make a difference where you are looking at alright. So, both of them are subsets of a C 2. C 2 is a vector space, but one is over real numbers the other is over complex numbers. So, the definitions of linear independence dependence do depend on set of a scalars alright. This is what you have to be careful about. I would like you to understand them fine.

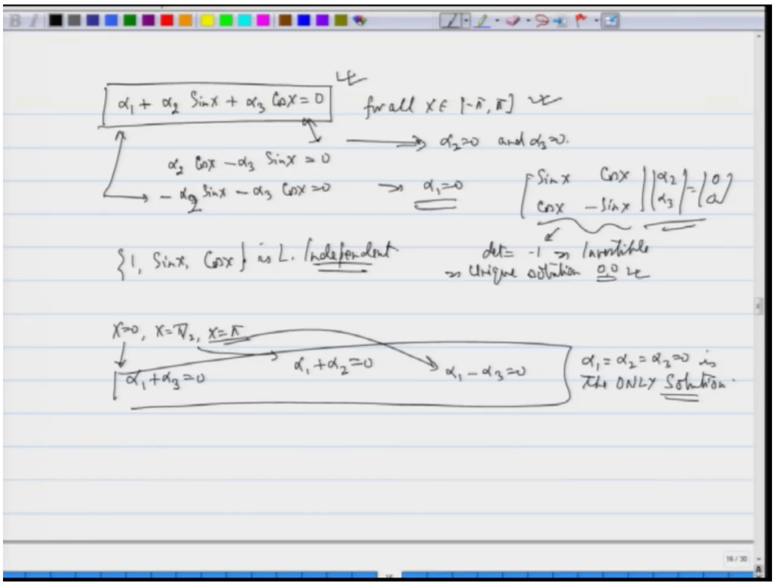
5th some example some more examples here for me 5th. Let us consider these alright. So, these are set of continuous functions, you can prove that V is equal to set of continuous functions with domain say minus pi to pi that is the domain alright. So, sum of two continuous function is a continuous function fine. If you multiply a scalar to a continuous function, it still remains a continuous function so, you can prove that set of continuous functions with domain whatever domain you take is a vector space.

So, v is a vector space over R. We are allowing the scalars to be real numbers. Now, in this set I would like to takes vectors say S is equal to 1, sin x and cos x. These are three continuous functions. I want to check whether they are linearly independent or not fine.

So, if you want to check whether they are linearly independent or not, I have 1 here, sin x here, cos x here they are three vectors so, I will have three unknowns x, y no so, x, y, z is a problem so, may be alpha 1, alpha 2, alpha 3 is equal to 0. So, I want to solve this system; solve this system for all x belonging to the domain minus pi to pi fine.

What we are saying is the left hand side is a continuous function for whatever alpha 1, alpha 2, alpha 3 you choose, they only need to be real numbers that is all you are saying and the right-hand side is a zero function, zero-continuous function. So, you are saying that two continuous function the 0 function is one side, the other side it is alpha 1 plus alpha 2 sin x plus alpha 3 cos x have to be 0 fine. What is the solution for this? I would like you to see that this thing leads to looking at following.

(Refer Slide Time: 30:41)



Alpha 1 plus alpha 2 sin x plus alpha 3 cos x is equal to 0 fine. Now, this is valid for all x belonging to minus pi to pi. So, that different things we can do it turns out that more than continuous it is also differentiable. So, I can differentiate both the sides.

So, if I differentiate both the sides, I can look at here alpha 2 cos x minus alpha 3 sin x is 0. I can again differentiate it to get minus alpha 3 sin x alpha 2 sin x minus alpha 3 cos x is 0. I can use this equation and this equation now just add of them to get that this will imply that alpha 1 is 0 fine. Now, I can use these two to show that this will imply alpha 2 is 0 and alpha 3 is 0 fine.

So, once I have put this, note that the equation is sin x, cos x, cos x, minus sin x times alpha 2, alpha 3 is equal to 0, 0. This is an invertible matrix minus sin square x minus cos square x is minus 1. So, therefore, the only solution is a trivial solution fine. This determinant is minus

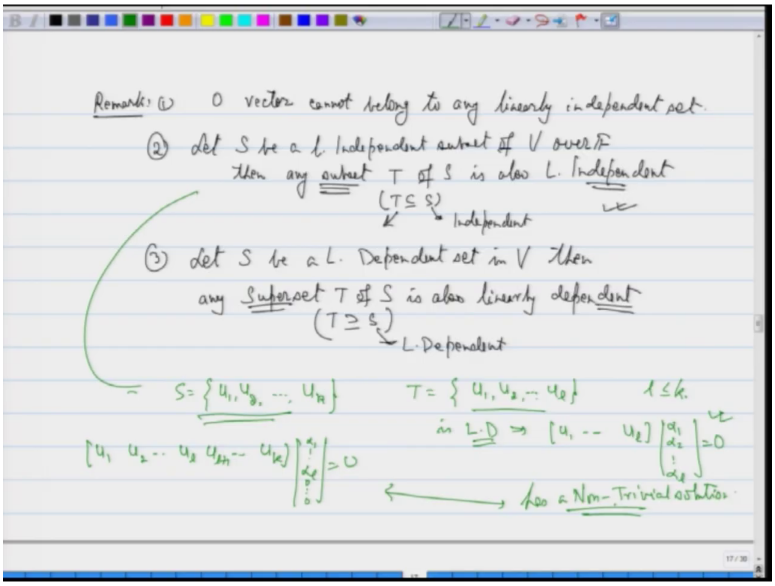
1 implies invertible implies unique solution 0, 0 fine. So, therefore, what we are saying is the set 1, sin x, cos x is linearly independent.

So, even though we had only one equation here, we have been able to generate three equations by differentiating alright because they are differentiable functions. We could have also done one thing that since this is valid for every x, I can evaluate these at different points. So, I can evaluate at x is equal to 0, x is equal to pi by 2 and x is equal to say pi.

So, do x is equal to 0 what do I get? Alpha 1 sin 0 is 0 alpha 1 plus alpha 3 is 0. At x equal to pi by 2 I will get alpha 1 sin of pi by 2 is 1 and cos of pi by 2 is 0 so, I will get this and at x is equal to pi I will get the third equation which is alpha 1 sin of pi is 0, cos of pi is minus 1 so, minus alpha 3 is 0.

So, I can solve this equation and prove that alpha 1 is equal to alpha 2 is equal to alpha 3 0 is the only solution fine. So, you can proceed in whatever way you want and try to prove things like that is that fine.

(Refer Slide Time: 34:01)



As the last part, I would just like to prove this theorem or just let me a state something first. So, remark. What we saw was that recall it. 0 vector cannot belong to any linearly independent set we saw this fine. 2nd remark: let S be a linearly independent subset of V over F, then any subset T of S is also linearly independent fine. So, T is a subset of that is important. T is a subset of S. S is linearly independent, this is bigger one is independent, then the smaller is also independent fine.

So, it when I am talking of independence dependence, we are assuming that our sets are non-empty sets fine otherwise vacuously we will have to say what happens to the empty set. 3rd one: let S be a linearly dependent set dependent set in V, then any superset fine. So, I have got T, T contains S alright. This is linearly dependent. T is a superset any superset T of S is also linearly is that.

So, you can prove it yourself fine because idea here is if you look at the idea here so, let us look at the idea here, I have S which is u 1, u 2, u k. Suppose T is say u 1, u 2, u l and l is less than equal to k there are some collections. This is linear dependent will imply that this sector u 1 to u l times alpha 1, alpha 2, alpha l equal to 0 has a non-trivial solution fine is that ok. So, you have a non-trivial solution here linear independent dependence will imply that.

So, if I want to look at here, I have the same thing here u 1, u 2, u l, u l plus 1 till u k. Again, I can patch up here with alpha 1 to alpha l and then 0's here fine. So, if I look at this, this has a solution which is non-trivial will imply that this will also have a non-trivial solution because alpha 1 to alpha l are the same, u 1 to u l are the same this will also be there. So, this will also have a non-trivial solution and therefore, there will be a contradiction fine.

I would like you to try the similar thing for 3. So, I would like you to understand these things which are very very important fine. In the next class, we look at some more examples of linear independence dependence alright to do with matrices and then, look at the theorems to understand these ideas because linear independence dependence will result into again solving system of equations and the ideas which were related with what are called the RREF of matrix.

Recall that we assume that RREF of a matrix is unique, we did not prove it, we did not do anything there. So, those ideas will be coming into play for us. We also had the idea that we have definition of row rank of a matrix, but we did not talk of column rank. So, we would like to look at row and columns at one go. It will take some more time for us, but all those ideas will get related with what are called independence.

So, when I say the rank of a matrix is k or rank of a matrix is 3, it means that matrix has exactly three linearly independent rows or exactly three linearly independent columns. We will get into all those things in the next class.

Thank you.