Linear Algebra
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Lecture – 22
Vector Subspaces and Linear Span

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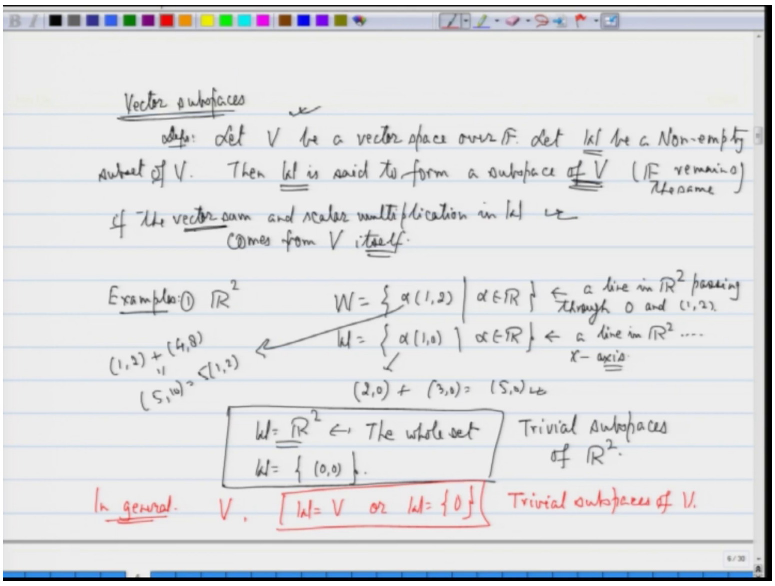

In the previous class, we learnt what are vector spaces, alright. Now, we want to an understand what are called vector subspaces, fine. In mathematics whenever we study some set, we talk of subset of it and try to look at what are the properties which are meaningful for the subsets that we are looking at fine; similarly here. So, we are going to look at vector subspaces. We will not be using the word vector again and again. We will just use the word space and subspace fine. So, so let us try to understand this. So, definition.

So, let V be a vector space over F. Let W be a non-empty subset of V, then W is said to form a subspace space of V. So, I am saying of V important F is not allowed to change. F remains the same remains I am not allowed to change it that is one thing.

Subspace of V or sub V over F you can again use the word over F. If the vector sum and a scalar multiplication in W comes from V itself , alright. So, what I am saying is that when something is a subspace when something is a vector space, I need to define what is called the sum of the two vectors fine. Now W is the subset of V.

So, every element of W is also an element of V and therefore, I can just pick all the elements of W, think of them as vector in V, add the two vectors in V itself and then come back to W, fine. So, again W is a set which is a subset of V. So, every element of W is an element of V itself. So, I can put everything there, add the two vectors in V. What I need is that the sum of two vectors even though it is an element of V again, it should be an element of W.

So, the vector sum has to be an element of W, but I am not allowed to change the definition of sum. The sum has to come from that of V itself and same thing with a scalar multiplication. Also I cannot change my scalar multiplication, is that ok? So, examples fine I have R 2 with me fine. In this I can talk of V or W as look at all collection alpha times 1 2 alpha belonging to R. I can also talk of W as alpha times 1 0 alpha belonging to R. So, let us see what they are.

So, this is nothing, but a line in R 2 passing through 0 and the vector 1, 2. What is this? It is a line in R 2 passing through so and so basically it is nothing, but the x axis, fine. So, I have summed this here. I am not changing the vector addition 1 plus. So, here if I look at 2 0 plus 3 0 is 5 0 for me alright. Similarly, here you are looking at 1 2 plus 4 8, then it is indeed equal to 5 10 which is 5 times 1 2 fine.

So, I am not allowed to change the addition similarly, the scalar multiplication. I can also think of W as R 2 itself, the whole set itself the whole set I can also take W to be just the zero vector 0 0 and nothing else, fine. So, these two examples that I am looking at where W is the whole set or W is just the zero vector, these two are called trivial subspaces of R 2, fine.

So, in general; so in general start with V, then take W to be V itself or take W to be is equal to just the zero vector, then they are called trivial subspaces of V because they do not give us any extra idea, fine. So they are important, but at the same time they are not giving us any new idea to us, fine. So, some more examples.

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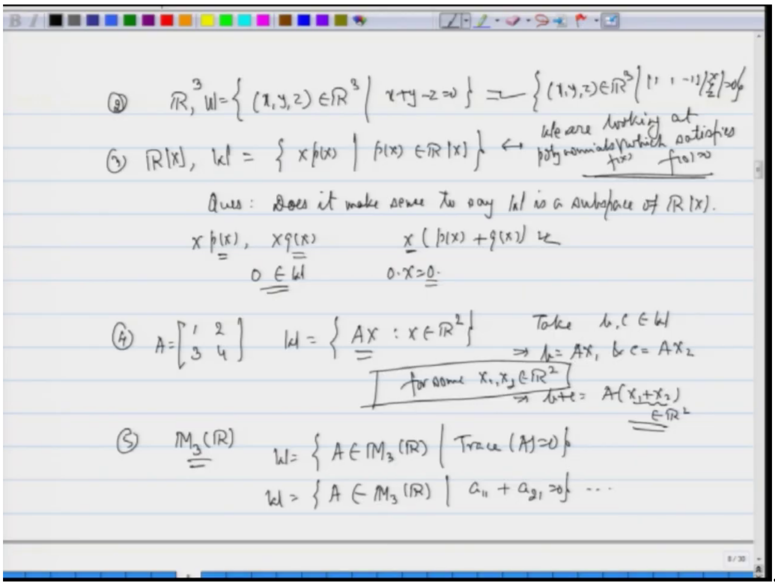

So, I do not know which example which was we need to in R n or R 3. You can talk of x y z belonging to R 3, such that x plus y minus z is 0 fine. I can also talk of sorry. So in R 3, I can talk of W is equal to this 3 I can also talk of. So, I want to look at R of x set of all polynomials. Here I can talk of x times P x where P x belongs to R x fine. So, here what we are saying is I am looking at only polynomials of degree greater than equal to 1 does it makes sense that is the question, alright.

So, question does it make sense to say R x? So, not R x this is the problem. So, in R x I am looking at W to say W is a subspace of R x fine. So, if I any two polynomials here say x P x and x Q x where P x and Q x are polynomial, I can talk of their sum. So, x times P x plus Q x. So, this is nice.

Zero vector is here, zero vector 0 is or 0 belongs to W because 0 times x is 0. So, it make sense to for us alright. There would not be any constants because x is here; everything is getting multiplied by x. So, therefore there is no problem as such. This may look that we are just doing something's, but what is important is that this corresponds to the idea that we are looking at that we are looking at polynomials which satisfies F of 0 is equal to 0.

We are be looking at polynomials F of x for which F of 0 is 0, alright. So, 0 is there, alright. All these polynomials have 0 as a root fine. Fourth I can look at. So, fix a matrix say I fix a matrix 1 2 3 4 and look at a equal to this. So, this corresponds to if I see I can think of this as x y z belonging to R 3, such that 1 1 minus 1 times x y z is 0, fine.

I can also talk of A times X; X belonging to R 2 alright. I can think of this, just have you look at it. So, I have A here. So, take any two elements here. So, take b comma c belonging to W, this will imply that b is equal to A X 1 and c is equal to A X 2 for some X 1, X 2 belonging to R and this will imply that b plus c is a times X 1 plus X 2 and this is again an element of R 2, fine. Is that ok?

So, therefore this is again a subspace zero vector is there because a times 0 is 0 things like that, fine. I can also talk of set of also I can talk of all matrices say M 3 of R as is there I can define W for me as all matrices A belonging to M 3 of R, such that trace of A is 0. I can also talk of this as all a belonging to M 3 of R such that a 1 1 plus a 2 1 is 0 and things like that. I can talk of that all of them will be vector spaces, fine. Now, what is not a subspace? That is more important, not a subspace.

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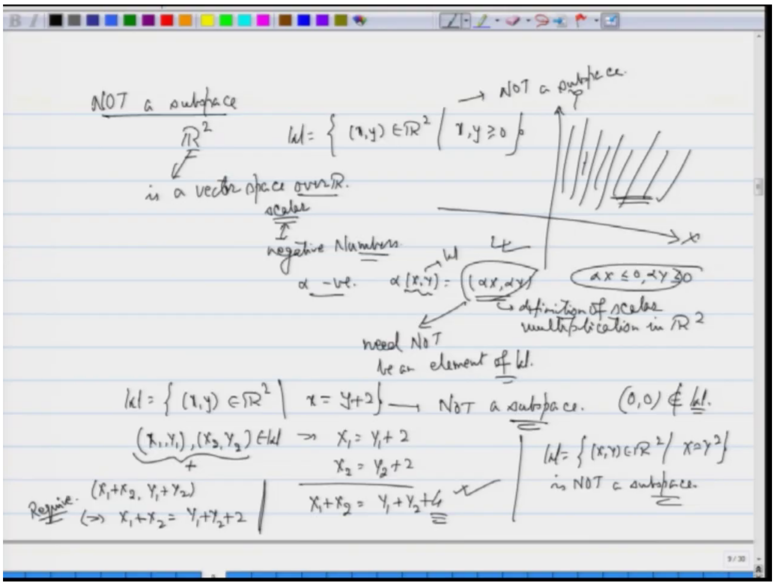

So, in R 2 let us look at some examples. So, look at defined W as all x y belonging to R 2 such that x and y are greater than equal to 0, fine. So, I am looking at the set which is this fine. This is the set that I am looking at. So, in this set if I look at, so this is my x, this is my y axis if I look at this set alright.

So, R 2 is a vector space over R or over scalars. Some scalar a scalars always contain negative numbers. So, now if I take alpha to be negative, alpha negative and look at alpha times x y which by definition is alpha x alpha y that is the definition in definition of scalar multiplication in R 2.

Then alpha x is less than equal to 0, alpha y is less than equal to 0 and therefore, I am in this part alright. So, I am changing. So, even though I am multiplying these two R, elements are supposed to be an element of W, but this part need not be an element of W. So, W is not a subspace, is that ok. So, not a subspace fine. Similarly, if I take W is equal to all x y belonging to R 2, such that x is equal to y plus 2 alright not a subspace.

Why it is not a subspace? Because note that the vector 0, 0 does not belong to W or if I take two element say x 1 y 1 x 2 y 2 belonging to W, then this implies that x 1 is equal to y 1 plus 2, x 2 is equal to y 2 plus 2.

So now if I add it, I get x 1 plus x 2 is equal to y 1 plus y 2 plus 4, alright but for vector for W to be subspace, what we need is their sum should be there. So, their sum is nothing, but x 1 plus x 2 y 1 plus y 2 and therefore, what I require that x 1 plus x 2 should have been y 1 plus y 2 plus 2 fine, but here I am getting 4. So, this is not a subspace.

Similarly you can show that W is equal to all x y belonging to R 2, such that x is equal to y square is not a subspace fine. So, there are restriction to be in a subspace. That is very important.

You need to understand that there are lots and lots of restrictions for getting a subspace, fine. There are also differences over complex numbers, but I will not get into that part for the time being. I will go to move to the next idea what is called the linear span; linear span of vectors, alright.

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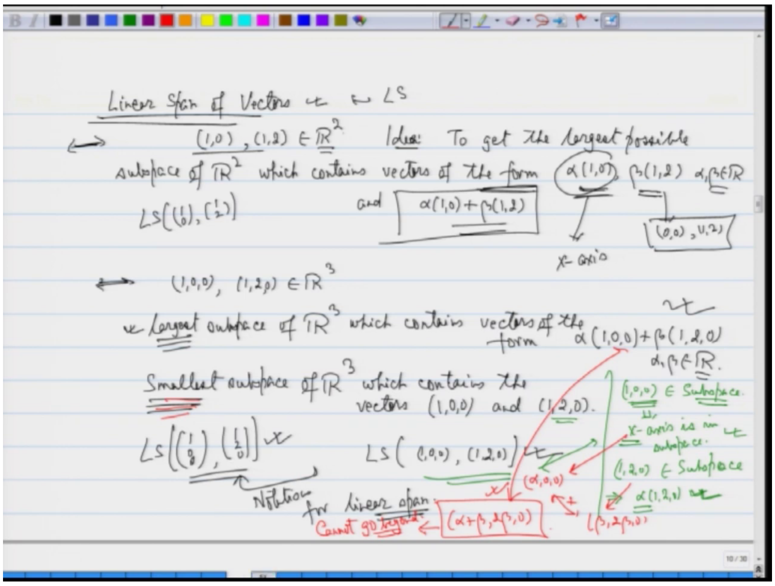

So, what we want is, that, so let us look at ideas that I am talking about. Suppose I have two vectors say 1 0 and 1 2 in R 2 fine. Idea is idea to get the largest possible subspace of R 2, R 2 which contains vectors of the form alpha times 1 0 beta times 1 2 for alpha beta belonging to R and their sums is that ok. So, understand what I want. So, given any collection of vectors I am allowed to multiply by a scalars. Whatever I want, I am allowed to multiply by scalars.

I want that collection to be inside that set itself whatever set I am looking at; I also want that I should be able to add this new collection alright. So, from two vectors I will have countable number of vectors because of alpha and beta being allowed to take any real number. So, there will be uncountable number of them. So, I have now huge collection.

For example, this collection is going to give me whole of x axis, this is going to give me collection of all the points on the line which passes through 0 0 and 1 2. So, the line containing these two points what I want is, I should be add those vectors also is that ok. So, I want the largest possible subspace which contains vectors of the form. This I am not saying that I want the biggest; I want only those vectors which have this form. Is that ok? That is very very important what I want.

So, this is the way one can think of it one way or the other way to think of is that in place of two it may three of them 1 0 0 and say 0 1 0 in R 3. Let me take previous one itself 1 2 0 itself fine. So, I want to get largest subspace of R 3 which contains vectors of the form alpha times 1 0 0 plus beta times 1 2 0 and alpha beta belongs to R fine.

This is one way of understanding the largest part. I want they also which should correspond to a smallest subspace of R 3 which contains the vectors 1 0 0 and 1 2 0. Is that ok? Here I am talking of a smallest subspace; here I am talking of the largest subspace.

So, please understand it. The very important concept what is called the linear span of a vector space we write it as the notation for that is LS, alright. So, here I am looking at LS of 1 0 1 2, here I am looking at LS of 1 0 0 and 1 2 0, fine. Sometimes I may just write LS of 1 0 0 1 2 0 fine, but as I said we want to look at only vectors as column vectors. So, this is the right one, but in a hurry or sometimes I will just write this to save a space for us, alright.

So, sometimes you need to save a space and therefore, I will use this, but that is the notation parts. So, these are notations alright notations for linear span, is that ok. So, understand it. Linear span means I want the largest subspace which helps this form. At the same time it is the smallest subspace which contains 1 0 0 and 1 2 0 fine. So, understand.

Now, if I am saying that 1 0 0 is there. So, I am saying that 1 0 0 belongs to a subspace, got it. What is it mean? Subspace means zero vector has to be there. If x is there, y is there, then x plus y has to be there fine. If x is there, alpha is a scalar, then alpha times x has to be there. So, once I am saying that 1 0 0 is there, it means that 2 0 0 is there, 3 0 0 is there fine alpha times 1 0 0 is there.

So, it implies that I need x axis is in subspace whatever subspace I am looking at, whatever these subspaces. What I want is since 1 0 0 is there, therefore x axis has to be there. Is that fine? Similarly, 1 2 0 is there fine. Since 1 2 0 is there in the subspace, it means what? This implies that and 0 is already there in that. So, 0 and this if I look at these two, I can look at their linear combinations or I can say that alpha times 1 2 0 is there. So, alpha times 1 2 0 is there.

So, 1 2 0 times alpha basically gives me the line passing through 0 0 0 and 1 2 0. So, this line is also there for you. So, I have got two lines; x axis and this line and what it says is that if I have a subspace sum of any two vectors is also in that, so an element from here and an element from there has to be there. Any element here is of the type alpha 0 0, any element here is the type beta 2 beta 0.

So, therefore what I get here is their sum will give me alpha plus beta 2 beta and 0. So, this has to be there with me. So, we can see that from one point of view this part gives you this part and this.

The smallest part tells you that I cannot go beyond this; cannot go beyond fine. So, linear span basically means time to understand things here, fine. So, this is important for you that you need to understand what are the linear span means. So, for example that itself.

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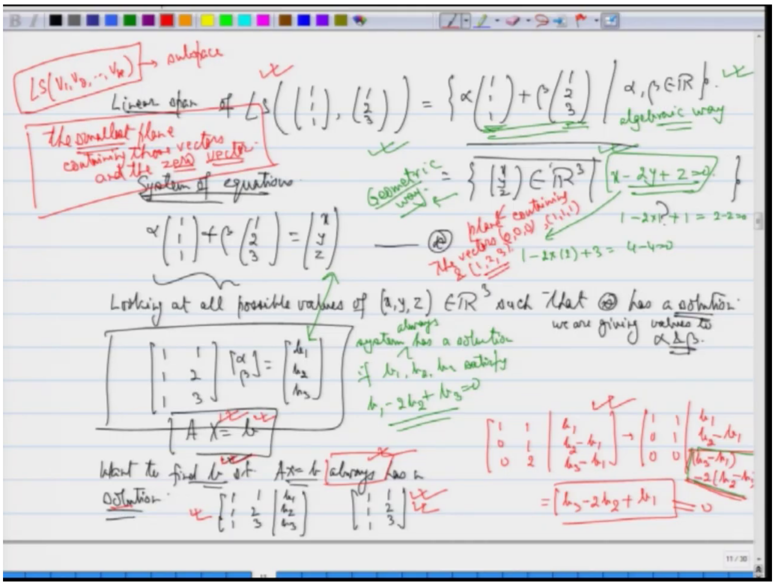

So, the linear span of, so let me take two vectors 1 1 1 and 1 2 3. What is the linear span of this? Let us try to compute this. So, the definition tells me that this is of the type alpha times 1 1 1 plus beta times 1 2 3 where alpha and beta they are real numbers, fine. So, we are saying that this is the linear combination. So, this word is used, but I am not yet say what is the linear combination I want to look at collection of all vectors of this type, fine.

So, what we are saying is that, so in terms of system of equations what I want is system of equations, alright. So, this if I look at this is nothing, but some vector x y z belonging to R 3 with some properties. What is this property that we need to find out, alright.

It has got three components. So, therefore, I have x y z. So, what I have is, I have got alpha times 1 1 1 plus beta times 1 2 3. I want to find what the sum looks like. So, in general it is has three components, but what exactly does it look like? So, let us go back to the system of equation and try to understand this.

What it tells me is that I want to generate to looking at all possible values of x y z belonging to R 3 such that alright such that star has a solution. Why I am saying has a solution? Basically because we are giving values to alpha and beta or we are saying that look at this system 1 1 1, 1 2 3 alpha and beta are unknowns and I have vector here say b 1 b 2 b 3, fine.

So, I am looking at this and what I want is, I look at the system that A X is equal to b has a solution for every b, alright. So, here I am saying that I want to look at the system A X is equal to b, what is given to me is that it has a solution. So, want to find b such that A X is equal to b always has a solution, fine.

So, when I am saying that want to find b, such that A X is equal to b always has a solution, it means that you call that this matrix which is b 1 b 2 b 3. I do not know what they are and this matrix 1 1 1 1 2 3, they should have the same rank. So, let us look at it again want to find a b; alright want to find a b such that A X is equal to b always has a solution. Since the word always has a solution has come it means that rank of the coefficient matrix should be equal to rank of the augmented matrix.

So, let us look at the rank of the augmented matrix fine. So, what we want here is 1 1 b 1, subtract it 1 minus 1 is 0 2 minus 1 is 1. Here it is b 2 minus b 1. I hope I am doing correctly. 1 minus 1 is 0 3 minus 1 is 2, b 3 minus b 1. At the next stage, it is 1 1 b 1, 0 1 b 2 minus b 1 or 0 0 b 3 minus b 1 minus 2 times b 2 minus b 1 which gives me b 3 minus 2 b 2 plus b 1, alright.

So, what we see is that augmented matrix rank is same as the rank of the coefficient matrix only when b 3 minus 2 b 2 plus b 1 is 0 alright, fine. So, that is this entry has to be 0 for us fine. So, what we are saying is that this system has a solution always has a solution; always has a solution; has a solution.

If b 1 b 2 b 3 satisfy, b 1 minus 2 b 2 plus b 3 is 0 alright. So, now if I relate these two ideas here in terms of x y z, what we are saying here is that x minus 2 y plus z should be 0. So, I would like you to check here, look at this vector 1 1 1, I get 1 minus 2 times 1 plus 1 which is 2 minus 2 which is 0. Let us look at this part again. X is 1 minus 2 times y is 2 and z is 3 which is 3 plus 1 is 4 minus 4 which is 0.

So, both these vectors they indeed belong to this, fine. So, what we are saying is this is the way to write the linear span that is the definition. This is called the algebraic way of writing it, algebraic way fine and this is called the geometric way; geometric way, so that we are able to say it what plane it represents. This is a plane.

So, recall that this is a plane containing the vectors 0 0 0, 1 1 1 and 1 2 3, fine. Again note I am writing it in terms of row vector itself basically because I do not have much space, fine. So, I will be interchanging this notations row and column notations, but be clear that I will be using only when the three comes. It will be only the column of vectors that are interested or that are important for me. Is that ok?

So, this was what I am saying is that when I am talking of linear span of something's, it means that I am looking at a plane or some space which contains those vectors alright. So, I am looking at the smallest plane; the a smallest plane containing those vectors; containing those vectors and the zero vector.

So, zero vector I am not putting here in general, but I want zero vector should always be there because I am talking of a subspace. Subspace means zero has to be there, alright. I am not proving here that linear span of any system is a vector subspace.

But I have given you an example to show here that whenever I have got a linear span of certain set of vectors V 1 V 2 V k, then this is a subspace and not only that it is a smallest subspace containing the vectors V 1 V 2 V k and the zero vector. Is that ok? So, keep track of that. In the next class, we will build up in this ideas further, alright.

Thank you.