Linear Algebra

Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 21
Vector Spaces

So, what we had seen was we have been able to understand how to solve a system of linear equation; how does the matrix multiplication helps in doing things, but what is more important is that there are lot of equivalent conditions which were playing some role or the other when we had square matrices, fine. So, I would like to go further on those ideas and then understand things that is one idea.

The other thing is that solving a system of equation also led us to the study of vectors in some sense in the sense that if I have a solution then there was some combination of vectors which gave me the right hand side. If I am writing at ax equal to b, then b was sum of certain vectors, fine. So, I would like to again go on those directions. So, let us understand what are vectors. So, the main idea for us will be now to understand what are vectors alright.

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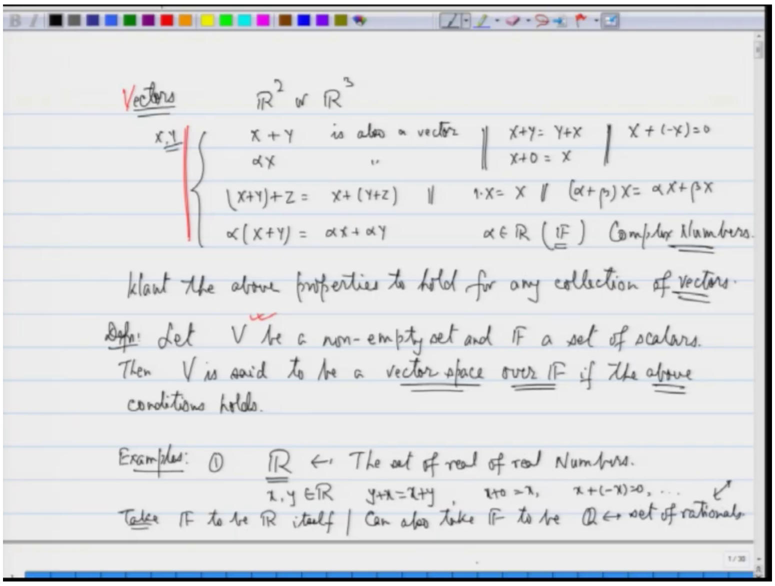

So, let us write what the vectors are. So, what we understand in R 2 or R 3; R 2 or R 3 that you have a vector X, you have another vector Y then you can add them; X plus Y is also a vector; is also a vector, fine. Alpha times X is also a vector. So, we had assumed that X and Y are vectors and therefore, all these two happens. Not only that, we have that X plus Y was equal to Y plus X; we had X plus 0 vector is same as the as the vector X for every X we had minus X such that this gave us 0.

Then there was a notion of what is called associativity hold that X plus Y plus Z was X plus Y plus Z, fine. We also have the notion that 1 times X was 1, fine and there is this notion of distributivity that holds that this is same as alpha times X plus beta times X. So, we would like to take them as our definition of what are the set of vectors, is that ok?

So, again the last part will be something like X plus Y times alpha is equal to alpha X plus alpha Y. So, when I am writing alpha here alpha is an element of R or in general it will be an element of F, fine because sometimes we may have complex numbers for us, fine.

So, the important thing to understand here is that whatever was true in R 2 or R 3 alright, there we had vectors and those vectors had certain properties. We want those properties to be satisfied for any set of vectors. So, we want the above properties to hold for any collection of vectors, fine.

So, for us definition is that definition let V be a non-empty set and F a set of scalars. So, remember a scalars could be real numbers, complex numbers, fine? It could be rational numbers, it could be Z p; if you remember Z p we had computed Z 5 it could be Z p, p a prime number fine and generalization of those ideas. But, mostly we will be looking at real numbers and complex numbers fine, but it could be anything alright.

So, let F be a non-empty set and F a set of scalars, then V is said to be a vector space over F. So, F does play a role that is important over F; V is said to be vector space over F, if the above conditions hold. Above conditions holds means I have an addition between two vectors.

So, given the set V; given the set V I have a definition of addition between two elements, I have a definition of a scalar multiplication there and with that all these conditions are valid that is more important for us. Is that ok? So, we already know that R 2 and R 3 are vector spaces let us look at more examples and there are hundreds of examples, but we will not look at all of them, we just a few of them alright.

For example, 1 let us look at R itself the set of real numbers, then I can add two elements of R. So, given any x belonging to R, y belonging to R, I can talk of x plus y. I have x plus y, I also have x plus 0 is equal to x, x plus y is same as y plus x, fine; we also have x plus minus x as 0 and so on. So, all the above properties are satisfied.

Here it turns out that I can take F to be R itself because the real number can be multiplied with real number to get a real number. I can also take; can also take F to be Q; set of rationals fine because rational times a real number is a real number. So, the multiplication by alpha where alpha is a rational number is a meaningful object and hence I can say that the set of real numbers is a vector space over real numbers as well as it is a vector space over rational numbers.

It turns out that these two are very very different objects, fine. Over rational numbers we will not be bothered about because this is out of syllabus for us, but this is very very important idea when you go for higher mathematics or higher places alright. So, we will not be bothered about, but this is very very important. It leads us to the study of rationalization and so on and lot of complicated things too.

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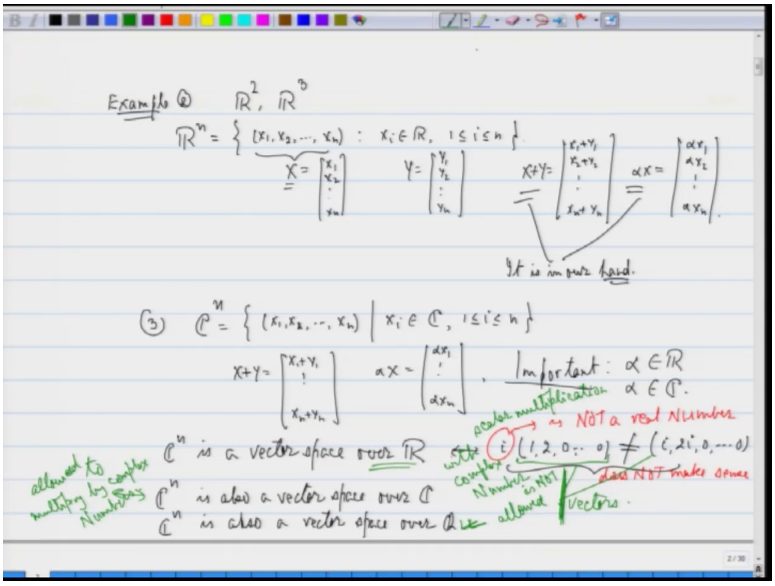

Example 2, you have already seen R 2, R 3 for you. We can also talk of R n; n is any positive number for us. So, I can take it as this R n is a set of collection of elements of the type X 1, X 2, X n such that each X i belongs to R 1 less than equal to i less than equal to n, fine R n. So, I can write this vector as X. So, generally I am not putting transpose here, but mostly we will be looking at as I said vectors are always column vectors for us.

So, it will look like this X 1, X 2, X n I will just write X in general; similarly, I will have Y which will be Y 1, Y 2, Y n. I can define X plus Y as X 1 plus Y 1, X 2 plus Y 2, X n plus Y n and alpha times X as I can define it as alpha X 1 alpha X 2 alpha X n alright. What is important is that this is the right way of doing things, but addition and multiplication is in our hand. It is in our hand.

We can choose these definitions in such a way that; that said that you are trying to study is a vector space alright. It depends on us how to define these objects. Nobody is going to dictate, it is you who has to decide which to take which not to take and why, fine depending on different issues. I will take some more examples.

Example 3: Let us look at R n we have already looked at complex number C n. So, now, here I will have vectors are going to be again X 1, X 2, X n. Each X i belongs to complex number one less than equal to i less than equal to n and again I can have the same thing X plus Y is X 1 plus Y 1 so on till X n plus Y n and you have alpha X as alpha X 1 till alpha X n, fine important. Alpha can be real number; alpha can be complex number fine.

So, what we are saying is that C n is a vector space over R. C n is also a vector space over complex number. C n is also a vector space over rational number and all of them they have different meanings, they have got different ideas. So, you have to be careful about them what you want to study and be careful about it.

For example, here in this case alright i times. So, i times if the vector 1, 2, 0, 0, 0 is not equal to i, 2i, 0, 0, 0 does not make sense that is important alright does not make sense. Why does not make sense? Basically, because i is not a real number alright. As vectors; as vector this is a vector, this is a vector, these two are vectors, they are fine , but this scalar multiplication is not allowed. A scalar multiplication with complex number is not allowed fine.

So, this is important when you are talking of complex number C n as vector space over R you are not allowed to multiply by complex number. You are allowed to multiply only with real numbers. Similarly, here you are allowed to multiply only with rationals, here you are allowed to multiply with everything allowed to multiply by complex numbers. But, there it is not allowed that is what you have to be careful about, fine. So, whenever you are looking at things be careful that is all it says nothing more than that. Some more examples.

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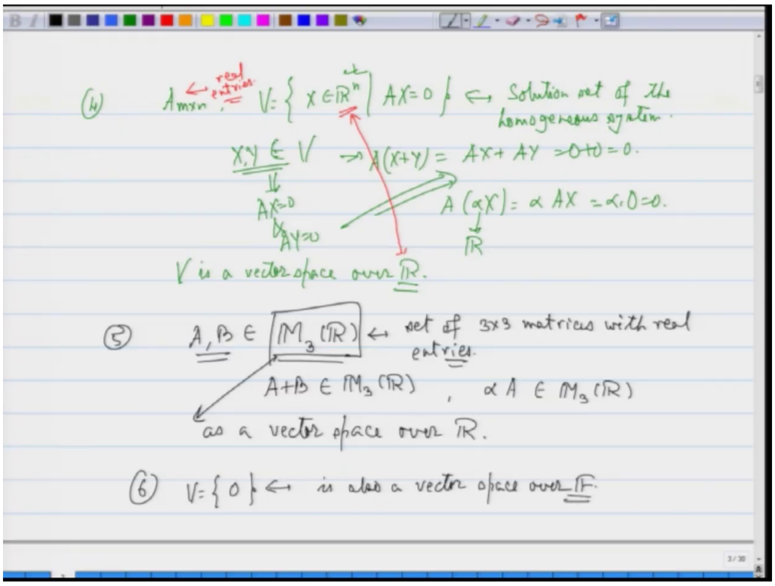

4th: we had looked at the solution system. So, suppose I have got a matrix a is m cross n. We have looked at the system all X belonging to R n such that AX is 0 alright this was called the solution set of the homogeneous system, fine.

So, look at here. So, if what we have already seen that if X 1 is a solution. So, let me write this as V. So, if X 1 and X 2 or X and Y let me write X and Y. So, if X and Y belong to V this implies that X plus Y times A is AX plus AY, fine. XY belongs to V implies AX is 0 and AY is 0 and therefore, this will together imply that this is 0 plus 0 which is 0.

Not only that A times alpha X is alpha times AX because alpha is a scalar say R I am looking at over R. So, R here so, this is same as alpha times 0 which is 0. So, therefore, if you see we are able to define vector addition and multiplication and you can check that all the conditions for V being a vector space is satisfied.

So, therefore, V is a vector space over R. Why over R? Because we looked at the system over R alright. So, this R is related with R here alright or this also related with assuming that this has real entries. If you are looking at complex numbers, it could have been over complex numbers fine.

I have this now let us look at this set of all 3 cross 3 matrices. So, I have A and B belonging to set of. So, remember M 3 of R means this means set of 3 cross 3 matrices with real entries, fine. So, given two matrices A and B which are 3 cross 3, I can add two matrices. I can talk of A plus B which will also belong to M 3 of R. We have a zero matrix, for every A we have a minus A and we also have a scalar multiplication alpha times A also make sense and this is a 3 cross 3 matrix itself.

So, therefore, I can think of this as a vector space over R. Similarly, if in place of the entries being real numbers if it was complex numbers I can talk in terms of complex numbers also, fine. One of the example that I have not given you is 6. Just think of a vector with having just a single element V to be 0 alright.

So, 0 plus 0 is 0 alpha times 0 is 0; 0 is a vector here alright. So, this is also a vector space over F, whatever F you want to take – real number, complex numbers, rational whatever you want you can do all of that fine.

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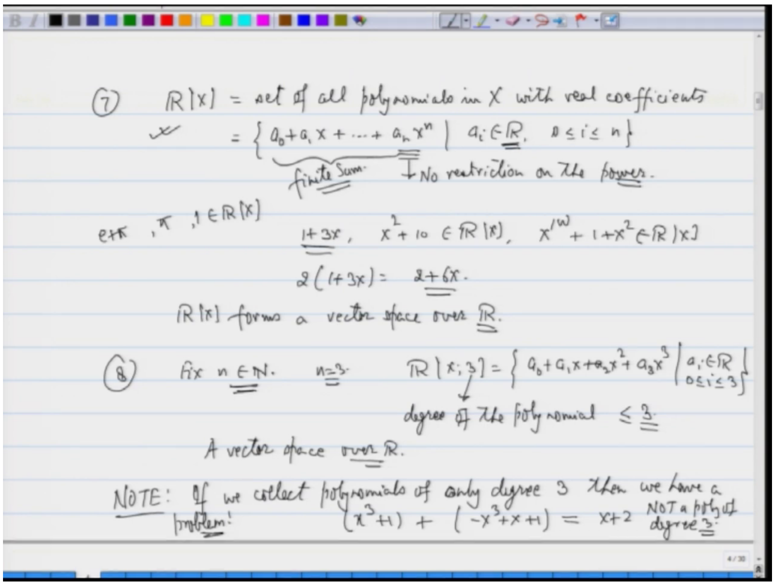

7: let me define the set R of X. R of X is set of all polynomials in X with real coefficients. So, at any element here looks like a 0 plus a 1 x plus a n x to the power n and a i belongs to R fine. 1 0 less than equal to i less than equal to n, fine. Note there is no restriction on n; no restriction on the power fine.

So, this set will contain constants. So, this will contain so, 1 belongs to R X, pi belongs to R X, e plus pi belongs to R X; we also have 1 plus 3X belongs to R X, X square plus 10 belongs to R X we also have X to the power 100 plus 1 plus X squared belongs to R X. So, all of them they belong to R X. So, any polynomial that you can think of polynomial means finite sum, there is no infinite sum here. There is a finite sum alright.

So, look at any polynomial; polynomial means finite number of monomials finite number of terms fine. So, that is an element of R X. So, given one polynomial, given another polynomial

I can add two polynomials to get another polynomial. So, therefore, R X makes sense; the addition of vectors addition of polynomials make sense. I can also multiply by a scalar. So, every number gets multiplied.

So, for example, 1 plus 3X times 2 is nothing, but 2 plus 6X. So, you are just multiplying it component wise in some sense and therefore, that also make sense. So, set of all polynomials in X with real coefficients make sense and this forms a vector space over R; R because the coefficients are real numbers alright. So, R X forms a vector space over R.

So, if I replace the coefficient by complex numbers we will again have complex numbers, rationals and so on coming into play alright. In the last two examples if you remember matrix multiplication is defined, similarly here polynomial multiplications are defined fine. So, matrix multiplication is a meaningful object A is a 3 cross 3, B is a 3 cross 3 matrix. So, I could multiply A into B to get a 3 cross 3 matrix, fine.

But, I am not bothered about those things alright. We are bothered about only addition and scalar multiplication. It has some extra things; let us not worry about those extra things, fine. We want to restrict ourselves to looking at only vector addition and scalar multiplication that is very important for us fine.

Another example; fix n belonging to natural numbers for example, take n equal to 3 alright, then I talk of R of I do not know what was my notation in my notes. Let me just look at it. Do I have a notation here? X and n yeah this is the notation. So, this is a 0 plus a 1 x plus a 2 x square plus a 3 x cube; a i belongs to R, 0 less than equal to I less than equal to 3 fine.

So, here we are fixing degree of the polynomial is less than equal to 3, alright. So, if I have two polynomials of degree less than equal to 3, then I can add those two polynomials and the degree of the polynomial is not going to increase because I am not multiplying, this is important. Since I am not multiplying, I am just adding two vectors so, I have got I have supposed to add only two polynomials of degree 3 or degree less than equal to 3 therefore it make sense fine.

Similarly, a scalar multiplication make sense. So, this is again a vector space over R, fine. Note, if we collect vectors collect polynomials of only degree 3, then we have a we have problem. What type of problem do I have? Alright.

So, let us take a polynomial x 3 plus 1; take another polynomial which is minus x cube plus x plus 1. These two are polynomials of degree 3. If we add it, fine this x cube x cube cancels out I am left out with x plus 2 not a polynomial of; not a polynomial of degree 3 alright. So, there is a problem, is that ok?

So, therefore, when you want to talk of polynomials we have to talk in terms of polynomials of degree less than equal to some number. So, that is n that we are talking about here. So, you can fix it and then proceed, fine. Now, one very important example you may be saying that what is new about it we are looking at the same things again and again that is true.

Most of the times we will be looking at the same polynomials or same matrices, same vector, same R n and things like that, but there are places where things are different. So, let me take an example here. So, that example was 8. Let me look at example 9.

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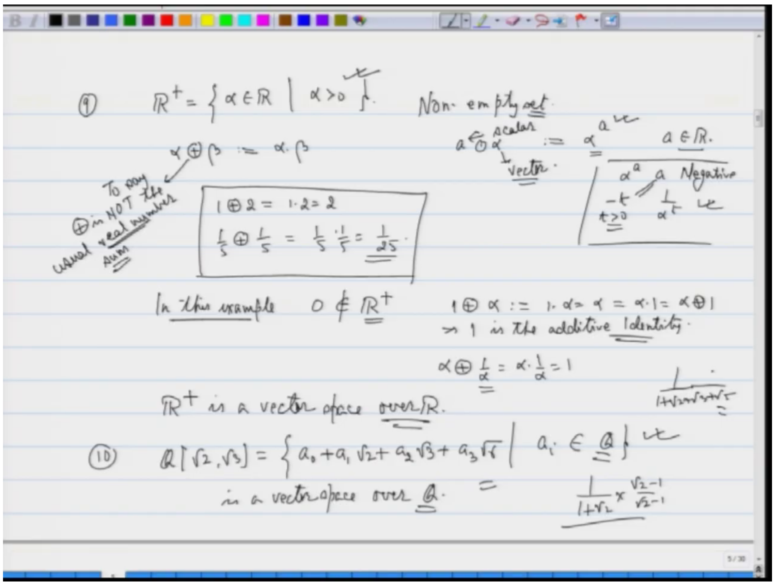

I am going to look at this set R plus. R plus is set of all alpha belonging to R, so that alpha is positive alright. This is my set this is a non-empty set. Take any two elements here alpha and beta I want to define the sum as not alpha plus beta, but something else. So, I put a plus sign here to say plus is this is not the usual real number sum. Real numbers are sum differently this is not the same alright.

So, here I am saying that this is I define this to be equal to alpha into beta. So, what I am saying is that 1 plus 2 is equal to 1 into 2 which is 2. 1 upon 5 plus 1 upon 5 is 1 upon 5 into 1 upon 5 which is 1 over 25, is that ok? Alright. The addition the vector addition that I am defining it has changed, fine.

So, here for example, in this example this example 0 does not belong to R plus. So, I cannot talk of 0 plus something, fine. But, I can talk of 1 plus any real number alpha and this by definition is equal to 1 into alpha is alpha. So, this implies 1 or this is same as equal to alpha into 1 which is same as alpha plus 1. So, 1 is the additive identity.

So, 1 is the one which works as something nice fine. Also, alpha plus 1 upon alpha is alpha into 1 upon alpha which is 1. So, the inverse the additive inverse is 1 upon alpha of alpha, fine and it make sense because alpha is a positive number here. Alpha is a positive real number, fine.

I have defined a scalar multiplication. So, let me take a as scalar fine and alpha as vector fine. I can talk of this as defined to be equal to alpha to the power a. Alpha is a positive real number, a is a scalar real or real number. So, whether it is negative or positive alpha to the power a makes sense alright. So, a can be our real number; a could be 0 also. So, any number to the power 0 is 1.

If a is positive number you will have positive powers; if a is negative so, a negative will imply I am looking at 1 upon alpha to the power a alpha to the power a will correspond to. So, a is negative; so, this is equal to say minus t. So, this will be alpha to the power t, fine t positive. So, accordingly you can make sense and everything is done.

So, you can show that R plus is a vector space over R. Again, you can make it a vector space over rationals and over anything. But, important this is a different vector space than all the vectors that I have looked at, fine.

Their other vector space also 10th, I will just give you the last one. I can think of this space Q of root 2 and root 3. So, this is the collection of all elements of the type a 0 plus a 1 root 2 plus a 2 root 3 plus a 3 root 6 and a i belonging to rational numbers. So, you can see that you can add two vectors; you can multiply by a rational number. So, this is a vector space over Q, fine.

So, we will be looking at all the examples till 8, we will also look at some more examples future whenever we come across them. These two 9th and 10th are given to you to make you understand that just looking at plane R 2, R 3, R n and items coming from there is not enough, fine. These are also very important ideas which give us different problems to solve at, to look at and they have got different applications.

For example, this will tell us that how do I rationalize numbers. So, can I talk of we used to talk of 1 plus root 2 multiplied by root 2 minus 1 by root 2 minus 1. What do I mean to say that I can look at 1 plus root 2 plus root 3 plus root 5 and so on. Here we do not have root 5 I can just include root 5 root 10 and so on and you can look at things, fine. So, that is all for now. We will look at vector subspaces, linear span and things like that in the next class.

Thank you.