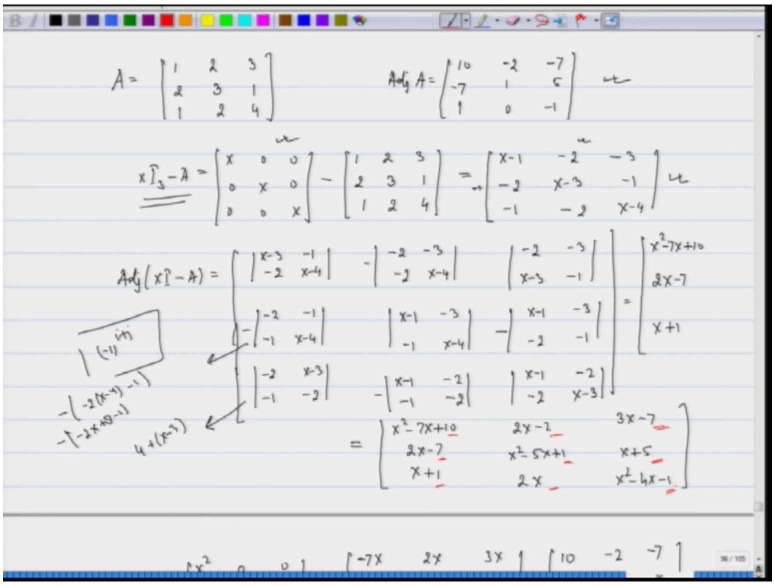
Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 20

(Refer Slide Time: 00:19)



Alright. So, recall that in the last class we had learnt what is A, adjoint of A, which were nothing but the transpose of the cofactor matrix. And in a hurry I just showed that A into adjoint of A is determinant of A into I and from there we got the inverse, alright.

So, what I would like now to do is that proceed with that idea itself and do something, so that there is a clarity of thought, fine. So, I take the same example, alright. So, let us look at the example that we were taken. We had A which was 1 2 3, 2 3 1 and 1 2 4. We had computed the adjoint of the matrix. I did it in a hurry, so we will do it again in some sense, but for a different matrix.

And then I want to compute what is x times identity minus A. So, x times identity is this matrix and minus A is (Refer Time: 01:05), so this is nothing but this matrix is equal to x minus 1, minus 2, minus 3; minus 2, x minus 3, minus 1; minus 1, minus 2, x minus 4. I want to compute the adjoint of this matrix.

So, what is adjoint of this matrix? Adjoint of x I minus A is nothing but, the first entry is this. So, adjoint is going to be determinant of this matrix. So, I have this determinant of this matrix which is x minus 3 minus 1 minus 2 x minus 4, fine.

Then, I have to compute the adjoint of this, alright, this entry I corresponding entry. So, if I want to look at this entry it is minus 2, minus 1, but I have to put it as. So, I am looking at this part, so I have to put it as a column here because it is a transpose cofactor transpose. I have to put it as minus 2 minus 1, fine. And then it is minus 1 x minus 4 here, fine. Similarly, for this I have to look at minus 2, minus 1 and x minus 3 minus 2.

Now, the next one I have to look at this part. So, this part gives me remove this and remove this, so it is minus 2 minus 3; minus 2 minus 3, minus 2 x minus 4. Then, the diagonal remove this remove this x minus 1, minus 3, minus 1, x minus 4. I hope I am not doing a mistake, fine.

This into this, so x minus 1 minus 2, minus 1 minus 2. When I do the calculation I will know the mistakes. Remove this remove this, so it is minus 2 minus 3, x minus 3 minus 1. Then this and this, so it is x minus 1, minus 3. This and this, so it is minus 2, minus 1. Here it is this and this, so it is x minus 1, minus 2; minus 2, x minus 3.

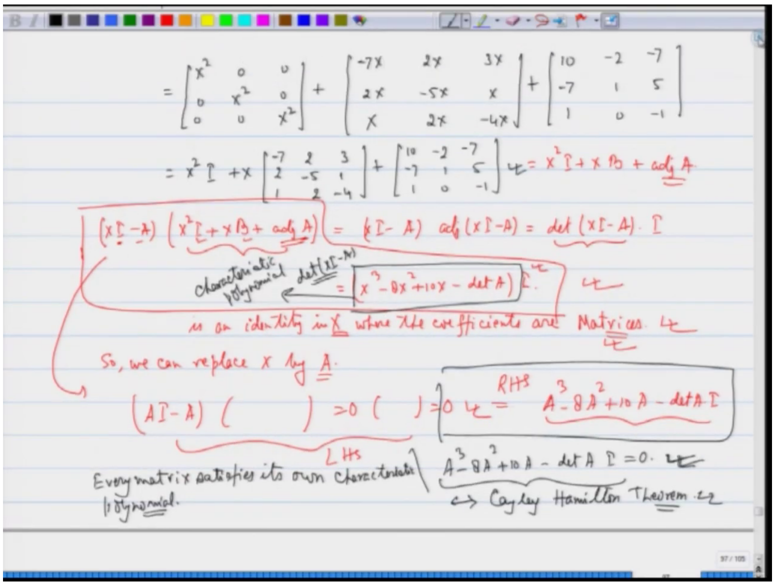
So, this I would like you to see that this is nothing, but x square minus 7 x and then plus 12 minus 2 is plus 10, alright. I hope its correct plus 10, fine and so on, fine. What is this entry? This entry is minus 2 x, alright. So, one thing I have forgotten it is minus 1 to the power i plus j.

So, it was plus sign here, there will be minus sign here, plus sign here, minus, plus, minus, plus, minus, plus, alright. Because it is minus 1 to the power i plus j that has to be taken care of. I did not take care of therefore, it was getting wrong signs, alright. So, as I said I always do mistakes.

So, if I look at this part it will give me minus of minus 2 times x minus 4 minus 1 which will be minus of minus 2 x plus 8 minus 1 which is 2 x minus 7, 2 x minus 7, fine. If I look at this part it is plus sign, so it is 4; minus minus plus x minus 3, so it is x plus 1, fine.

So, I would like you to compute it yourself. So, this turns out to be equal to let me complete it, so that I use it afterwards 7 x plus 10, 2 x minus 7, x plus 1; 2 x minus 2, x square minus 5 x plus 1, 2 x; 3 x minus 7, x plus 5 and x square minus 4 x minus 1, fine. So, this is the way I write it down. So, this is this.

(Refer Slide Time: 05:49)



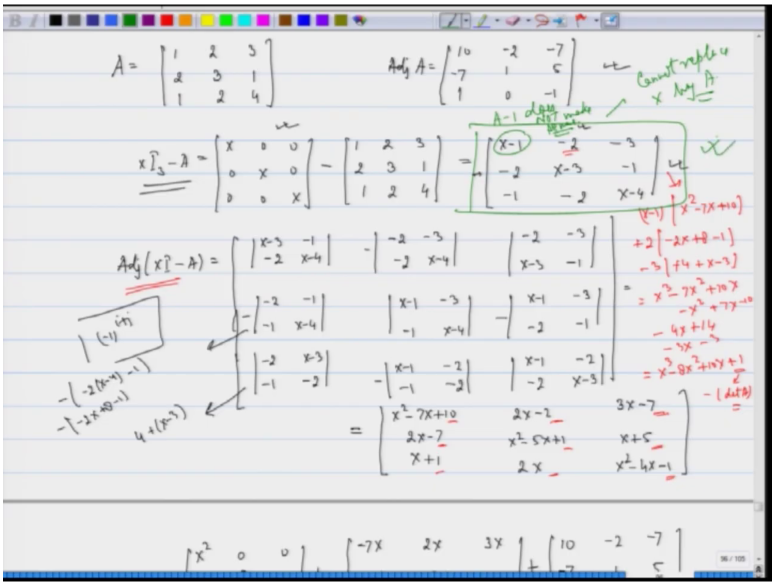
Now, what is important is I can go and write this also equal to just see it nicely. It is x square, there is no x x square there is no x square it is 0 0, 0 x square 0, 0 0 x square. Let us look at coefficient of x. Coefficient of x comes from or here I could have written as let see. So, coefficient of x comes from minus 7 x, 2 x, x; 2 x, minus 5 x, 2 x; 3 x, x, minus 4 x; plus there is a constant which is 10, minus 7, 1; minus 2, 1, 0; minus 7, 5, minus 1.

So, this is same as x square times identity plus x times a matrix which is minus 7 2 1; 2 minus 5 2; 3 1 minus 4 plus 10 minus 2 minus 7; minus 7 1 5; 1 0 minus 1. I would like you to see that this matrix, just let us go back. This matrix is same as adjoint of A, 10, minus 7, 1. So, you can say here. So, if you look at here 10, minus 7, 1 is the constant; minus 2 is the constant, 1 is a constant, 0 is a constant here; minus 7, 5 and minus 1, alright.

So, I would like you to see that I get here this is as x square identity plus x times some matrix B, alright. I do not know what that matrix B, but there is some matrix plus adjoint of A; is that ok. Now, let us multiply. So, this matrix with x I minus A. I want to multiply x I minus A with this matrix, x square I plus x B plus adjoint of A, I want to multiply this, fine. So, what we are doing here is I looking at x I minus A. Now, what is this matrix? This matrix was adjoint itself. So, this was adjoint of x I minus A.

So, with what we have learnt that A into adjoint of A is determinant of A into I. So, this will give me, it will be determinant of x I minus A into identity, fine. So, we need to compute what is determinant of this and then do the calculations; is that ok. Or you can just multiply it here and see what is happening to you, fine. So, whatever the calculations. So, this turns out that let me calculate the determinant directly now, fine.

(Refer Slide Time: 08:53)



So, let me compute the determinant of this matrix. So, determinant of this matrix is x minus 1 into x square minus 7 x plus 10 that we were looked at, then this plus 2 times minus 2 of this, fine. So, minus 2 plus 2, so minus 2 x and minus minus plus 8, this minus 1 minus 3 times minus 4 plus x minus 3, I hope it is correct.

So, this is same as x cube minus 7 x square plus 10 x, minus x square plus 7 x minus 10, minus 4 x plus 14, minus 4 minus 3, I think this is wrong I think. I hope minus 4, let me just look at. This is plus 4 and then, x and minus, so this is plus 4.

So, it is x plus 1, so x plus 1, so minus 3 x minus 3. So, this is nothing, but x cube minus 8 x square. Let me check, 10 plus 7 x minus 7 x is 0, so plus 10 x, fine. And then it should be minus 1, so let see minus 10 minus 13 plus 1, so it is plus 1. So, this is same as minus of determinant of A, alright, fine.

So, I would like you to see that. So, this is equal to x cube minus 8 x square plus 10 x minus determinant of A whole times identity, fine. Now, what I would like you to see here is that, this part this is an identity in identity, is an identity in x where the coefficients are matrices, alright. I have matrix I here, A here, identity B and A, adjoint of A is again a matrix.

Here also the I can think of this as a whole x cube times identity, so the coefficient of x cube is identity, coefficient of 8 x square is also minus I, coefficient of 10 x is identity and there is a constant of which is minus determinant of A. So, it is again an identity as far as matrix identity is concerned. Since, it is an identity in matrix I can replace. So, we can replace x by A itself, by A, alright. So, if I do that then I see here that I am going to get A times I, this times A into something which will be 0 times something which will be 0, fine.

So, the important thing here is that in this expression, since this is an identity in x and the coefficients are matrices, I am allowed to replace x by the matrix A, fine. And therefore, what I see is that this is this, this is a left hand side. What about the right hand side? The right hand side is A cube minus 8 A square plus 10 A minus determinant of A into identity, there is no change here because there is no x. So, this is equal to this.

So, what we are trying to say from here is that A cube minus 8 A square plus 10 A minus determinant of A times identity is 0, fine. This part is called what are called the Cayley Hamilton theorem. It says that so, what the Cayley Hamilton theorem says? That every matrix satisfies its own characteristic polynomial.

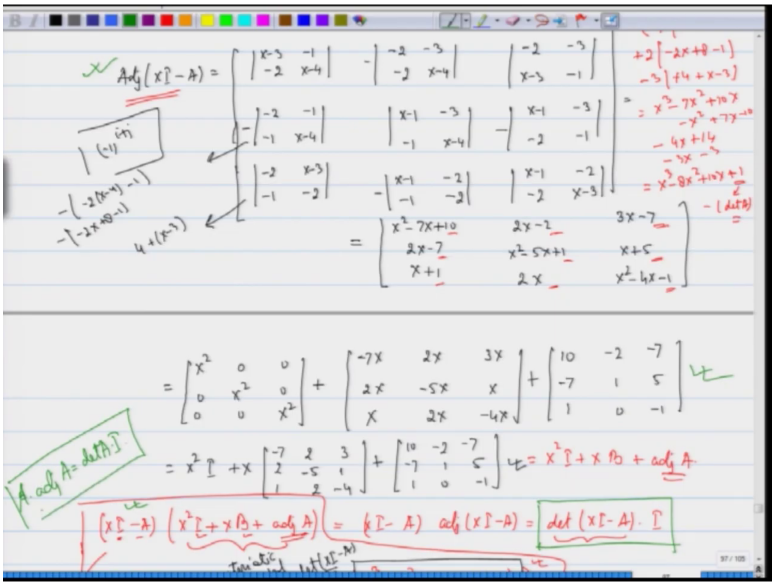
So, in this example, alright, this is the characteristic polynomial or which is same as determinant of x I minus A, alright. So, determinant of x I minus A is called the characteristic polynomial. And Cayley Hamilton theorem says that every matrix satisfies its own characteristic polynomial, so just replace x by A, so A cube minus 8 A square plus 10 A minus determinant of A into identity is 0. This is what it says, alright.

Now, the important thing to note here is that I cannot replace x in this expression. So, let us go back here from where did I get. Here, alright, so in this expression if I look at this expression here, alright, cannot replace x by A. This is not allowed here because A minus 1 does not make sense, alright.

So, to make sense of things from this I had to go to adjoint of A, rewrite it as a matrix identity, rewrite it as this, in terms of 3 matrices, alright. In general, if there are n electrolyte, n matrices and then do your multiplications and so on; is that ok. So, from there we will get the ideas.

Now, here just to spend some time I would like you to understand that why did I get this expression, alright. Because there also I did not explained you much. So, let me explain it here to you.

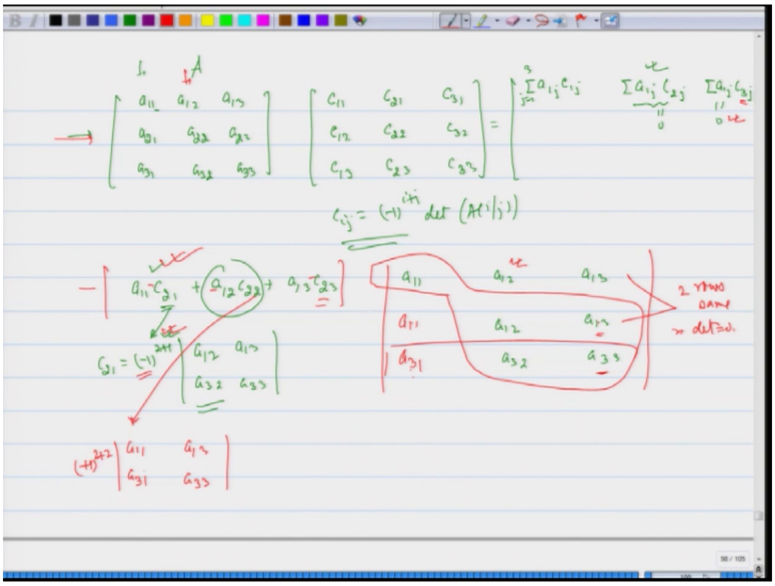
(Refer Slide Time: 15:30)



So, how do I get that x I minus A or A into adjoint of A, adjoint of A is determinant of A into I. How do I get this? So, let me try to make you understand this in this example here, alright.

So, I had this matrix with me, this was A, this is adjoint of A. So, let us look at what exactly we are doing. When I multiplying this matrix with this, so I multiplying x minus 1 with this one, so I will just write it down So, I multiplying x minus 1 to this, fine, that is minus 2 times this part, fine.

(Refer Slide Time: 16:16)



So, just let us look at things a 11, a 12, a 13; a 21, a 22, a 23; a 31, a 32, a 33. This is my A. Now, when I am looking at the adjoint, fine what I have is c 11 with a minus plus sign, fine, then I have got c 12 here with a minus sign c 13, then I have c 21 with a minus sign, c 22, c 23; c 31, c 32, c 33. So, when I multiply it the first part is summation.

So, look at here this one, alright 11, 11, 12, 12, 13, 13. So, it is a 1 j, c 1 j, j equal to 1 to 3 and minus sign is also coming into play again as usual, I do mistakes. So, there is no minus here this is this because this itself comes to the minus sign, alright. So, c ij itself comes with a minus sign, minus 1 to the power i plus j determinant of this it itself comes, fine. So, there is no minus sign here.

Now, this into this part if I look at this into this, it will be here. So, it will be a 11, 21, 12, 22, 13, 23. So, here it is a 1 j, but it is not 1, but c 2 j, alright. Similarly, the last one will be a 1 j, c

3 j, fine; c 31, 32, 33. Now, question is why is this 0, why is this 0, alright. So, let us look at this what exactly we are doing.

So, when I am looking at a 11 c 21 plus a 12 c 22 plus a 13 c 23, I have got the matrix a 11 here, alright, fine. With a minus a 12 I can put here minus here, here, so or I can replace this by just a 12 and then we will see what exactly we are doing I have a 13.

Now, what is c 21? c 21 is nothing but minus 1 to the power 2 plus 1, determinant of I remove the second row and remove the first column, second row and first column I have to remove it. So, I am looking at a 11, a 13 this goes off and this goes off sorry 21, so a 12 and this. And then a 32 a 33. So, I have got here this, so when I am computing the determinant this I am not bother about for the time being minus sign just look at this. When I am computing the determinant this is supposed to appear here a 12, a 13, a 32, a 33. So, this into this part is going to be this, alright. So, what I am saying is this part is this one, fine.

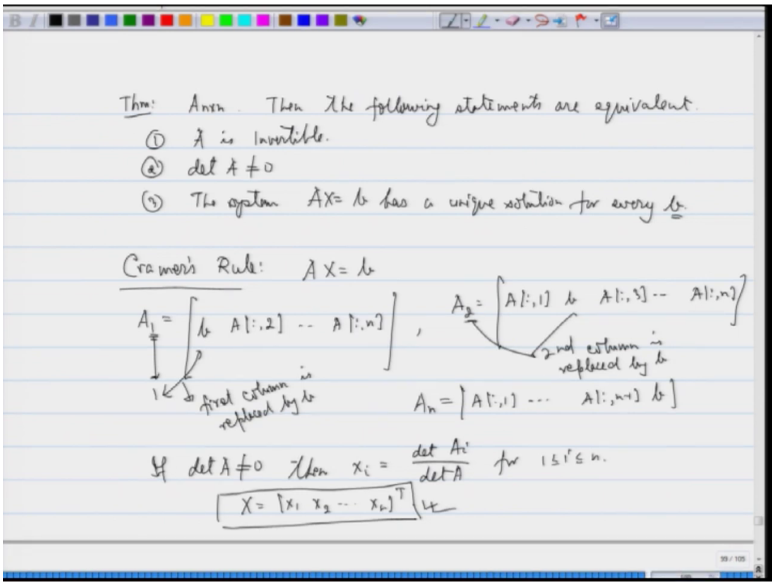
Now, I want to look at a 12 into c 22, there is a minus sign let us not worry about the for the time being; a 12 into c 22 c 22 is nothing, but c 22 means remove this and remove this. So, a 11, a 31, a 11, a 31; a 13 and a 33, fine there is a plus sign here minus 1 to the power 2 plus 2, so there is a plus sign here, fine.

So, now understand the where does minus and where does the plus come. Here I had a minus sign extra, fine, is that in this I had a minus sign extra. Now, if I look at here it is c 22 has no sign, but when I want to multiply here if I write here a 11 and a 31 it will minus a 12 times this matrix, this matrix, this matrix, this matrix, alright.

So, there is a minus which is coming into play. So, I could have done here, I could have put a minus sign here somewhere and then multiplied here minus, I could have multiplied minus here, fine; is that ok. And, similarly minus here, not in this, but in c 31 minus and looked at the whole thing.

So, what we see is that when I am computing this basically I am looking a determinant of this matrix in which there are two rows are same and therefore, the determinant is 0, alright. So, the same argument goes here. Here see I am looking at 3, so again this part will get replaced by a 21, a 22, a 23, but the first two rows will be the same hence, so therefore, they will become 0 as such, fine.

(Refer Slide Time: 21:33)



So, the last part today, alright what is called the Cramer's rule or I will just remind you of this theorem and then. So, a n cross n, then the following statements are equivalent. 1, A is invertible. 2, determinant of A is not 0. 3, the system AX is equal to b has a unique solution for every b, alright. So, there are some more conditions 5 more or 4 more, write them down and complete it, fine.

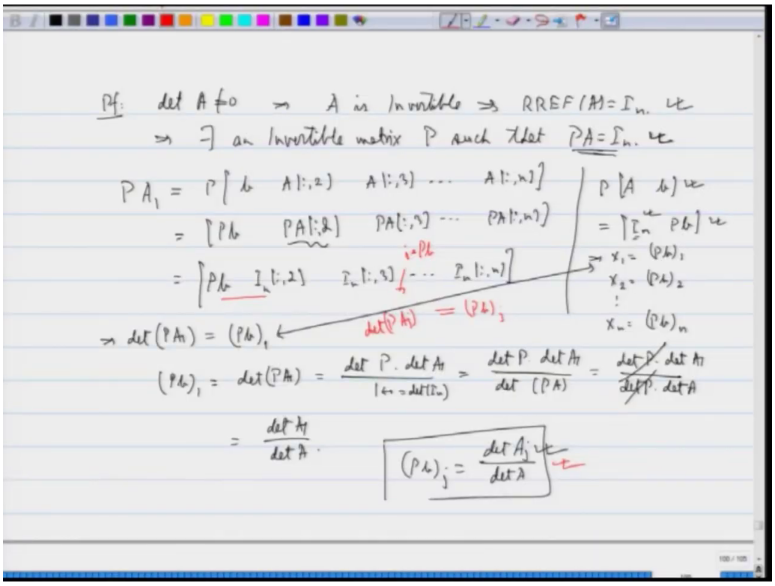
Now, let us come to what it called Cramer's rule. So, what we are doing is we are going to look at solid system AX is equal to b, and then there is a formula which you need to look at. So, we write what is called the matrix A 1 as I have the matrix b here, fine. Look at the second column, look at 1 here, so second column and so on till the last column. This is my matrix A 1. Matrix A 2 is just write the first column, alright, then b here, then the third column and so on till the last column, fine.

So, the important thing is this is 1, so the first column is replaced is replaced by b. 2, so second column is replaced by b. So, in general when I look at A n it will be first column will remain till the last, but 1.

So, n minus 1 column will remain b will come here, fine. So, it says that if determinant of A is not equal to 0, so we know that then x i is equal to determinant of A i upon determinant of A, fine for 1 less than equal to i less than equal to n and our X is x 1, x 2, x n transpose because our vectors are always column vectors therefore, I need to put a transpose here, fine.

So, this is what the Cramer's rule says. Let us try to understand Cramer's rule now, fine.

(Refer Slide Time: 24:39)



So, proof. Now, I have been given that determinant of A is not 0; implies A is invertible implies the RREF of A is identity implies; RREF of A is identity implies there exist an invertible matrix P such that P A is identity, fine.

So, what do mean P is identity? It means that, alright. So, I have got; so, let us look at A 1, I want to compute what is P times A 1. P times A 1 will be P times A 1 is b, then the second column of A, third column of A, so on till the last column of A.

So, we know what is P times A is identity. So, I want looking at P of A 1. So, this is same as P times b, P times A of this, P times A of this, P times A of this. So, this is same as P b. What is this?

This is supposed to be the second column of identity, this is the second column of identify, this is the third column of identity and so on this is the nth column of identity. So, therefore, determinant of P times A 1 will be equal to, alright; look at the 1 1 entry of the first entry of P b; is that ok, fine.

So, let us again understand what we are doing. So, I have P times A is identity, I am looking at the augmented matrix this, fine. Apply P to it what I get here is identity and P of b, fine. So, if there is an identity here, it means that x 1 is nothing but the first entry of P b, x 2 means second entry, x n is equal to the nth entry, fine, important. Again understand that determinant of A is nonzero implies RREF of A is identity, it means that there is invertible matrix P such that P of A is identity.

So, just multiply P to the augmented matrix, I get I and P b. And what you know is that since there is an identity here therefore, the system of equation has a unique solution given by x 1 is equal to the first element of P b, x 2 equal to the second element of P b and so on. So, what I see here is that determinant of P of A 1 is also same as first element of this; is that ok. So, therefore, I get that determinant of P A 1 is this and therefore, x 1 is nothing but this, fine.

So, what we know is the, so P b of 1 is equal to determinant of P of A 1 which is same as determinant of P into determinant of A 1 divided by 1 which is same as determinant of P into determinant of A 1 divided by determinant of P times A. Because I gives me determinant of identity which is same as determinant of P into determinant of A 1 divided by determinant of P into determinant of A. This part cancels out, you get this is equal to determinant of A 1 upon determinant of A, fine.

The same idea you can use to prove that in general P b of j is equal to determinant of P j, so determinant of A j divided by determinant of A, fine. The idea being same. The only thing is that you are looking at the jth part here, alright, since you are looking at the jth part here everything will remain same, the jth component here will become P b, fine and since jth component is P b the determinant of this matrix will be nothing but P b of j, fine; is that ok.

So, when you look at P of A j determinant of that it will turn out to be this because every where else it is identity, fine and therefore, you will get this.

So, I would like you to keep track of this. Mostly we are not going to use Cramer's rule, but you should know it, fine. So, this finishes our chapter on system of linear equations.

And in the next class, what we look at is what are called vectors, the general set of a vectors. What we had learnt was recall that when we our we solve a system of equation, who have been able to solve it and then get some relationship between lines, planes and so on, but there was a hidden relationship in vectors. So, we need to understand the vectors in a proper way that is the first thing.

Second is solving a system of equation also resulted in trying to understand rows and columns, in the sense that it told us the determinant of A is not 0. Means the columns are in certain sense giving me some information, rows are giving me some set of another set of information's, can I relate those ideas because they are basically looking at column size vectors, you can also look at rows as vectors, fine and then build upon those ideas.

So, from next class onwards, we will not look at system of equations, but we will be using those ideas, system of equations ideas will be used again and again to get our results. To look at new ideas, new definitions and so on all the things will be based on solving the system AX is equal to b, unique solution, no solution and infinite number of solutions. They lead us to different ideas what are called linear combination, linear span, fine, linear independence and dependence and so on. So, please read it before you come to the next class.

Thank you.