Linear Algebra
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Lecture – 18

Alright so we had learnt quite a few things about invertibility and so on, a square matrices and system of equations. Now, today we would like to look at what are called determinant of a matrix, determinant of a square matrix.

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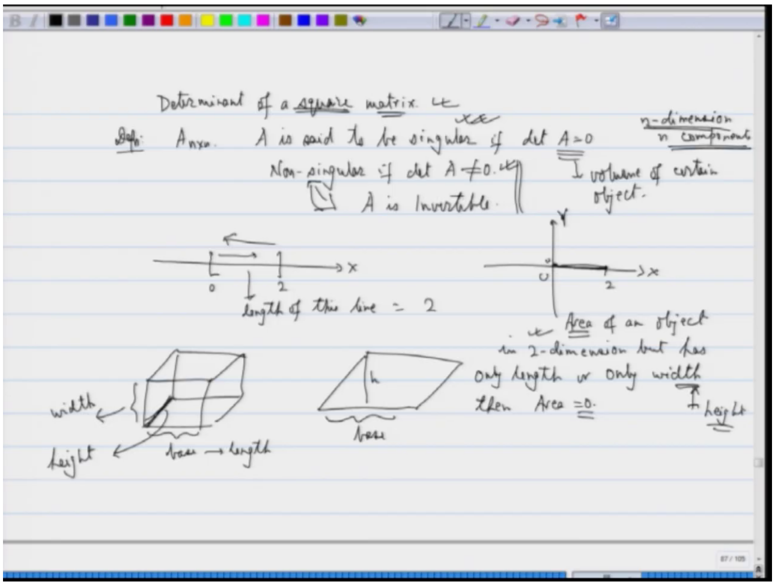

And then we relate it with invertibility and so on. So, what it turns out is that, A is singular, this is notion of singularity and this is a notion of non singularity. So, there is this definition, I will just write it down within; we have not computed what the determinant is, but I will just write it.

So, A is n cross n, A is said to be singular if determinant of A is 0 and is said to be nonsingular if determinant of A is not equal to 0. So, this non singularity is equivalent to saying that. So, since it is a square matrix, we can talk of invertibility; so A is invertible.

So, all these notions of invertibility having system of equation having a unique solution or system of equation having a solution for every choice of b and so on, they are all related in some sense, alright. So, therefore, we need to understand this, what is called non singularity of matrix or determinant of a matrix? I would also like you to know that determinant of A is nothing, but volume of certain objects; it is nothing, but volume of certain object, what is called a parallelepiped, alright, fine.

So, for example, when I talk of singularity basically it means that, when I am looking at say a length of a line alright fine; length of a line is a meaningful object when I am looking at a 1 dimensional object, fine.

So, if I have the. So, if I am looking at the interval. So, I have an x axis here and I have the interval say 0 to 2; then length of this line, length of this line is 2, fine. If I am going from here to here, it counts twice; if I am going in the opposite direction, it is minus 2, alright.

So, it is 0 minus 1 and things like that, fine. But suppose I have a figure here in r 2 we have x and y fine and I have 0 to 2 here; but the height here is again 0 itself. So, height is 0 itself. So, in that case we get only one line and I am looking at just this, area of this portion.

So, area, if I want to talk of area of an object in 2 dimension; but has only the x component or the y component, but has only length or only width , then area is equal to 0, alright fine. Because we need, for area we need. So, even if I am got a parallelogram here, I need, it should have certain height and you should have a base, fine.

So, when I say that. So, width is nothing, but the height that I am talking about here, fine. So, what we need is base as well as height. So, we need two components, same 2 dimension two components. If I am looking at 3 dimensions; for example, a cube, then I have got three things, alright.

So, if I am looking at a cube fine, if I am looking at a cube; then I need a base or the length, then I have this which is the width or height whatever we want to say depends on you, and then I have third component which is this part, fine.

So, there are three objects that we are looking at and this I can say it as height, whichever you want to say height depends on you. So, there are three components. So, in 3 dimension, there are three components. So, in general when I am talking of determinant say in dimension n alright, n dimension, then I should have n components, fine. So, we have got n components.

So, again I what I am trying to say again is that, if I am looking at a 3 dimensional object; but there the height is 0, volume will be 0, but the area it is still has some area. Similarly when I am looking at 2 dimensional, object area will be 0; but there will be at least one length, whether the length will be there or there will be a width or something will be there, fine.

So, in which dimension you are looking at that matters. So, singularity basically means that, I am looking at something in say 10 dimension and singularity means that from 10 dimension, I am going to 9 dimension or 8 dimension or something like that fine.

So, whenever there is a notion of singularity, it basically means that I have not got full dimension alright; the object that I am looking at does not represent the object in the full dimension that I am required to look at that is very important, alright.

So, with this notion with this understanding, let us look at some notations and then proceed to get our definitions, alright. And also I will not give you the actual definition; because actual definition is a bit difficult to understand at this stage, but I will give you some ideas, fine.

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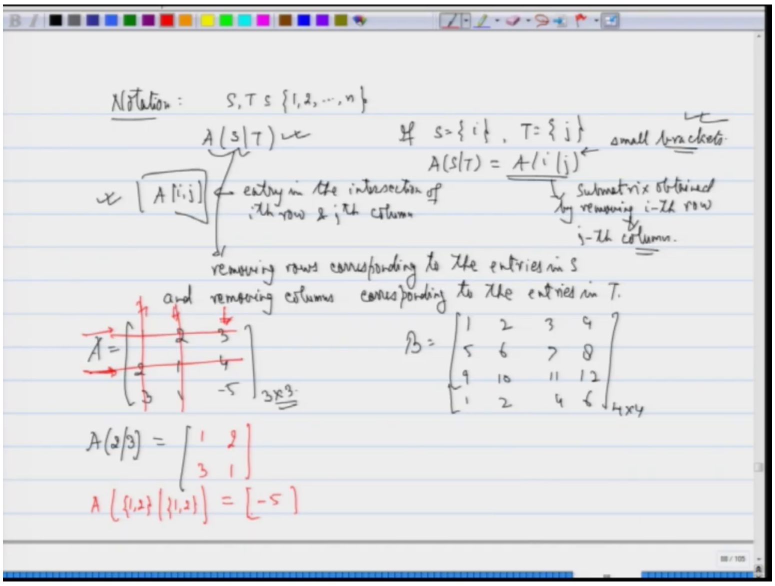

So, some notations, notation. So, suppose I have got S and T two subsets of. So, I am looking at an n cross n matrix.

So, I was set 1, 2, n and S and T are subset of this. So, there is a notion of what is called A of S T like this fine, as a subsets. If S consist of single objects like i and T consist of single object j; then I will write A S T just as A of i j, alright. The important thing here is these brackets are what are called a small brackets.

So, let us not get confused with the idea of A i j, here this A i j was entry in the intersection of intersection of ith row and jth column, alright. But here this means, sub matrix obtained by removing i th row and j th column. So, there are two different things; here I am removing the things, here I am looking at the corresponding, there is a i comma j and there is a i and then vertical line j, alright. So, this definitions are important; here it is A S vertical line T, alright.

So, I am removing rows corresponding to S. So, this part corresponds to removing rows corresponding to the entries in S and removing columns corresponding to the entries in T. So, let us take an example to understand it better.

Suppose I write A as 1, 2, 3; 2, 1, 4; 3, 1, minus 5, I have A. I write B as a 1, 2, 3, 4; 5, 6, 7, 8; 9, 10, 11, 12; 1, 2, 4, 6 suppose I have this. So, it is 4 cross 4 here, it is 3 cross 3 here the square matrices; these definitions can make sense even if it is rectangular, similarly I have to take S and T as subsets.

But here if I am looking at A of say 2 comma 3, so I am removing the second row. So, I am removed the second row; remove the third column, remove the third column. So, let me I think I should do it with some other ink, there is a problem otherwise.

So, I am looking at this part and this part. See, if I am remove this, what I am get out getting is 1, 3 entry here; 2, 1 entry here. So, this is my entry, alright. So, I have got removed the second row, the second row and third column. If I am removing say 1, 2 comma 1, 2 alright; then this is equal to.

So, remove 1, 2 remove the first and second row and remove the first and second column. So, if I do that, this goes off, this goes off, this goes off, this goes off; I am left with minus 5 only, is that, fine. So, this is what you have to be careful about.

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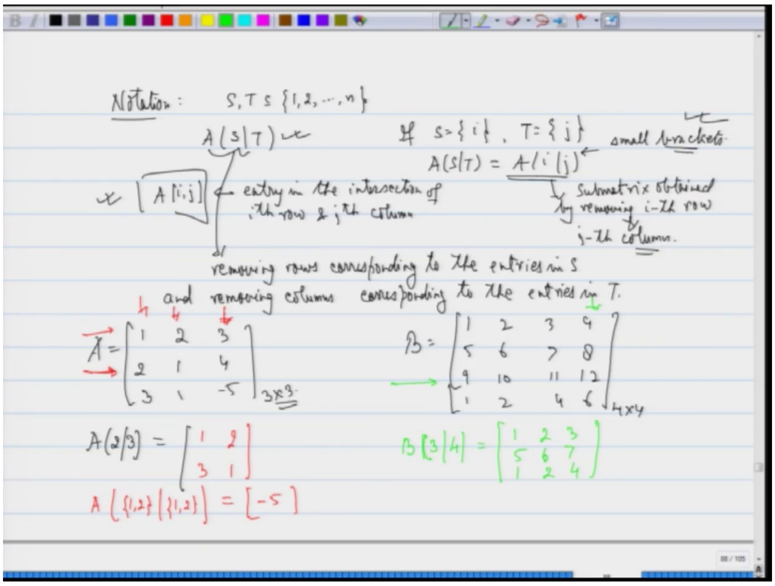

Similarly, when I am looking at here, I can look at B of 3, 4; I am removing the third and fourth. So, I am doing that third and fourth it is and this is the fourth one. So, it is 1, 2, 3; 5, 6, 7; 1, 2, 4 is that ok. So, you can remove and then you can look at what is left out and things like that, fine. So, with this notations let me now define, what is a determinant of a matrix?.

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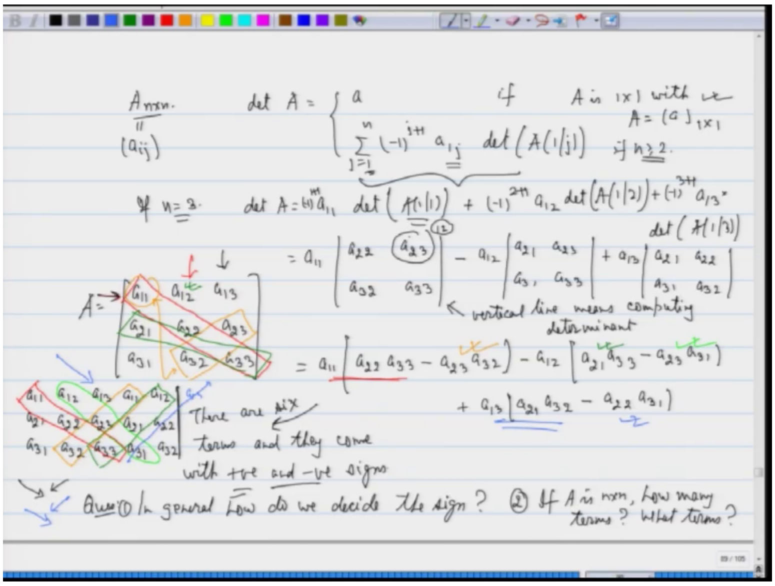

So, determinant. So, A is n cross n, determinant of A is a itself; if A is 1 cross 1 with A as just a alright, it is 1 cross 1, fine. And for the general set up it is. So, I have got A as a i j fine; j equal to 1 to n n cross n, there is a notion of, alright. So, let me write first.

So, j plus 1 a. So, I am looking at 1 here, 1 j, alright. So, j will go from 1 to n. So, a 1 1, a 1 2, a 1 3 and so on and then I remove the corresponding 1 is the first row. So, I remove the first row and j for the jth column, is that ok. So, what basically what I am trying to say here is that, if n is 3, n is 3, I am looking at determinant of A as a 1 1; I should have written determinant here A of 1 j into determinant of A of 1 1, 1 plus 1 is.

So, let me write that also 1 1 plus 1 plus minus 1 to the power 2 plus 1 a 1 2 into determinant of A of 1 2 plus minus 1 to the power 3 plus 1 a 1 3 into determinant of A of 1 3. So, which is same as a 1 1.

So, what is determinant of this? It is nothing, but. So, I will write like this. So, a 1 1 remove the first and second. So, it is a 2 2, a 2 3, a 3 2, a 3 3. So, this vertical line means determinant, vertical line means computing determinant, fine. This minus 1 here a 1 2 times remove the first row and second column.

So, first row, so a 2 and second also. So, what should I do? So, let me write it, so that there is a no confusion here; a 1 1, a 1 2, a 1 3; a 2 1, a 2 2, a 2 3; a 3 1, a 3 2, a 3 3 this is the matrix that we are looking at. So, I have supposed to remove 1 and 2. So, I am supposed to remove the first row and second column, fine. So, I am left out with this part a 2 1, a 3 1. So, I am left out with a 2 1 and a 2 3, a 2 1, a 2 3; similarly a 3 1 and a 3 3 plus a 1 3 into.

Now, I am supposed to remove the first row and third column. So, now, I am supposed to remove this column and this row. So, this goes off, this goes off; I am left out with this part, alright. So, I am left out with a 2 1, a 2 2; a 3 1 and a 3 2. So, this is again a 1 1. Now I can again apply induction here. So, this is true for if n is greater than equal to 2 fine; n equal to 1 I had already given with the definition and this was for n greater than equal to 2. A 1 1, again it is a 2 2.

Now, from here remove this row first and first, so it is a 3 3 left out. Again I have to remove this, but now look at here, this is supposed to be the 1, 2 entry. Since this is the 1, 2 entry; therefore I need to look at minus 1 to the power 2 plus 1 which is negative. So, minus of a 2 3 into a 3 2 minus a 1 2 into, again a 2 1 into a 3 3, a 2 1 into a 3 3 minus a 2 3 into a 3 1 plus a 1 3 into a 2 1, a 3 2 minus a 2 2, a 3 1.

So, the important thing here is that, this is the formula. So, you just have to remember it the way you compute, fine. There are six terms in these. So, that is the more important, there are six terms and they come with positive and negative signs, fine. How do I get in general?

Question is question; in general how do we decide the sign, fine? And that is one question. 2, if A is n cross n, how many terms do I have, how many terms, alright?

And what are those terms? What terms, fine? So, we need to understand these things. So, let us go back to the first part. So, if I look at this part, it says that I am looking at this diagonal. So, there is this diagonal that I am looking at; this is the first one, fine. This part is looking at a 1 1, then I am looking at this part and this gets associated with this one here, this is the second part, fine.

Now, when I am looking at this one fine, it is a 1 2 is coming from here and then it is looking at a 2 1, a 3 3, a 2 1, a 3 3 fine, is that ok. That is the way it looks like, it looks quite complicated. So, to make it easy, let me write it in a different way now, alright. So, we have want to proceed, it becomes complicated. So, let me write it different. So, I will write it like this a 1 1, a 2 1, a 3 1; a 1 2, a 2 2, a 3 2; a 1 3, a 2 3, a 3 3.

Now, I am looking at row, so I am going to expand it further right; this terms again and again, alright. So, a 1 1, a 2 1, a 3 1; a 1 2, a 2 2 and a 3 2, let me write all these terms, fine. Then the first one is looking at this one, fine. The second one we have already seen what was that; second one was a 1 1 into a 3 2, a 2 3, alright. So, this was there, fine.

The next is a 1 2, a 2 1, a 3 3. So, let us look at this 1 2, then it is 2 1 and 3 3, is that ok. So, this is what you have to be careful about; how do I get this term here, fine? Is that ok. So, you can see here that, there is a 2 1 here and a 3 3 is here, is that fine? So, a 1 2 look at here, a 2 1, a 3 3. So, there is this green term here, fine.

So, you can use this idea also to get all the terms fine; then you have a 1 2, a 2 3, a 3 1. So, this part, this one is a 1 2, a 2 3, a 3 1. So, a 1 2, a 2 3, a 3 1 is here, fine. You can also note that this term that I wrote alright, I could have got it differently also; this I could have got through this diagram, fine.

So, once I go through this means. So, once I need to go in this direction, another I need to go in this direction at each term. So, let us look at here what happens here, the last one; 1 3 alright, 2 1 and 3 2; 1 3, 2 1 and 3 2 I have got something like this. I want a corresponding thing back here, so I should have written one more term here; it means a 1 3. So, look at a 1 3, a 3 1 and then I have this term here which is this part a 1 3, a 3 1, alright.

So, all the six terms are there; once going in the direction which is this, the other I am going in this direction, fine. And signs we have to be a bit careful to understand later. So, let me look at some examples to proceed further, alright.

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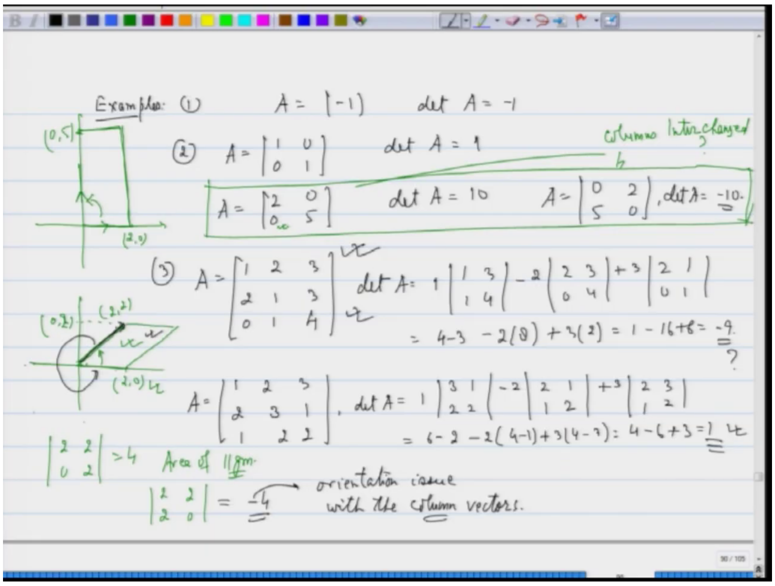

So, examples, 1; so if matrix A is just minus 1 determinant of A is minus 1. 2, if A is 1, 0; 0, 1 determinant of A is 1, if A is 2, 0; 0, 5 determinant of A is 10. If A is 0, 5; 2, 0 determinant of A is minus 10.

3, if A is 1, 2, 3; 2, 1, 3; 0, 1, 4 determinant of A is 1 times 1, 3; 1, 4 minus 2 times 2, 0; 3, 4 plus 3 times 2, 0; 1, 1 which is same as 4 minus 3 minus 2 times 8 plus 3 times 2 which is same as 1 minus 16 plus 6 which is minus 9 I think.

So, what you have do is that, you have to calculate yourself; because I do lot of mistakes, be careful and see that everything is nice. I cannot verify it here whether I have to done it correctly or not; but I would like you to verify it yourself whether the calculation is correct or not, fine.

Another example, so that I know I have done correct work; A is 1, 2, 3; 2, 3, 1 and 1, 2, 2 then determinant of A is 1 times 3, 1; 2, 2 minus 2 times 2, 1; 1, 2 plus 3 times 2, 1; 3, 2 which is same as 6 minus 2 minus 2 times 4 minus 1 plus 3 times 4 minus 3 which is 4 minus 6 plus 3 which is 1. So, this is indeed correct; I do not know whether this is correct or not, please find it out yourself, is that, ok.

So, this is the way we compute determinant. So before I go to the next definition, next idea; what I would like you to see here is that, look at this part, alright. What I have done is, I have just interchange rows and columns, fine. So, I have just interchange; look at here, the columns have been interchanged, columns interchanged, alright.

And the determinant gets multiplied with minus 1. What is more important from my point of view is, look at things here; my vector 2, 0 was, this was my vector say 2 comma 0 fine and 0 comma 5. So, this is 2 comma 0 vector, this is 0 comma 5 vector; so 0 comma 5 is going to be somewhere here, fine.

I am looking at area of this; my vector this is the first vector, this is my second vector and I am going in this direction, counterclockwise direction. And it make sense and I am getting a rectangle and its area is 10 that you can compute, that you know yourself.

Similarly, if I have a vector here say 2 comma 0, I have a another vector here say 2 comma 2; I go like this, I form a proper something here, fine. So, if I look at here the height is 2, is not it. See look at this part, the height this vector corresponds to 0 comma 2. So, if I look at. So, my first vector is 2 comma 0, the second vector is 2 comma 2, its determinant is 4 which is nothing, but the area of the parallelogram, fine.

But if I written my vector as 2, 2 and 2, 0; then it will mean that I am starting at this point and going like this, fine. And therefore, if I want to make a parallelogram using these two; I am not getting this parallelogram, but something else.

So, I will have to go in a negative direction and then build up the idea, fine. So, either I shift here and make a parallelogram like this or I go down here and then make a parallelogram here. So, I have to take a negative direction to get a proper parallelogram, and therefore I get this as minus 4; this minus is coming because of the direction or the what is called an orientation issue with the column vectors.

So, I am using the word column vectors, because initially I had said in the first class itself that all my vectors are column vectors. You can use row vectors, there is no problem and accordingly the definitions will go on, fine. But what is important is that, you are looking at negative sign coming from some sort of orientation issue; otherwise it is nothing, but the volume, the area itself here, fine.

So, here also if you want to look at, you got minus 9 here in this one; so there must be issue with the direction. Try to find it out yourself where is the issue; whether the third vector is in the positive direction of the plane generated by the first two or not and things like that, fine. So, that is all for now, alright yeah.