Linear Algebra

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Lecture – 17

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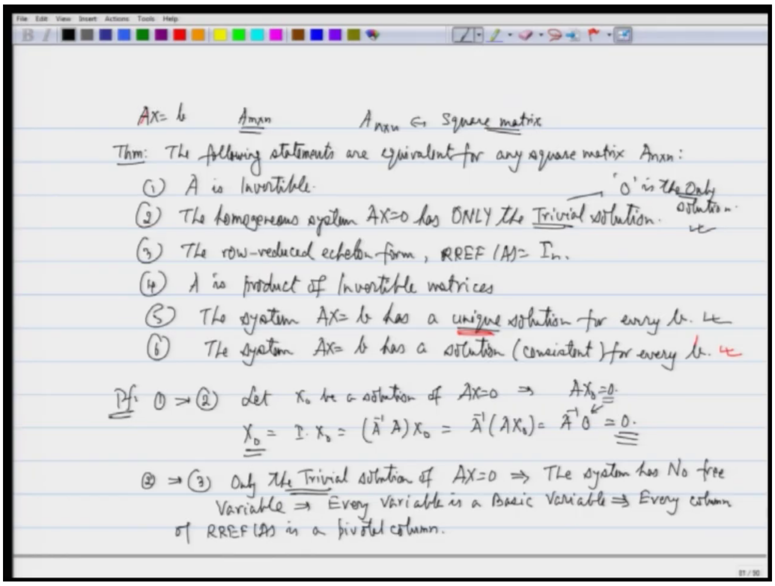

So, we learned solution space of the system A X is equal to b. We looked at A X is equal to b, where A was m cross n fine. So, there was about the rectangular matrices. Now, we want to concentrate only on a square matrices.

So, for us in this lecture, we are going to have A as n cross n alright, a square matrix and want to understand whatever we have done till now as far as this matrix is concerned alright. So, there is the theorem which I would like you to understand. The theory is very very important theorem, which relates all the ideas that I had till now about matrices fine.

So, the following statements are equivalent; flowing statements are equivalent for any square matrix A of m cross n fine. So, first statement is A is invertible, we had learned this what is invertible. Second, the homogeneous system genius system A X is equal to 0 has only the trivial solution; only the solution. So, recall that trivial means, ONLY the trivial means '0' is the only solution alright, is the only. So, there is no nonzero solution alright.

Three, the row reduced echelon form RREF of A is identity. Fourth, A is product of invertible matrices. Five, the system A X is equal to b has a unique solution; unique solution for every b. Six, the system A X is equal to b has a solution or consistent for every b alright. So, let us look at the proof of this. What would like to say here is that see if I look at five and six, five, it is a unique solution. The second says I do not have to worry about uniqueness alright; I do not have to worry about a uniqueness.

Basically we are saying these because what happens is that in lot of problems, it is very difficult to prove that something is unique fine. But it may be easy to find the solutions for in each case alright. So, showing uniqueness can be difficult, but if I can show that there is a solution for every choice of b, then I have that A is invertible right.

So, these are basically to help you get things without doing much work alright. Let us try to prove them. So, 1 implies 2; so, 1 implies 2. So, we are saying that A is invertible implies the system A X equal to 0 has only the trivial solution fine. So, A is so let X X naught be a solution of A X is equal to 0 fine. So, this implies A X naught is 0 fine.

So, now let us look at X X naught. X X naught is same as identity times X naught. Now, we are saying that A is invertible. So, I can write identity as A inverse A times X X naught, which is same as A inverse of A X naught, which is same as A inverse of 0. This is what we get from here which is 0 alright. So, X X naught has to be 0. You do not have a choice other than that. So, it has only the trivial solution fine. Now, let us look at 2 implies 3. The 2 says that the system has only the trivial solution fine. What does it mean?

Only the trivial solution means trivial solution of A X is equal to 0. What does it mean? This implies that the system has; the system has no free variable alright. Because if a system has a free variable, then it has infinite number of solutions; but here, it says that it has only the trivial solution is only one solution. So, there is no free variable and this implies every variable is a basic variable; is a basic variable and now, where are the basic variables? The basic variables that come from pivotal columns alright.

So, we are saying that every variable is a basic variable means that every column is a pivotal column, implies every column of RREF of A is a pivotal column fine. There are n of them and there are only n equations and therefore, what you get is so alright.

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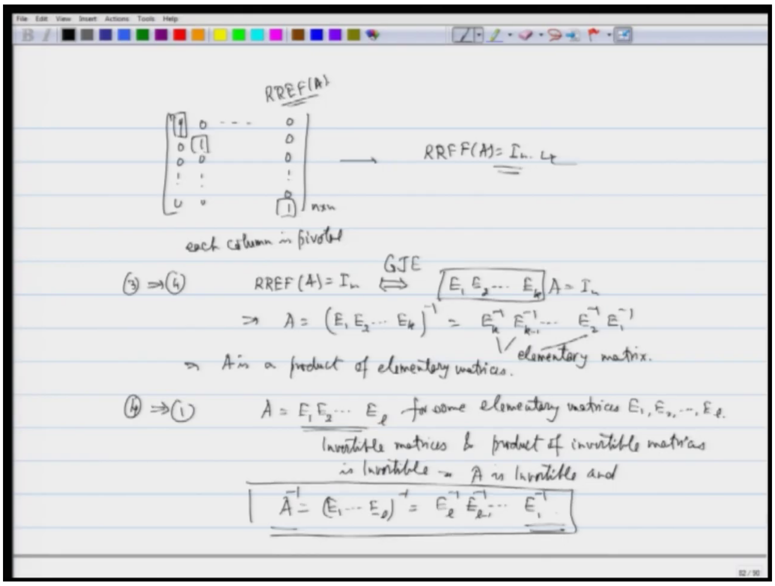

So, everything is pivotal means I can write this as 1 0 0 0 0 1 0 because every column is a pivot alright 0 0 0 and 1. So, this is a pivotal column, this is pivotal column, each of them is pivotal column, every. Each column is pivotal fine and therefore, so this is the RREF of A and this corresponds to looking at that RREF of A is equal to I n fine. So, this is want we want. The rule is to claim form of A is I n. Is that ok?

So, again let us go back, understand it. We want to prove that 2 implies 3. So, 2 says the homogeneous system has only the trivial solution. Since, it has only the trivial solution, it means that the system has no free variables. It has no free variables means it has only basic variables. So, there are n basic variables. Now, each basic variables corresponds to a pivotal column and therefore, each column is a pivotal column and therefore, this is a matrix that you are going to get because the matrix is n cross n fine, which is a I n fine.

Now, once you have that, so this what it was. The (Refer Time: 08:05) form of A is I n. So, we have proved that part. Now, let us look at 3 implies 4. So, we are saying that RREF of A is I n, but what is RREF of I n? To obtain RREF of I n, we need to multiply by elementary matrices on the left, that is the way we get.

We have to apply the Gauss Jordan algorithm elimination method. So, what we are saying that we have got E 1, E 2, E k times A is identity and this implies that A is nothing but inverse of this matrix which is E 1, E 2, E k inverse alright. A is inverse of this which is same as E k inverse, E k minus 1 inverse so on E 2 inverse E 1 inverse. Is that ok?

Now, what we know is that if E is an elementary matrix, its inverse is also an elementary matrix. So, each of them is an elementary matrix and therefore, we have shown that A is a product of this implies A is a product of elementary matrices. Is that ok? Now, let us prove 4 implies 1 fine. So, what we four says that A is product of elementary matrices. So, A is equal to some E 1, E 2, E l for some elementary matrices E 1, E 2, E l. We have to prove that A is invertible.

Now, what we know is that each of them is invertible because they are elementary matrices, invertible matrices and product of invertible matrices and product of invertible matrices is invertible implies A is invertible and nothing but A inverse is equal to E 1 to E l inverse which is E l inverse E l minus 1 inverse till E 1 inverse. Is that ok? So, we have been able to show that A is invertible because you could write A inverse as product of invertible matrices. Is that ok? Fine. Now, let us try to prove 1 implies 5 and 5 implies 6.

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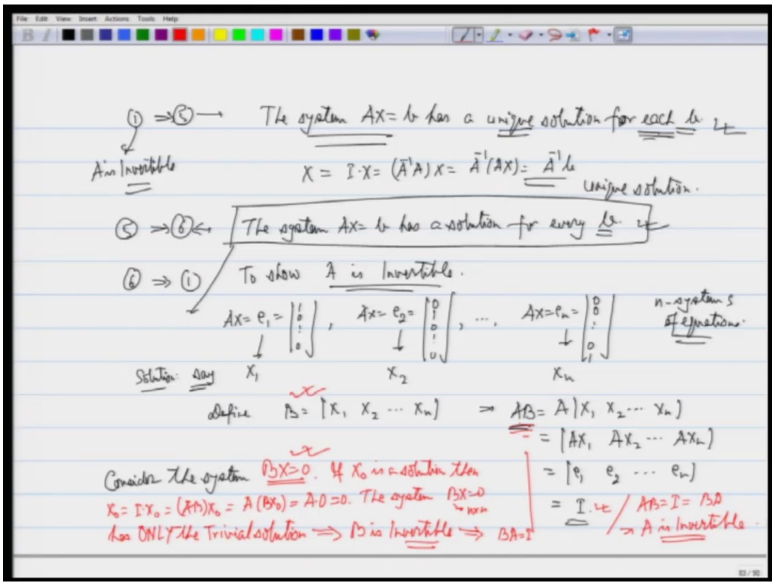

So, 1 implies 5. So, 1 says that A is invertible that is given to me and 5 is that the system A X is equal to b has a unique solution for each b alright. So, let X not be a solution. So, let so here, we already know that X is nothing but A inverse or X is equal to I times x. Now, A is invertible means I can multiply it by A inverse A of X which is same as A inverse of A X which is A inverse of b. So, this is the unique solution fine. X X equal to A inverse of b is the unique solution.

Now, 5 implies 6. 5 says that the system A X equal to b has a unique solution for every b, 6 is the system A X is equal to b has a solution for every b, what does it mean? We are not talking of a unique solution, we are just saying that there is a solution. Five says that it has a solution and not only it has a solution, the solution is unique fine.

So, 5 directly implies 6 because is the same statement other than removing the uniqueness fine. So, 5 is more a stringent as compared to 6 and therefore, 5-6 directly follows from 5. This is just the same a statement after removing uniqueness alright. So, it is even a lighter one fine.

Now, let us try to prove 6 implies 1. So, we have been given that the system A X equal to b has a solution for every b. We have to prove that to show A is invertible fine. So, here we will use the idea that the homogeneous system A X equal to b has a unique solution or things like that fine. So, we have to show that A is invertible. So, let us consider this part that is given to me. So, now, let us consider the system A X is equal to E 1 which is 1 0 0; A X is equal to E 2 which is 0 1 0 and A X is equal to e n which is 0 0 0 1.

So, let us consider this n system of equations; n systems of equations. Is that ok? So, you are not considering one system, we are considering n of them, where the b is e 1. These are e 1, e 2, e n are the different choices of b that I have fine.

Now, what we have been given that the system, this has a solution for every b. It means that there is a solution for e 1 say X 1. So, this is the solution fine. For e 2, b equal to e 2, I again have a solution X 2. For b 3, I have X 3; b 4 X 4 and for b equal to e n, I have X n as my solution. So, what we are looking at is I define b is equal to X 1, X 2, X n.

Then, A times B will be equal to A times X 1, X 2, X n will be equal to A X 1, A X 2, A X n which is nothing but e 1, e 2, e n which is equal to A times or which is same as identity fine. So, we have got AB is identity fine.

Now, from here I would like you to see that A is invertible fine. So, how do I go about it? So, consider the system B X is equal to 0; consider the system B X is equal to 0. I am not looking at the system A X is equal to 0 that is important, I am looking at the system B X is equal to 0 fine.

So, here what you see is that if X naught is a solution, then X naught is same as I times X naught. Now, but we just proved that I is A times B. So, I is A times B of X naught which is A of B X naught which is A times 0 is 0. So, what we are saying is the system B X is equal to 0 and B is a square matrix, system is equal 0 has only the trivial solution. Is that ok?

Since, it has only the trivial solution, so by what we are prove that 1 implies 2, 2 implies 3, 3 implies 4 and 4 implies 1, the first part alright implies that B is invertible alright. So, what we have done is that we started with A we have got a B and we have shown that B is invertible.

Now, since B is invertible, its inverse is unique and since A times B is identity, so the invertibility of B will imply that BA is also identity because AB is identity alright and these now together, so together that AB is equal to identity is equal to BA implies A is invertible fine.

So, what we have done is we could not prove 6 implies 1 directly. We have to go from A to B. So, we formed another system B X is equal to 0 and use the previous part to prove that A is invertible. Is that ok. Those are the things a small-small tricks that you need to understand.

But what it says is which is very important is that you do not have to worry about trying to prove AB equal to identity equal to BA whenever A or B is a square matrix, just one side is enough to conclude that everything is nice as far as the invertibility is concerned. Is that ok?

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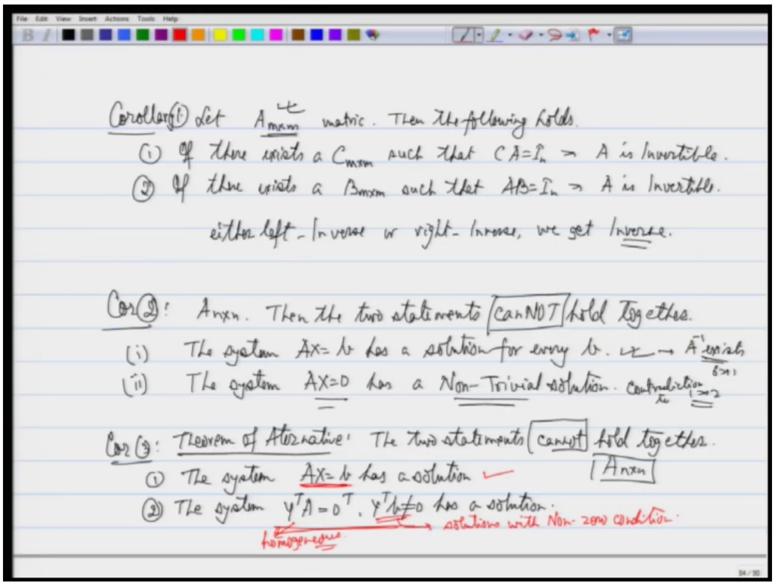

So, now some corollaries related with these ideas which are very important corollary. So, first corollary is what I have just stated. So, let me write it again. So, let A be m cross m matrix m matrix over c or r that we are not bothered, then the following holds; one, if there exists a C n cross n, sorry; yeah, it was m. So, m cross m; m cross m such that CA is identity implies A is invertible. Two, if there exists of B m cross m such that A B is identity implies is invertible alright.

So, the idea here is that whether you can get the left inverse or the right inverse; so, either left inverse or right inverse, we get inverse alright. So, the idea here is we are looking at system of equation, it is a square matrix and therefore, everything goes nicely fine. Another corollary.

So, again I have the square matrix A X, I have a square matrix A n cross n fine. Then, the two statements; the two statements cannot hold together; cannot hold together. What are the statement? The first statement is the system A X is equal to b has a unique solution or has a solution for every b.

And b part, so let me write 1 and 2. I think 1 and 2 like this, the system A X is equal to 0 has a non trivial solution alright. So, understand that when I say the system A is equal to b has the solution for every b; it implied that A inverse exists. This is what we have proved. 6 implies 1, this was 6 implies 1 alright.

Now, here say the system A is equal to 0 has a non-trivial solution, what we had shown is that if A has if A is invertible, then we have only the trivial solution that was 1 implies 2. So, this is a contradiction to contradiction to 1 implies 2. Is that ok? So, that is the way, it helps things to alright.

Another corollary is what is called the theorem of alternatives, alternatives that only one of them can happen alright. Again, the two statements cannot hold together. The two statements cannot hold together; cannot hold together, is that ok? So, what are the statements? One, the system A X is equal to b has a solution. Again, important A is n cross n and everything so square matrix is only. Two, the system Y transpose A is equal to 0 transpose and Y transpose b not equal to 0 as a solution.

So, we are talking about two system of equation; one is looking at the system A X is equal to b. So, I am solving the system from the right, I look at Ax is equal to b and the second is looking at the solution of Y transpose A equal to 0 transpose homogeneous system.

Again, this is a homogeneous fine; but we are also requiring that I want only those solutions for which Y transpose b is nonzero alright. Solutions with nonzero condition alright and this we will also come back to this when will look at what I called fundamental species, then and do some things there.

What we are saying is that if I want a solution of the system A X is equal to b and at the same time a solution of the system Y transpose A is 0 transpose or the homogeneous system with Y transpose being nonzero, that is Y and b are not orthogonal, they are not perpendicular alright. Then, there is a problem, I cannot get both of them together and the proof, the proof is simple.

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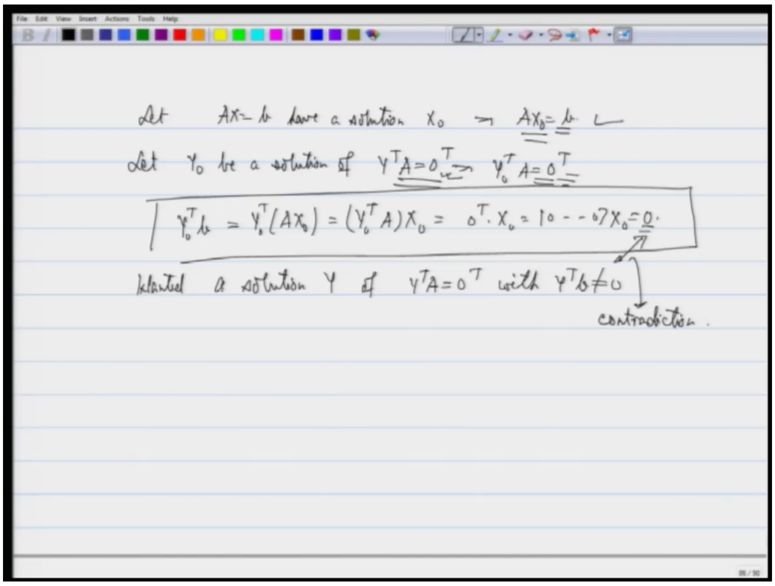

So, let me write the proof. So, what we have? So, let A X is equal to b have a solution X naught. So, this implies that A X naught is b fine. Let Y naught be a solution of Y transpose b, Y transpose A is equal to 0 transpose fine. So, now I have a solution here. So, this implies that Y naught transpose A is 0 transpose. Let us compute Y naught transpose b alright. So, Y naught transpose, what is b? b is A X naught which is same as Y naught transpose A of X naught, but Y naught transpose A is already 0 transpose times X naught.

So, which is 0 0 times X naught which is 0. Is that ok? So, what we see is that as soon as I have a solution of X naught here and a solution Y naught here, their dot product or their product Y naught transpose b is 0 fine. But we needed, wanted a solution a solution Y of Y transpose A equal to 0 transpose with Y transpose b being nonzero alright.

So, that is a contradiction. Is that ok? So, this is the thing why it is called the system of alternatives? Basically because of this that, so this is called system of alternative because what it says is that if you want both the solution, then both the solutions are not possible.

Either you have a solution of A X equal to b or you have a solution which looks like this, which is solution of the homogeneous system, but of Y a transpose not of A x; is that ok? There is difference between Y transpose A and A of time A times X alright. So, be careful; this idea of three alternatives appears again and again at different places fine. So, this is the end of this lecture and we look at the next time in what are called determinants and try to solve the system using determinants and look at equivalent conditions. That is all.

Thank you.