Linear Algebra
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Lecture – 16

So, have you learned RREF and so on how they are used and so on. Let us now go back to system of equation that is the basic idea for us trying to understand everything for system of equation.

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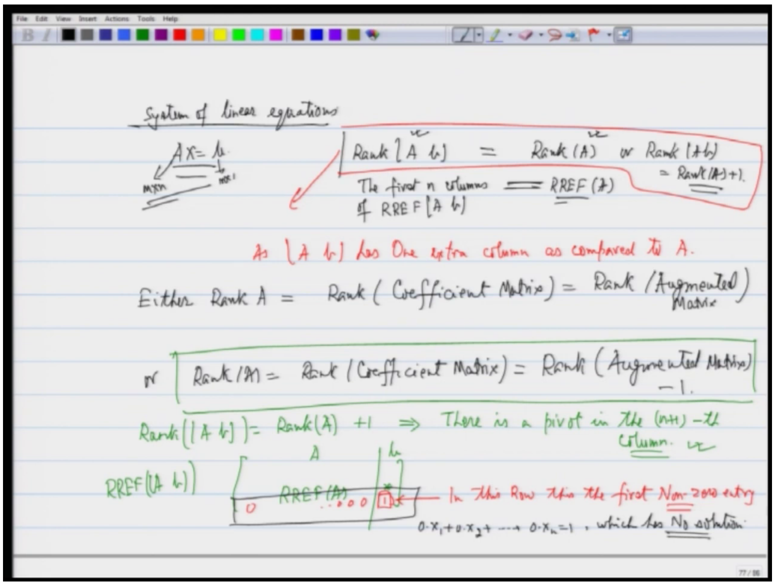

So, let us look at some examples again before I give you the main theorem. So, example about system of equation you are going back to system of linear equations. So, our system of equation is AX is equal to b, fine. We can talk of rank of A b and rank of A so, what can be the relation? What I know is that RREF of A is nothing, but what we learnt was this is equal to the first.

So, A is m cross n, b is m cross 1, fine. The first n columns of RREF of A b this is what we learned. Since A is nothing, but the first n columns of A b, therefore rank of A will have to come from RREF of A which is nothing, but what we know is that RREF of A will be equal to the first n columns of the RREF of A b itself, fine.

So, therefore, rank of A will be either equal to this or rank of this will be equal to rank of A plus 1 fine. There is no other choice other than this; because we have added one column this, fine. So, what we are saying here is that we are going to have this as A b has one extra column as compared to A, fine.

So, rank of A that is the rank of the coefficient matrix can be either equal to the rank of the augmented matrix. So, what we are going to write is; so either rank of A is equal to rank of which is same as rank of coefficient matrix is equal to rank of augmented matrix or rank of A which is same as rank of coefficient matrix will be equal to rank of augmented matrix minus 1, fine.

So, would like to look at this case first, alright. So, let us look at this case first. When the rank of augmented matrix is rank of A plus 1, alright. So, when I say that rank of A augmented matrix is rank of A plus 1, what does it mean? alright. It means that this will imply that there is a pivot in the n plus 1th column is that ok? So, this you have to be careful you have understand this nicely.

What we had learnt was that RREF of A is same as the first n columns in the RREF of A b alright. Therefore, we had proved that theorem that if I pick the first s columns the RREF is just taken from there itself, fine. So, here what we are saying is that I have this matrix A b. So, A is here, b is here the RREF of A will remain in this part itself. There is not going to be any change when I look at the RREF of so, I am looking at RREF of A b, I am looking at this.

So, this part is nothing, but RREF of A and this is something that I am having I am not bothered what exactly it has. But, what it says is that the rank is more so, there has to be a pivot at the n plus 1th column. Now, there is a pivot means I have an entry here which is 1 somewhere I do not know where 1 is, fine because there is a pivot.

So, there is a 1 and since there is a pivot it means that in this row this is the first nonzero entry alright, fine. This is what the definition of pivot was. What was the pivot? Look at any nonzero row, the first nonzero entry in that row was the pivot. So, therefore, if this is the pivot it means that the corresponding entry here in that row is 0, fine.

And, this tells me that this equation corresponds to looking at 0 times X 1 plus 0 times X 2 plus 0 times X n is 1 which has no solution; which has no solution alright, fine. So, this is what you have to understand that I will not have a solution as soon as rank of A b is greater than a rank of A or rank of the augmented matrix is greater than this.

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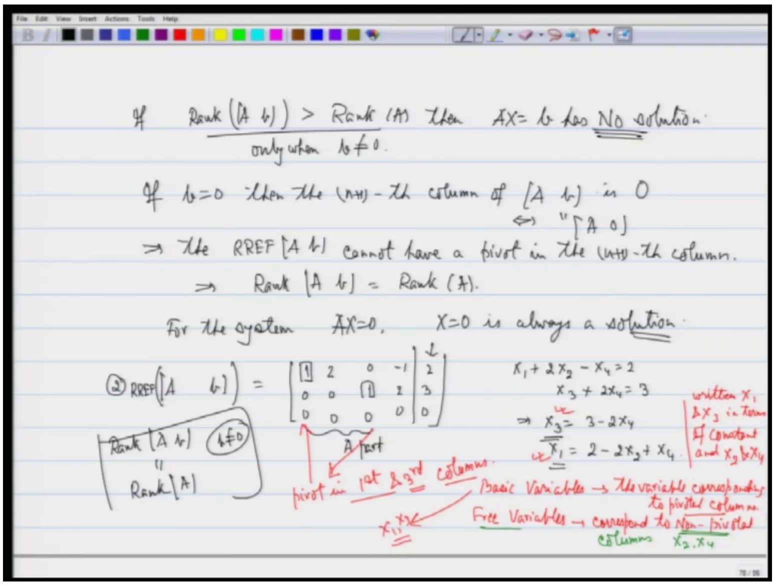

So, let me write it down. So, as a corollary or as a understanding of what we have done, if rank of the augmented matrix is greater than rank of A, then AX is equal to b has no solution, is that ok? Fine. So, this is when b is nonzero, alright. So, this can happen only when only when b is not 0, fine.

If b is equal to 0, then the last column cannot have any non; then the n plus 1th column of A b then the nth column of A b is 0 or which is same thing as saying that A b is same as A 0 implies the RREF of A b cannot have a pivot in the n plus 1th column. And, this will imply that rank of A b will be same as rank of A and we already know that for the system AX is equal to 0, X is equal to 0 is always a solution fine, alright.

Now, let us take another example to proceed further alright. So, that was first example second example alright. So, let me write a small example and then build up on that idea. So, I will take a larger also let me take A b as I have some matrix A b I am looking at RREF of A b, fine.

So, RREF is going to look like so, now, I am assuming the take the case that. So, case rank of A b is same as a rank of A, fine. I am taking this case and b is not 0 and then I will go to b equal to 0, fine. So, let me write this suppose I have got this. So, one is a pivot so, I got 0 here, 0 here there is another pivot at. So, I have got say 0 here, 1 here, 0 here and I have got say 2 here, 0 here, minus 1 2 0 0. So, this is my A part alright A part and the b part is say 2 3 0, fine.

So, you can see that rank of A so, this is satisfied that you can see, but this is the A part here there are two pivots and there is no pivot in the fifth column, alright. So, this equation if I look at this corresponds to looking at X 1 plus 2X 2 minus X 4 is equal to 2 and X 3 plus 2X 4 is equal to 3 from here I can write here that this implies X 3 is equal to 3 minus 2X 4 and X 1 is equal to 2 minus 2X 2 plus X 4, fine. This is very very important, alright have a look at it nicely, fine.

So, what exactly we are doing is I have written X 1 and X 3 in terms of so, we have written. So, let me write it written X 1 and X 3 in terms of constant and X 2 and X 4, look at the pivots. The pivots are here. This is a pivot and this is a pivot. Pivot is there pivots in first and third columns alright what we know is that every column corresponds to a variable. So, pivot 1 corresponds to the first variable X 1; the second pivot in the third column corresponds to the variable X 3.

So, what we are doing is that we are writing the variables corresponding to the pivots in terms of non-pivots. So, this so, there is an notion here what are called basic variables and what are called free variables. So, what are basic variables? The variables corresponding to pivotal columns, alright. So, in this example pivotal columns are 1 and 3. So, the basic variables are X 1 and X 3.

What are free variable? They correspond to correspond to non-pivotal columns alright. So, non-pivotal columns those, what are non-pivotal? second and 4. So, therefore, you have X 2 and X 4 here, fine and for the free variables they where X 1 and X 3, is that ok?

So, what we have done here is that we have written you have to be careful here we have written free variables in terms of a constant which are 3 and 2 which are coming from the side b part and the free variables. That is always important because free variables can be given whatever value you want and still the system of equation will have a solution.

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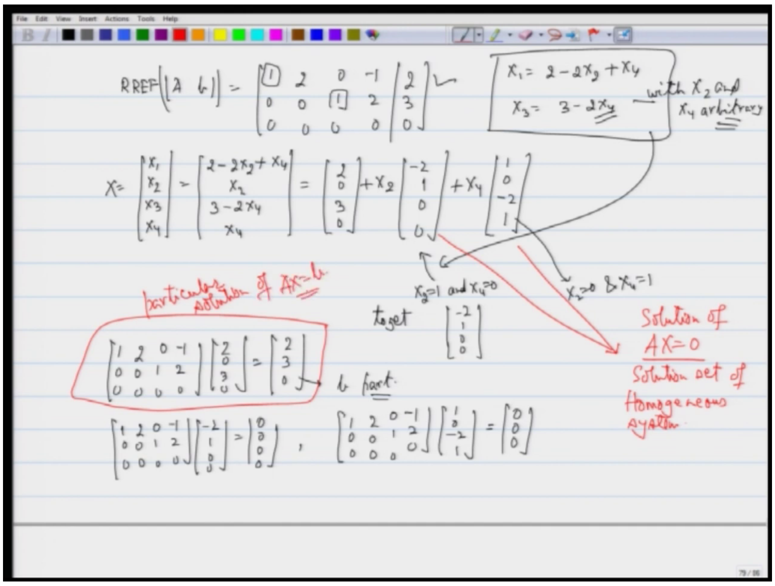

So, let me rewrite this in the next page now. So, what I have is my RREF of A b was 1 0 0 2 0 0 0 1 0 minus 1 2 0 2 3 0, fine. X 1 was 2 minus 2X 2 plus X 4 X 3 was 3 minus 2X 4. So, let us go back and rewrite it nicely now, alright. So, let me at the solution X as X 1, X 2, X 3, X

4. What we see here is that X 1 is 2 minus 2X 2 plus X 4 X 2 there is no change in X 2 because it is just a free variable.

X 3 is 3 minus 2 X 4 and X 4 is also free I can leave it as it is. So, this is same as 2 0 3 0 plus X 2 times minus 2 1 0 0 plus X 4 times 1 0 minus 2 1, alright. So, look at this solutions here look at this X fine this is obtained by. So, this is the solution I have this with X 2 and X 4 arbitrary. These are solution I have, fine. I have written this in terms of this vector.

Now, how do I get this? So, here I have taken from here to here if I want to go I have taken here X 2 to be 1 and X 4 is equal to 0 to get minus 2 1 0 0, this is the thing I got. Look at here you put X 2 to be minus 2, forget about this constant part you had minus 2 here 0 here and so on X 2 is 1, fine. So, here other than the constant we are looking at the remaining part. This corresponds to looking at X 4 put X 4 to be. So, X 2 is equal to 0 and X 4 is equal to 1.

So, X 4 is 1 in the first one. So, you get 2 here X 2 was already free variables that is 0 already for us and then minus 2 coming from X 3 and this one is that ok. So, I would like you to check here that this times 1 0 0 2 0 0 0 1 0 minus 1 2 0. These times 2 0 3 0 gives you this just multiply here 2 times this, 0 times this, 3 times this. So, you get 2 here 3 here 0 which is coming from the b part, fine.

What else? 1 0 0 2 0 0 0 1 0 minus 1 2 0 this times minus 2 1 0 0 gives me 0 0 0 0. Similarly, 1 0 0 2 0 0 0 1 0 minus 1 2 0 this times 1 0 minus 2 1 will also give you 0 0 0, fine. So, these two alright these two vectors they correspond to the solution of AX is equal to 0. AX is equal to 0. So, they are what are called corresponding to solution set of set of homogeneous system alright and this was what is called a particular solution of AX is equal to b, fine.

Let me take a slightly more complicated one bigger size and then I will write the theorem.

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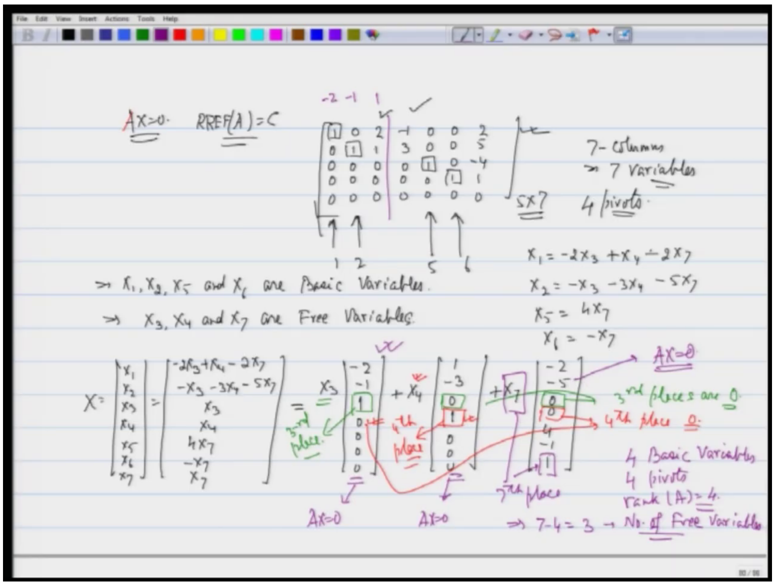

So, now, I am looking at only AX is equal to 0. I am not looking at I am looking at AX is equal to 0, AX is equal to b I am not bothered about because AX is equal to 0 gives me all the ideas alright. So, suppose my corresponding matrix a RREF of A is C and my matrix C looks like 1 1 2 3 2 plus 3 5 1 2 3 4 5 0 1 2 1 minus 1 3.

Suppose this is the matrix that I have, alright. So, it is 5 cross 3 6 1 7 5 cross 7 alright. So, how many variables I have? Since it has 7 columns implies 7 variables or 7 unknowns, fine. What are the pivots? This is a pivot, this is a pivot, this is a pivot, this is a pivot, 4 pivots alright.

And, if you look at the pivots they are in the first column, second column, first, second, fifth and sixth. So, therefore, this implies that X 1, X 2, X 5 and X 6 are basic variables, fine. This also implies that X 3, X 4 and X 7 are free variables, is that ok? So, when I want to write X as X 1, X 2, X 3, X 4, X 5, X 6, X 7 what we have is look at X 1 from here the first part.

So, X 1 is it is coming from here. So, minus 2X 3 then it is plus X 4 and minus 2X 7; X 2 is another basic variable. So, X 2 is minus X 3 minus 3X 4 minus 5X 7; then the next is X 5, X 5 is plus 4X 7 and X 6 is minus X 7 alright. So, let us write it back minus 2X 3 plus X 4 minus 2X 7; X 2 is minus X 3 minus 3X 4 minus 5X 7; X 3 there is no change, X 4 there is no change because there are free variables.

X 5 is 4X 7 and X 6 is minus X 7 and X 7 is again a free variable. So, I keep it as it is. So, this is same as X 3 times minus 2 minus 1 1 0 0 0 0 plus X 4 times 1 minus 3 0 1 0 0 0 plus X 7 times minus 2 minus 5 0 0 4 minus 1 1, fine. So, now, some observations.

X 3 was a free variable, alright. Since X 3 is a free variable this one appears at the third place is that. Look at the other places here. Third places are 0, alright third places are 0 fine. Let us look at the next, here. You have X 4 one at the fourth place fine. So, since it is the fourth variable X 4 is the fourth variable it corresponds to the fourth column, the fourth entry here is 1, here it was 0, here also it is 0, fourth place 0. Is that ok? You can see here this also is 0 here, fine the last one this is X 7.

So, look at the seventh place there is a 1 here, alright and seventh place here everything is 0. So, when I am looking at the solution here fine, there were four basic variables or basic variables or four pivots or rank of A is 4, fine. So, therefore, 7 minus 4 which is 3 was number of free variables. Each of these free variables have given me a solution set in the sense that I have got this vector which is the solution of the above system. You can see that minus 2 minus 1 1 times that will be 0.

So, what I am saying is just look at minus 2 times the first column minus 1 times the second column plus 1 times the third column just add these three minus 2 plus 2 is 0, minus 2 plus 2 is 0, minus 1 plus 1 is 0. So, you get a solution of AX is equal to 0 similarly verify that this is also solution of AX is equal to 0 and this is also a solution of AX is equal to 0 is that ok.

So, what we are seeing here is that they have some nice properties and not only that they again they correspond to in some sense 1 0 0 0 1 0 0 0 1 alright. But, now it is with respect to the rows that we are looking at we are forming the rows 1 0 0 as the first row coming from X 3, coming from X 4 is 0 1 0 and from X n you are getting 0 0 1, fine.

So, now let me conclude this by a stating given the statement of the theorem.

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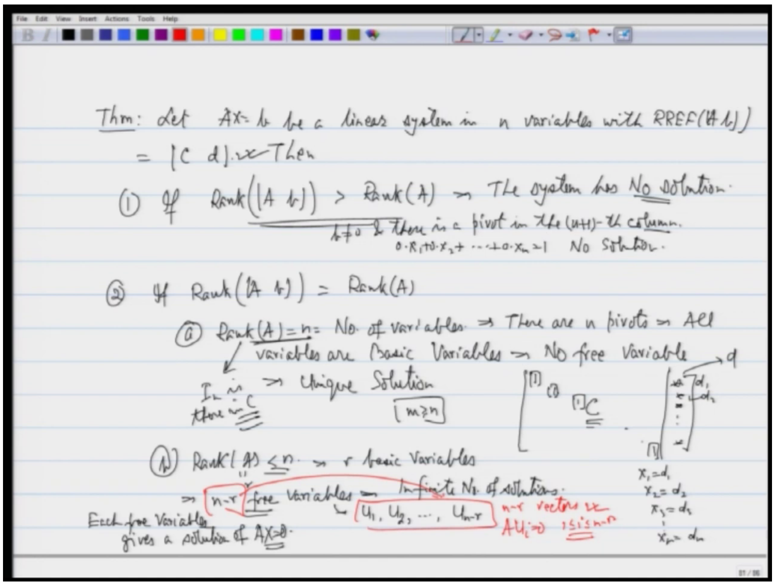

So, what is the theorem? Theorem let AX is equal to b be a linear system in n variables, fine with RREF of A b is equal to C d C d then 1, if rank of A b is greater than rank of A. So, rank of the augmented matrix is greater than the rank of coefficient matrix implies the system has no solution.

So, note that when I am saying that this is greater it means that b is nonzero directly because you are saying that there is a pivot at the bth place. So, we already seeing that b is not equal to 0 and there is a pivot in the n plus 1th column alright and therefore, the other system 0 times X 1 plus 0 times X 2 plus 0 times X n is equal to 1 no solution, fine.

So, now, that case is when they are equal. So, if rank of A b is equal to rank of A. Now, here we have two cases – case a rank of A is n which is number of variables, alright. So, since the rank of A is n it means that there are n pivots implies all variables are basic variables implies no free variable free variable implies unique solution.

So, look at it from the this point of view b is there. So, b will have some entries fine now you are saying that rank is n. Rank is n means there are n pivots and not only that rank is n also means that I have got I sub n. So, I sub n is there in C alright because C d was the matrix RREF. So, this is my C part and we are saying that I n is there.

So, I n is there means I have got 1 here, 1 here, 1 here. So, this one every column has a pivot, alright. If every column has a pivot it means that I already have X 1 is equal to this, X 2 is equal to this. So, if this was d 1, this was d 2 we are already saying that X 1 is d 1, X 2 is d 2, X 3 is d 3 and so on till X n is d n whatever it is fine because this matrix was d for us, this was the C for us, fine. So, there is a unique solution and this also means that m is greater than equal to n alright.

So, that we are not saying, but since we are saying rank of A is n that implies that part alright the third part is not the third part, but the b part of this is rank of a is a strictly less than n suppose this is r. So, this will imply that there are r basic variables and implies n minus r free variables this will have infinite number of solutions and each free variable is going to give me a vector. So, each free variable free variable gives a solution of AX is equal to 0.

So, suppose the free variables they give you vectors so, u 1, u 2 u of n minus r alright. So, there are how many free variables? n minus r free variables and therefore, you have got n minus r vectors, is that ok? n minus r vectors.

They have some nice form in the sense that they have got if I look at u 1 will be have will have 1 0 0 somewhere u 2 will have 0 1 0 0 0 and so on. So, there will be n minus r vectors and A of u i will be 0 4 1 less than equal to i less than equal to n minus r, is that ok?

We will look at the implications of these ideas in the next class, alright. So, I want you to understand that whenever the system of X equal to b whatever m size the whole thing the whole argument is the independent of choice of m. It is only dependent on rank of augmented matrix rank of A and the number of unknowns. It has nothing to do with m. That is all for now.

Thank you.