Linear Algebra
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Lecture – 15

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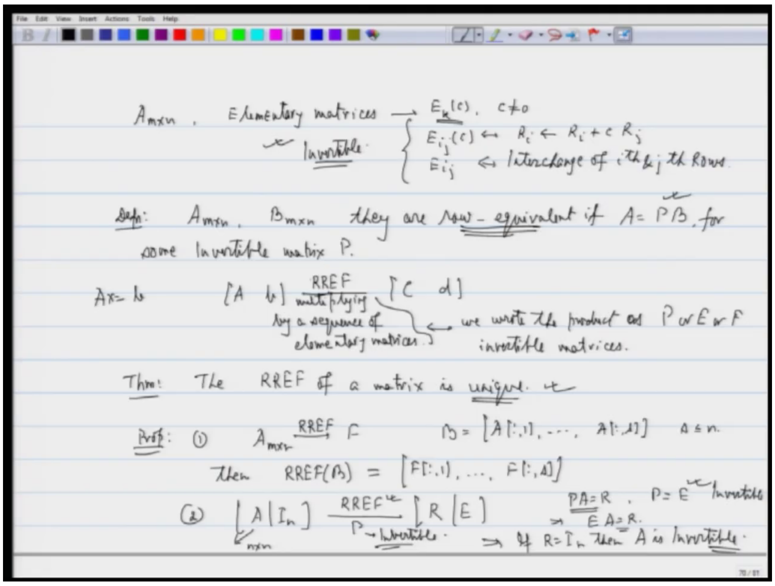

So, let us recall what exactly we have done till the previous class. The idea was that given a matrix A which was m cross n. There was a notion of elementary matrices and what were they? They were of three types E k of c, c is not 0 so, I am multiplying the k th row by c of identity.

Then, I had E ij of c that what you are doing here is you are replacing the i th row by i th row plus c times the j th row alright and then, there was E ij itself which was the interchange of i th and j th rows. So, these are the three elementary matrices that we had, and they were invertible. So, they are invertible as well.

We also had this definition where we said that definition was that there are two matrices A is m cross n, B is m cross n. So, they are row equivalent; they are row equivalent if A is equal to P times B for some invertible matrix P alright fine.

So, the idea was that row equivalence was an important role for us and we got it through the invertible matrix P. What turned out was that elementary matrices are invertible. So, when we try to solve the system, we could write the augmented matrix A b so, we are solving system Ax is equal to b, then we know the corresponding augmented matrix we went to RREF and got the matrix C d and going to this was basically multiplying by a sequence of elementary matrices alright.

And if I take a finite sequence of elementary matrices, each of them is invertible so, the product is invertible. So, basically, they gave me an invertible matrix and we wrote it at P itself so, we wrote it as P. We wrote the product as P or E or F whatever it is invertible matrices. This is what we did fine.

The main theorem that I did not prove was theorem was the RREF of a matrix is unique. So, I did not prove it because it requires going out of the system and then coming back fine. So, therefore, we just assumed it.

Based on that, we could do lot of calculations, we could prove the first thing that so, one of the thing that we could prove was or propositions or results whatever it is was that if I take A is m cross n its RREF is F, take B which has columns 1st column till s columns alright and s is less than equal to n, then RREF of B was nothing, but the 1st s columns of F itself you could do this alright fine.

And from there, you could also do this that if so, we looked at the system A and identity. Here, we took A to be n cross n, looked at its RREF I got it as R and E fine. So, if I relate the two so, RREF basically means multiplying by a matrix here something so, multiply by P which is invertible, then what we see here is that two matrix are equal if they have the same component. So, that implies that P times A was R and P was (Refer Time: 05:43) E itself and therefore, here I could say that E times A is R fine.

Now, E is product of elementary matrices and therefore, it is invertible and this implied that if R is identity, then A is invertible else we could not say what will happen to the next part alright. So, if R was not identity, it means that A is not invertible.

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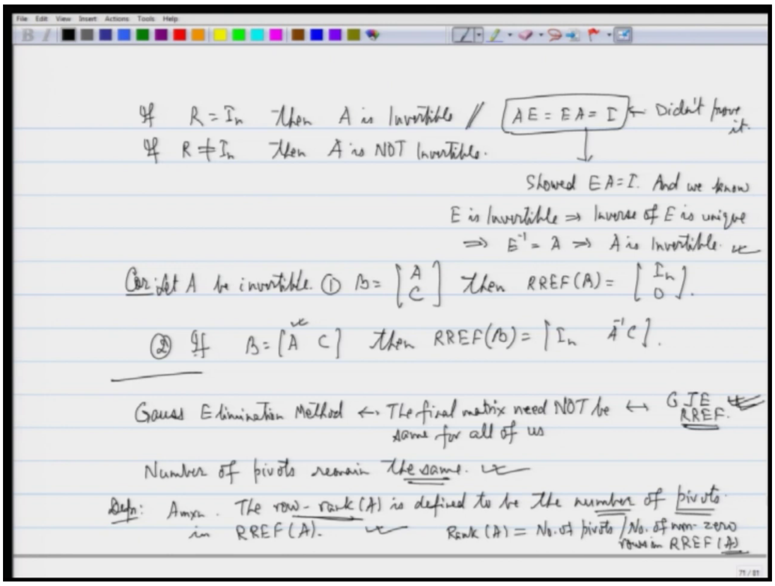

So, this what we said that if R is I n, then A is invertible. If R is not I n, then A is not invertible alright. What you have to keep in mind here is that to show that A is invertible, we did not proved that AE is equal to EA is equal to identity did not do this, did not prove it.

What we did instead was we just showed E times A was identity and we knew and we know E is invertible and therefore, it is so, this will imply that its inverse is unique, inverse of E is unique and therefore, this gave me that E inverse is A and this implied that A is invertible alright. So, in place of showing that A is invertible directly that is to show that this happens we went round about and then, got the result is that ok.

Now as a corollary of this ideas, I will just estate some theorems here so, corollaries. First corollary is that if A is invertible, let A be invertible. Suppose I have matrix A B which looks like A and C here alright.

Then what can I say about B then RREF of B is I n and 0 fine. 2, if B is A time C, then RREF of B is I n and whatever it is so, since I am multiplying by A inverse to get identity here, it will be A inverse of C fine. So, you have to keep track of this so, that those ideas can be used at later stage fine. I will not prove it; we will just take it as it is fine.

Now, what we learnt was that when we apply the Gauss elimination method; elimination method, we got different matrices. The obtain matrix, the final matrix not be same for all of us alright. So, this was a problem for us. So, from there we went to the Gauss Jordan method or the RREF which gave us uniqueness fine.

But we call that the number of pivots; number of pivots remain the same fine. So, now, I can do one thing that this is this matrix RREF is unique for all of us, we can use the number of pivots there to define what is called the rank of a matrix. So, we define definition A is m cross n fine. The row rank of A is defined to be; is defined to be the number of pivots so, it is number of pivots in RREF of A fine.

So, to define the row rank of a matrix, I was forced to go to RREF because, we needed somehow to say that number of pivots are going to be the same that I could not guarantee when I am looking at the Gauss elimination method. But using the Gauss Jordan method where I know that the matrix is unique for all of us, it is the same for all of us we could say that the number of pivot there will be the same and therefore, the number of pivots is same so, I can talk of that as a object that is a number and therefore, I could define row rank of a matrix fine.

But if you want to find the row rank of a matrix, you just start to do the Gauss elimination method because as we said the number of pivots do not change when I go from Gauss elimination to Gauss Jordan fine.

Now, at this stage I am using the word row rank basically because my calculations are based on only rows. I am multiplying the matrix A on the left by elementary matrices fine. Question arises that if I had started with applying some column transformation, applying elementary matrices on the right, will this number remain the same fine.

So, I do not know at this stage whether they are same or not, but what I can say is that I am also going to get certain number of pivots and I can define what is called the column rank of A because I am multiplying matrices on the right fine.

It turns out we will prove it when we go to vector spaces and so on prove that row rank is same as the column rank fine. So, for the time being, I will not use the word row rank even though I have defining row rank, I will just use the word rank of A matrix rank of A is nothing, but number of pivots or number of non-zero rows in RREF of A.

So, you have to compute RREF or you have to call get the REF the reduced equivalent form alright the row equivalent form to get your rank fine.

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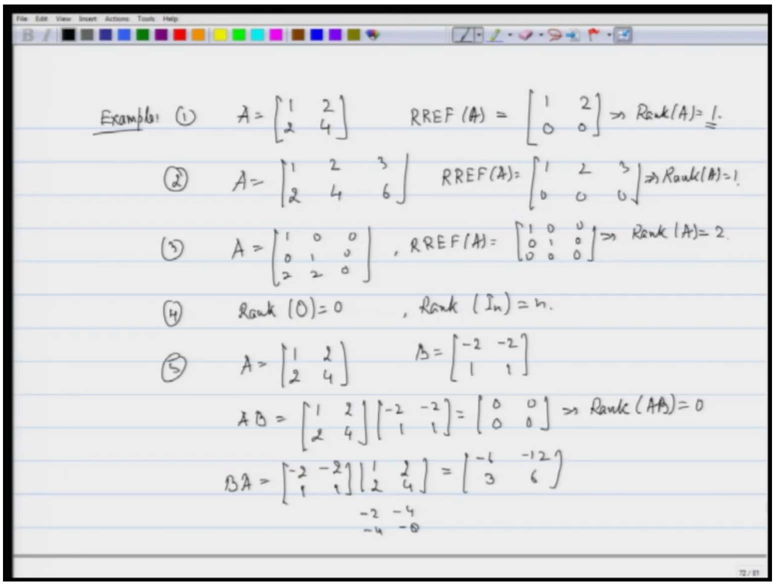

So, let us take up some example to understand it. So, example 1. So, I have this matrix A which is 1, 2, 2, 4 fine. So, if I at the RREF of A, it will be I can just use this to make this one 0 so, it is 1, 2, 0, 0 and this implies rank of A is 1 fine. 2 take A equal to 1, 2, 3, 2, 4, 6 again the RREF of A is 1, 2, 3, 0, 0, 0 and therefore, rank of A is 1 itself.

If I take A as 1, 0, 0, 0, 1, 0, 2, 2, 0 fine, then the RREF of A is 1, 0, 0, 0, 1, 0, 0, 0, 0 implies rank of A is 2. In general, you can see that rank of a 0 matrix is 0, rank of identity matrix of size n is n fine.

Another thing that I would like you to understand is that suppose I take so, I had my A as 1, 2, 2, 4 fine I take B as alright so, let us take B as say minus 2, 1 fine and say minus 2, 1 suppose I take this. Let us multiply what is A times B so, A times B is 1, 2, 2, 4 times minus 2, minus 2, 1 and 1.

So, this turns out to be this into this multiply this by minus 2 one times minus 2, minus 2 and two times this will give you 0, 0 similarly this into this will give you two times this minus 4, minus 4 so, 0, 0 implies rank of AB is 0.

What about BA? If I look at BA, minus 2, minus 2, 1, 1 times 1, 2, 2, 4 this is equal to multiplied this by minus 2, multiply this by minus 2 so, you are multiplying this by minus 2 so, you are looking at minus 2, minus 4, minus 4 and then it is minus 8 so, minus 6, minus 12 and this turns out to be 1 plus 2 is 3 and 6 fine.

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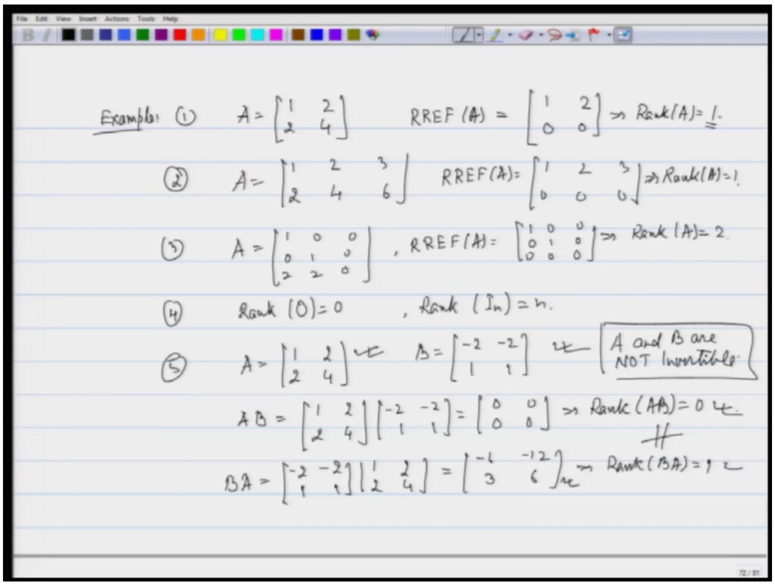

So, if you look at this, again I can multiply this by 2 and add I will get 0. So, this will imply that rank of this matrix BA is 1. So, here I see that rank of AB need not be equal to the rank of BA fine. So, these are important ideas that you need to understand that I can have issues with AB and BA commutativity does not hold not only that rank also may not be the same fine.

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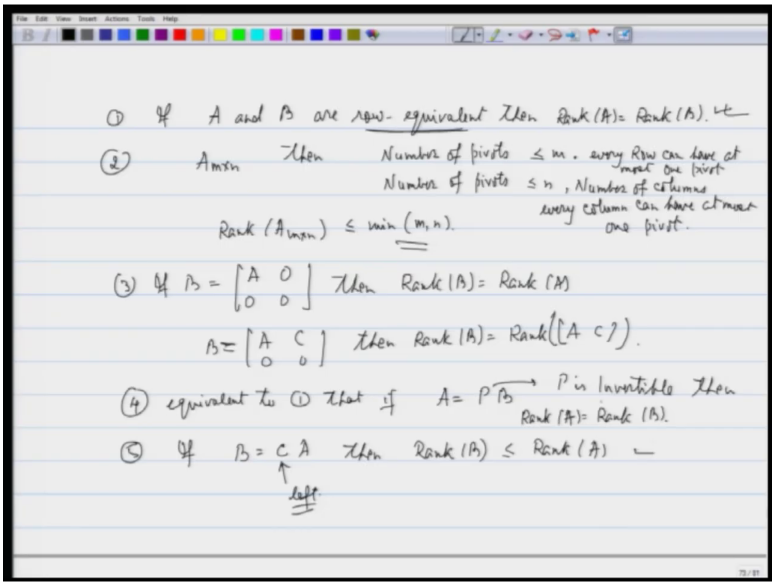

So, let us now build up some theory again. I will just write the statements which you can see easily fine. So, 1: if A and B are row equivalent, then rank of A is same as rank of B. 2: if A is m cross n, then number of pivots has to be less than equal to m because in each row I am supposed to have at most one pivot also number of pivots will be less than equal to n number of columns as every column can have at most one pivot.

So, here it was every row can have at most one pivot. Here it is number of columns again here every column can have at most one pivot alright. So, therefore, what we see is that rank of A which is m cross n is less than equal to minimum of m and n fine.

3: B is equal to if B is equal to A, 0, 0, 0, then rank of B is same as rank of A alright or in general, if I want to write B is equal to say A, C, 0, 0, then rank of B is equal to rank of A, C. So, whatever you want to say you can build up on those ideas. 4: so, this part the first part says that they are row equivalent. Row equivalent means I am multiplying by invertible matrices.

So, when I am multiplying by invertible matrices, the rank does not change. So, what we are saying is indirectly which is same as 1 equivalent to 1 1 that if A is equal to P times B and P is invertible, then rank of A is same as rank of B fine.

We already saw in the previous example, let us look at go back 72 here that look at A and B, the rank of these two matrices is 1 they are not invertible. So, A and B are not invertible in this example alright. When they are not invertible, the rank can either remain the same or they can also reduce fine.

So, both the things are possible. So, if B is equal to C times A, then rank of B will be equal to rank of less than equal to rank of A I am multiplying C on the left is that is important. So, depending on what it is things can change. I already given you some examples alright.

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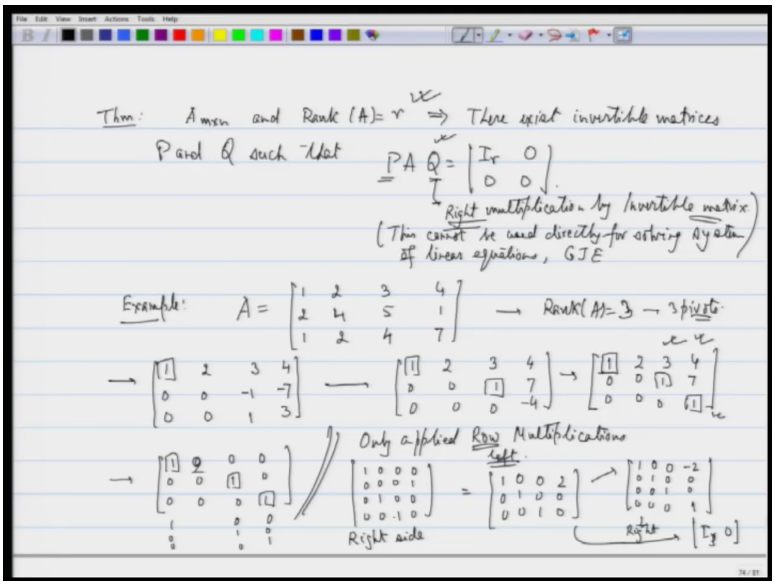

One more thing that I would like you to say is that so, this part is there, this is there now one very important theorem now theorem and I would like to prove it, but the 2nd part becomes a bit difficult in the sense that you have not seen it, but I would like you to understand it, it may take time.

So, suppose that I have matrix A which is m cross n everything is real number or complex numbers and rank of A is equal to r fine. This implies there exist invertible matrices P and Q such that PA Q is I r, 0, 0, 0 alright. So, the important thing is till now we have been doing only left multiplication alright. Here we are multiplying left as well as right. So, there is right multiplication also right multiplication by invertible matrix alright.

So, this idea cannot be used since I am multiplying on the right, this cannot be used for system of equations. So, this cannot be used directly for solving system of linear equations or whatever algorithm that you are doing that Gauss Jordan algorithm and so on, we cannot use these ideas fine.

But this rank idea can be used and done something else when I am not solving a system of equation or when I am just looking at the matrix A, the row space of that, column is space all those thing I have not defined, but looking at A itself and trying to understand the matrix A fine, but we are doing everything with respect to rank.

So, let me first take an example and then I will give a proof of this example. Suppose my matrix A looks like 1, 2, 3, 4, 2, 4, 5, 1, 1, 2, 4, 7 suppose I have this matrix with me fine. I want to compute the RREF of this matrix. So, if I want to apply the so, let us do some things here. So, I just leave it as 1, 2, 3, 4 multiply this by 2 and subtract you get 0 here, 0 here, 5 minus 6 is minus 1, 1 minus 4 is alright. So, 1 minus 8 is minus 7 this is 0, 0, 4 minus 3 is 1 and 7 minus 4 is 3 I have this.

At the next stage, I can take it as 1, 2, 3, 4 there is this pivot that I have these were the pivot 0, 0, 1, 7 this is my pivot, 0, 0 I can just 0 here and just adding it so, minus 4 at the next stage, I have 1, 0, 0 1 is a pivot, 2, 0, 0, 3, 1, 0, 4, 7, 1 alright. So, I have applied the Gauss elimination method and also multiplied by 1 upon minus 4 to get it 1 fine.

Now to proceed further, I have to use this 1 to make these two entries 0, use this 1 to make this entry 0 alright. So, we can do that at one go, I have not write the matrix here, so, what we have here is 0, 0, 0, 1, 0, 0, 1, 0, 1, 2, 0, 0 I get this fine. So, this was all applying to till this stage only applied row multiplication. Our row multiplication means left multiplication on the left fine.

Now, what I would like to recall that I had said that if the number of pivots is r, the RREF will have I sub r sitting inside it. Since, the rank is r so, the number of pivots is r and therefore, we will have I s sitting there alright. Here, in this case, it turns out that the rank of A is 3 this is what you see there are 3 pivots 3; 3 pivots and therefore, I have got I 1, 0, 0, 0, 1, 0 and 0, 0, 1 sitting here fine.

So, therefore, what we can do is that we can multiply something on the right here. So, for example, in this example, let us interchange so, I keep the first column as it is I do not do anything. I want to make the third column as my second column alright. So, I want my third column to be my second column. So, it is 0 times this plus 0 times this plus 1 times the third column will give me this part fine.

I want to make this as my third 0, 0, 0, 1 and I keep this is. So, if I multiply this matrix here on the right side fine, then what I will get here is just have a look at it 1 time this will give me 1, 0, 0, 0, 0, 1 will give me 0, 0, 1 which is 0, 1, 0, the third one will give me 0, 0, 0, 1 it will give me 0, 0, 1 and this one will shift to here 0 fine.

Now again on this, I can look at this column minus this alright. So, fourth column so, they I am not doing anything to the first column, I am not doing anything to the second column, not doing anything to the third column and fourth column is being replaced by fourth column and minus two times the first column; minus two times the first column alright.

So, look at this, you have multiplied this on the right again, this matrix will turn out to be I r 0 fine. So, it is I 3 and 0 here fine. So, what I am trying to say is that this is the way I am going to get the matrix Q. I am going to get somethings here fine.

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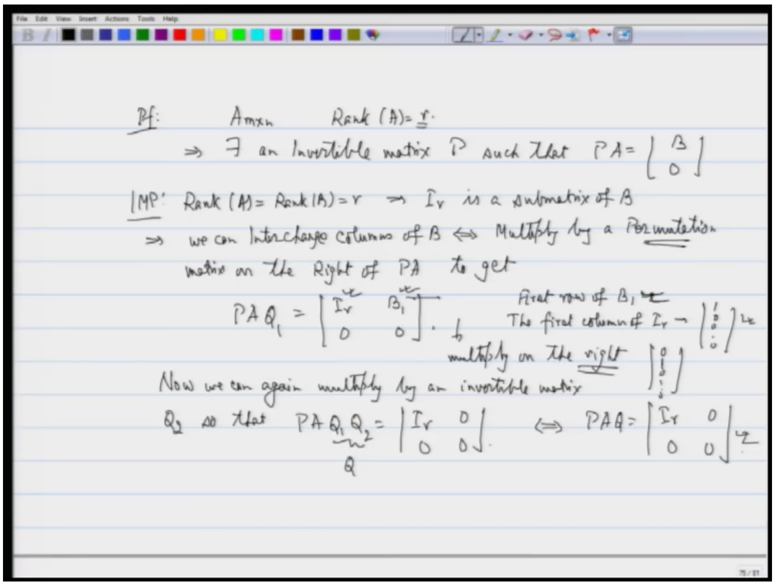

So, let me go back and give the proof of that. So, proof of proof I have A was m cross n rank of A is r. So, this implies there exist an invertible matrix P such that P times A is going to give me some B here 0 here fine. Important rank of A is same as rank of B which is equal to r implies, I sub r is a sub matrix of B.

So, this implies we can interchange columns of B alright or which is same thing as saying equivalently multiply by a permutation matrix; permutation matrix on the right of PA to get PA Q 1 will be equal to now I r will be there, there will be some B 1 here, 0 here, 0 here fine.

Now, alright once I have got this, then I can make all these entries 0 using this I r. So, I can so, look at the first row of B 1, I can make this 0 using the first; using the first column of I r which was 1, 0, 0, 0. I can use this to make every entry here to be 0 fine as I did in the previous example alright.

So, again, but I have to since I am doing column transformation, I am using the first column to make this entry 0, I am going to multiply on the right so, multiply on the right; right is that ok. So, therefore, what we are saying is that similarly to make the second row of B 1 to be 0, I have to use the second row of I r which will be 0, 1, 0, 0, 0 fine. So, the this way I can make all the entries of B 1 to be 0 or I can just multiply by some matrix and then get them to be 0.

So, what we are saying is that now we can again multiply by an invertible matrix Q 2 so, that PA Q 1 and Q 2 is I r 0, 0, 0 fine and this Q 1 was the matrix Q that we looked at because Q 1 is a permutation matrix. Permutation matrices are invertible not only that they are inverse that transpose itself and Q 2 is invertible so, Q 1, Q 2 is invertible. So, therefore, we have got here PA Q is equal to I r 0, 0, 0 alright.

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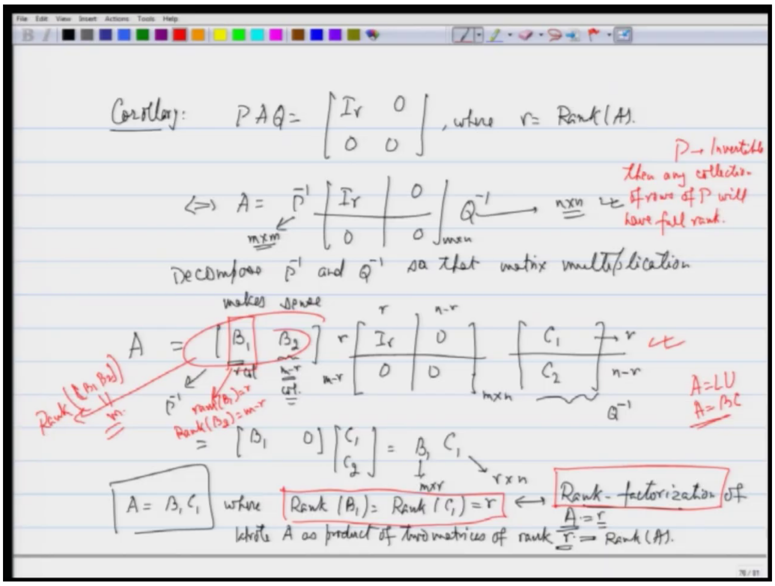

I will take five more minutes to just explain you what is the implication of this. So, now, I can write so, since once I have got this so, as a corollary of this; corollary of this corollary. So, I have, I wrote PA Q as I r 0, 0, 0 where r was nothing, but rank of A fine. So, what would like to do is so, from here what I can get here is that at A is nothing, but P inverse I r 0, 0, 0 Q inverse.

Now, I want to decompose. So, decompose P inverse and Q inverse so that, matrix multiplication make sense alright. So, let us see here I have I r here, 0 here, 0 here, 0 here. This matrix was m cross n fine. So, there are r rows here so, there are m minus r rows here, there are r columns here, there are n minus r columns here fine.

What I know is that this Q is n cross n matrix so that, the matrix multiplication make sense. P is m cross n. So, let us decompose it accordingly so, I can write Q inverse if you look at Q is n cross n there are n columns here so, I would like to break it accordingly so, I want to write it as say C 1 here, C 2 here what is the size of C 1? I want to multiply this into this. So, there has to be r rows here and there has to be n minus r rows here fine.

Similarly, here I can break P inverse as say alright so, I think B 1 has already been used already (Refer Time: 34:43) collates B 1, B 2. Now here, this is r, this is m minus r so, this has to be r, this has to be m minus r columns alright so, r columns here and m minus r columns here fine.

Once I have done that, so, this is writing as this I wrote it as P inverse is this and this is Q inverse for me so, A remains as it is so, this is same and let us multiply it you get here this into this is B 1 fine, this into this is 0 times C 1, C 2 which is equal to B 1 C 1 fine. So, now, let us look at this. So, B 1 is of size m cross r, C 1 is of size r cross n.

So, therefore, we have got A as B 1 C 1 where rank of B 1 is equal to rank of C 1 is equal to r and this is called the rank factorization of A alright. Because the rank of A was r and we have written A as product of two matrices each of rank r. So, wrote A as product of two matrices of rank r which was the rank of A itself is that ok? So, this is called the rank factorization as I said this is called the rank factorization and this was one of the things that we wanted to do as our aim.

Our aim was to look at L U decomposition that we have done earlier. This is the rank factorization which was the second step and after this what are the factorization we are going to look at is going to be quite complicated and therefore, we will have to understand what are called vector spaces, inner product space, linear transformation and so on fine.

So, we have done the first two things that is writing A as LU and A as BC where was the A equal to BC was the rank factorization here I would like you to see that why is rank of B equal to rank of C is equal to r.

I have not given you an argument, but the simple argument is that P inverse is invertible. So, if a any matrix A let me not write A here say any P which is invertible fine, then any collection of rows of P will have full rank alright. Similarly, any collection of columns will give me full rank alright that is important.

So, here look at this the rank of B 1 will be r and rank of B 2 will be m minus r because the rank of this whole matrix rank of B 1, B 2 is m alright. So, it means that every row has a pivot, every column has a pivot is that because the rank is m so, every row every column has a pivot.

Similarly, Q inverse has a full inverse so, it is invertible. Therefore, all the rows have pivots so, there are r pivots therefore, you get it is that ok? So, that is for now. We will look at the some ideas of this in the next class.

Thank you.