Linear Algebra
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Lecture – 14
RREF and Inverse

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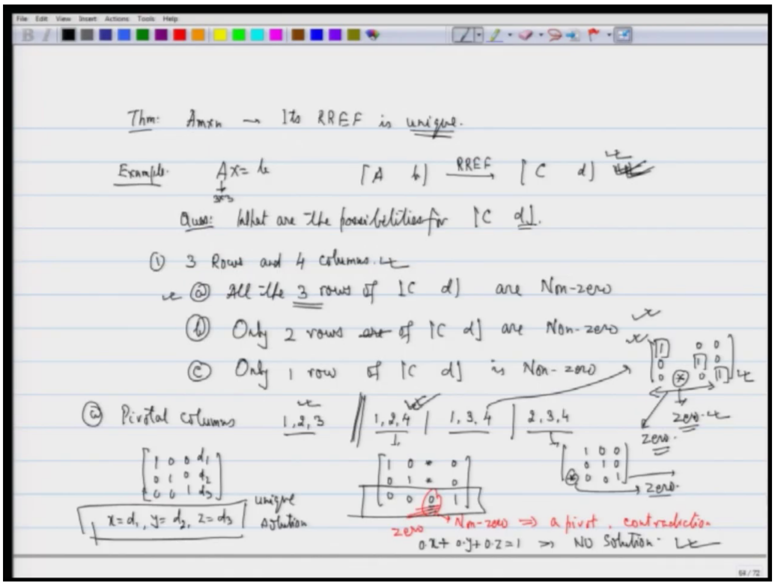

So, let us recall that where this theorem which says that given a matrix m, it is RREF is unique; EF is unique alright, fine. So, all of us are going to get the same matrix finally and therefore, this is plays a very important role in understanding the things. So, let us try to understand this using some example first example. So, suppose I have a system Ax is equal to b, where A is 3 cross 3 and x is 3 cross 1, b is also 3 cross 1.

So, I have the augmented matrix as A b fine and suppose my RREF is C d fine. Question is what are the possibilities for C d? So, question is question, what are the possibilities for C d alright, fine. So, let us go by one by one, try to understand it. Now, this matrix has 3 rows alright and 4 columns. Now, since it has got 3 rows, how many pivots can it have? Fine.

So, let us try to understand that. What the definition of pivot was that, every non-zero row has a pivot alright. It is not about A, the pivot is with respect to RREF. So, when you are saying that, I am looking at RREF of A b, so I am looking at C d. So, I would like to see to see that in this matrix C d, the RREF how many non-zero rows are there.

It may happen that all the 3 rows are non-zero or it may happen that so, all the 3 rows are nonzero; 3 rows of C d are non-zero that is the first choice that I have; second choice only 2 rows of C d are non-zero. And, the C is only 1 row of C d is non-zero.

So, I am not taking the case here, when the whole matrix A as well as b is 0, that I am not taking. I am assuming also that for every variable or the unknown x, y, z I have at least one non-zero entry with me, fine. So, that assumption is there with me that, all the variables x, y and z or all the unknowns x, y, z have at least one non-zero coefficients with it. Is that ok? So, that I can talk of things here fine.

So, I have got three cases. This matrix has A has 3 rows augmented matrix joins with me. So, there are 3 rows and 4 columns as per as C d is concerned; C d is an RREF. So, it can have all the 3 rows as non-zero, only 2 rows as non-zero and only 1 row as non-zero.

So, in the first case what happens is I am assuming that all the 3 rows are non-zero. So, what are the possible pivots? So, the possible pivots are the pivots, pivotal columns will be in A either the first column is a pivot, second column is a pivot, third column is a pivot or first is a pivot, second is a pivot, fourth is a pivot fine or it can happen that the first is a pivot, third is a pivot, fourth is a pivot or second is a pivot, third is a pivot, fourth is a pivot, fine.

So, there are 4 columns, out of that any 3 can be a pivotal column. So, pivotal columns could be 1, 2, 3; 1, 2, 4; 1, 3, 4 and 2, 3, 4.

So, let us write the corresponding matrices for us. So, if this is the pivot 1, 2, 3; then, I need to get identity with me. This is what we had understand; understood that in the previous class that whenever we have a pivot, each pivot gives me some form of identity. So, since there are 3 here.

So, I will get identity here in the first 3 columns and therefore, d will be something like d 1, d 2, d 3 something here and this will give me that, the solution as x as d 1, y as d 2 and z as d 3. So, I get a solution directly here and this is a unique solution. Is that ok?

So, I have a pivot, there are 3 pivots and these 3 pivots, they are at position 1, 2 and 3. So, they correspond to the variable x, y, z and therefore, I get a unique solution. What happens at 1, 2, 4; 1, 3, 4 and 2, 3, 4? Let us try to understand that part fine. So, here it says that 1, 2, 4; so, 1 0 0, 2 is 0 1 0 and the fourth one it says that 0 0 1 fine. So, C again here that pivotal or the RREF is so important, that the once they are going to the right fine, the ladder like idea is coming into play fine.

So, what will happen to the third one here? Third one here could be anything here, but it has to be a 0 here because if it was a nonzero entry, if this entry was nonzero, so if this was nonzero, this would have become a pivot; nonzero will imply a pivot alright. So, that will give you a contradiction.

So, therefore, this entry has to be a 0, there is no other choice for us alright fine. And, this entry whatever it is that we are not bothered about; but this is how it is going to look like.

So, here I would like you to see that this equation, it corresponds to saying that x times, 0 times x plus 0 times y plus 0 times z is 1 and this implies no solution. I cannot find any value of x, y and z such that 0 times x plus 0 times y plus 0 times z is 1 fine. So, in this case, I do not get a solution. What happens to 1, 3; 4?

Now, let us look at 1, 3; 4. So, I have got 1 here; 3 is the second one and fourth is the last one. Again, I would like to understand what is this entry? Can I say that this entry has to be a

0; because if this entry was nonzero, again this will become a pivot and that will again contradict that C d is in RREF alright.

So, this is important that we are doing everything with respect to C d. So, this has to be 0. Because if it was non-zero, this becomes a pivot. This has to go up and things like that fine. This will not be a ladder like. So, this is a not a pivot. So, it has to be a 0 here, because this is the pivot that I have got here; these are the pivots.

So, this is the pivot, everything on the left of that is supposed to be 0, fine. This is what the idea was. In every non-zero row, the first nonzero entry is a pivot alright. So, if you see again, I get 0 times x plus 0 times y plus 0 times z as 1 and therefore, no solution here is that, ok. So, again the same argument.

Now, what happens to this one that case 2, 3, 4? Here, 2 is the first one. So, it is 1 here, 0 here, 0 here; 3 is the second one and this is the third one that I have got. Here again, if I look at, this has to be 0, because if this was not 0, then again this should I become a pivot and would have contradicted that C d is in RREF alright. So, therefore, what we see here is that this again this case again leads me to the idea that 0 times x plus 0 times y plus 0 times z is 1, which has no solution alright.

So, we can see that from RREF, I am able to conclude even the system will have no solution has such, fine. There are other ways also. Now, let us look at what happens to the case b and C. So, the case b is we have got only 2 rows which are non-zero. So, there are only 2 pivots fine.

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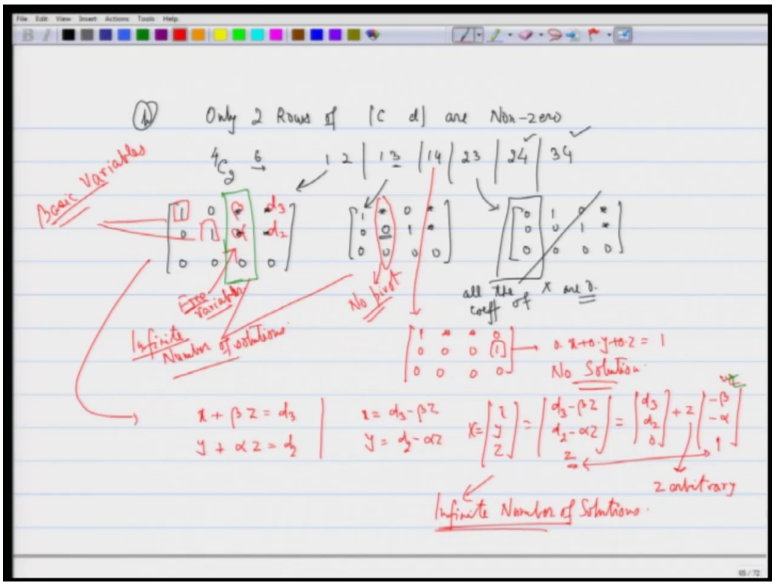

So, now, case b only 2 rows of C d are non-zero. So, again there are 4 columns. So, out of 4 columns, any 2 columns can be pivotal columns. So, the pivotal columns will be 6 of them. So, either 1, 2 or 1, 3 1, 4 2, 3 2, 4 and 3, 4. So, let us look at these 6 separately.

So, the what, I know is that there are only 2 rows which are non-zero rows and the 0 row are supposed to be the bottom. So, last one is 0 here. 1, 2 here basically means that, I have got 1 0 here, 0 1 here there no more pivots. So, there are some entries here that, we do not know what they are fine.

Similarly, we are look at 1 3, I have got 0 0 0 0, 1 0 here fine. I also know that 3 is in the pivot. So, 1 is here, there is a 0 here. I do not know what these are, can I say that this entry has to be 0 because if this was non-zero, this would have become a pivot. So, this has to be 0, but I do not know what it is here, fine.

Similarly, for this entry 2 3, again look at it the last one has to be 0 fine. Here, there is a 0 here; because there is no pivot at in the first column; second column, there is a pivot and third column, there is a pivot here fine and I do not know what they are. These are choices that I have; but we had also assumed, if you recall we had assumed that every variable comes with a non-zero coefficient alright.

Since, we had assumed that every variable comes with a nonzero coefficient, so this case is not allowed basically because of this part, because it says that all the coefficients of x, all the coefficients of x are 0, which is not allowed for me.

So, I can forgot that part alright. Similarly, for this 2 4 and 3 4, I will have to forgo because, that is not allowed for me fine. So, now let us look at what is 1 4 for me. 1 4 basically means I have got 0 here, 0 here, 0 here, 0 here. There is a 1 here and there is a 1 here, fine.

Now, this is a pivot, it means that all other entry in that row is 0 and therefore, this again gives me 0 times x plus 0 times y plus 0 times z is 1, so I have got no solution fine. And here, these entries could be anything that I not bothered, but this is what is going to happen.

So, I would like you to see that, I have this issues that I need to take care of fine. And, in the cases when I have got this and this fine, you can see that I will have infinite number of solutions. Why will have infinite number of solutions? Let us look at it nicely. So, let me put these numbers here, some numbers here. So, let me put alpha here, beta here, d 2 here and d 3 here, alright.

So, this equation tells me that, x plus beta z is equal to d 3 and the second equation tells me it is y plus alpha z is equal to d 2 fine. These two together, I can write it as solution as x is equal to d 3 minus beta z and y as d 2 minus alpha z.

So, therefore, if I write x as x, y, z. This gives me x as d 3 minus beta z, d 2 minus alpha z and z which gives me d 3 d 2 0 as one solution plus z times minus beta minus alpha 0 fine and this z is arbitrary and therefore, I can keep will replacing the value of z by 0 1 2 e pi whatever real number, I can think of and therefore, this gives me infinite number of solutions.

This is very important for us to observe and understand, that things become very nice when I have got RREF form. I am able to say just by looking at the matrix itself the RREF, whether it will have infinite number of solutions or not. Because, look at this part the third one beta and alpha, this does not have a pivot. Similarly, here if I look at, there is no pivot here, alright.

So, when you look at the solution set will talk about it there. I will also like you to understand here is that, look at how we have got this beta and alpha fine. So, in this expression, there was beta, alpha and 0 coming into play.

Here, we have got minus beta minus alpha and this 0 is coming from the z part that we have here fine; sorry, this should have been 1. So, there is a mistake there. So, let me rewrite it correctly, there has to be a 1 here; there has to be a 1 here because it is z is here. So, look at here it is z here. So, you have to be careful.

So, z here when we look at the systems, will say that the pivotal columns; these are the pivotal columns, they will correspond to what are called basic variables alright and these which are free variables. So, these which are not pivot they will called free variables.

So, will come to that afterwards. Let us not get into that part. But what we are saying is that this gives us all the ideas about solution set in general alright. So, given a system, go to the augmented matrix, compute its RREF and you have the solution as it is. I have not looked at the case when there is only one non-zero row, please do it yourself alright.

So, now from here, I would like you to understand this small small theorems which are very important.

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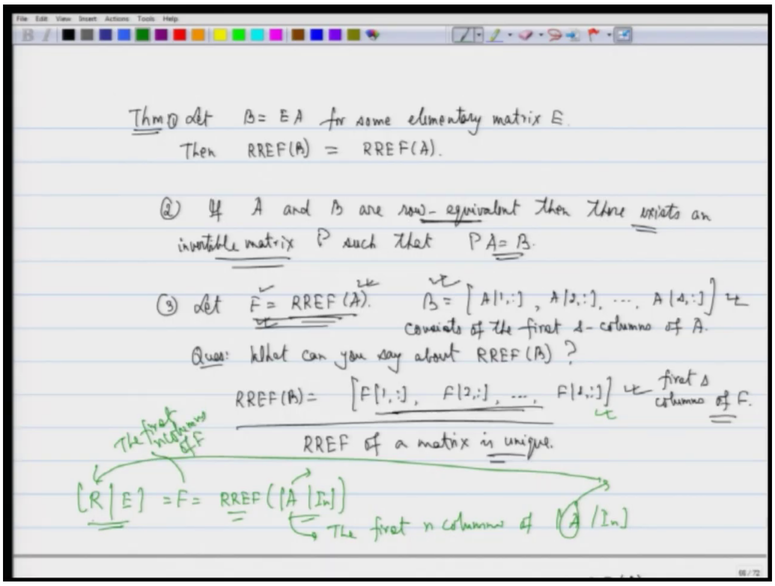

So, let me write that theorem and they follow from the idea that, the RREF of a matrix is unique alright, fine. Let b is equal to EA for some elementary matrix E alright. Then, RREF of B is same as RREF of A. So, whenever I multiply by an elementary matrix alright, RREF does not change fine.

Because, B and A are row equivalent. We had just said that, if 2 matrices are row equivalent, their RREF is the same. This what we are writing and nothing else fine. You can prove it yourself. This is 1, 2.

Now, the next thing that we would like to say is that, I already said that if A and B are; if A and B are row equivalent, then there exists an invertible matrix P, such that P times A is equal to B.

We did it for the RREF, because when I am saying that, they are row equivalent, I can obtain one from the other by a multiplying elementary matrices, the elementary matrices are invertible. So, P is product of elementary matrices which are invertible, alright.

So, therefore, P is also invertible. So, we have able to get one invertible matrix fine. This is the most important part which will help us to define the rank and things like that and proceed us.

So, let F is equal to RREF of A, all entries given to me and I have a matrix B which has not all the rows of A, but not all the columns of A, I have got only certain columns of, is that ok? So, I have just picked up some s-columns, the first s-columns alright. So, B consists of the first s-columns of A fine. Question is what can you say about RREF of B? Alright, that is the question that we have here. I know that RREF of A is F.

B is nothing but, the matrix which comes as the first s-columns of A. I have just picked up the first s-columns. So, what I am doing is that I have an augmented matrix A B, I am just looking at the coefficient part, I am forget about B in some sense alright.

So, what we are here saying here is that if I pick the first s-columns, first is very important, I am not picking any s-columns. It is the first s columns that I am picking up, in that case the RREF of B will be nothing but, so RREF of B will be equal to just look at the first s-columns of F alright.

Then, this will be happen. Why this is true? Because again, the idea is that RREF of a matrix is unique alright. So, note that since F is RREF of A, F has all the properties or this sum matrix which consist of the first s-columns; first s-columns of F also has all the properties of being an RREF and therefore, this result follows. Is that ok?

So, I would like you to keep track of this. This is very important idea that you have fine, that we are just picking up certain things, certain first certain columns and everything is nice for you alright. So, next thing would like to see is that, how do I get the inverse and then, relate invertibility with RREF and so on fine.

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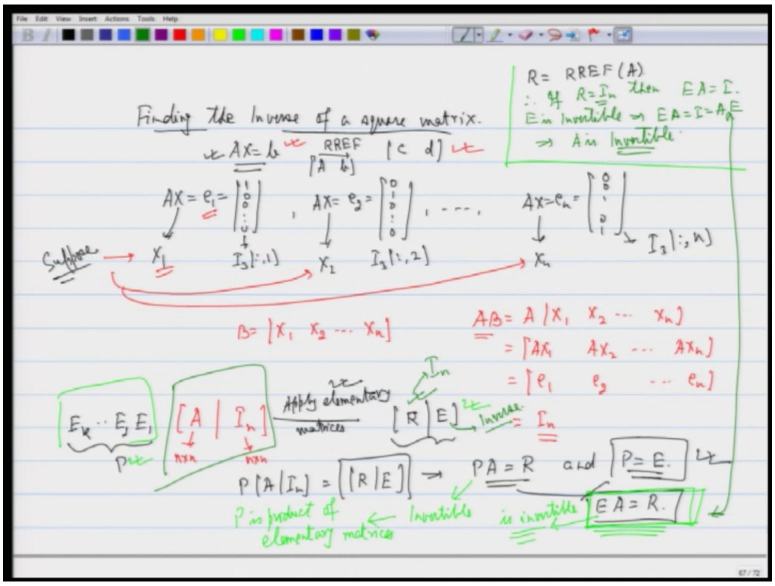

So, finding the inverse; finding the inverse of a square matrix. So, what we have learnt here is that given a system A x is equal to b, we can apply RREF to it and get a matrix C d fine.

So, applying this, applying it to the augmented matrix and I am getting C d here fine and I can solve it as it is. There is no issue of that. So, in place of just solving a system A x is equal to b, I would like to solve the system A x is equal to e 1 which is 1 0 0, the first column of the identity matrix.

Similarly, I would like to solve the system A x is equal to e 2 which is nothing but, 0 1 0 0 0 the second column of I 2 and so on till the last A x is equal to e n which is 0 0 0 and 1 fine which is nothing but, last column. I wrote it wrongly, this here. So, these are the columns not the rows, but anyway they are same; but transpose of that.

So, first column, second column and so on fine. So, let us try to understand what exactly I am trying to do. What I am saying is that I want to solve a system which is A x is equal to e 1, A x is equal to e 2, A x is equal to e l, I want to solve n system of equations.

Suppose, the solution here is x 1, solution here is x 2, solution here is x n suppose alright. Suppose, these are the solutions for me; solution is x 1, x 2 and x n fine. Let us write the matrix B as x 1, x 2, x n. I am writing B as whatever the solutions are there for each one of them. I want to compute what is A times B? Fine, so A times B is nothing but A times x 1, x 2, x n.

Now, by matrix multiplication, it is same as A times x 1, A times x 2, A times x n which is same as x 1 was a solution of e 1, A x is equal to e 1. So, A x 1 is e 1, A x 2 is e 2 and A x n is e n and what were e n? e 1, e 2, e n they were nothing but the columns e 1 was the first column for identity, e 2 was the second column, e n was the last column. So, I do get back identity here fine.

So, what I see here is that A times B is indeed identity fine. So, let us try to understand this what exactly I am trying to say. So, what I am saying is that, in place of solving the system C d, sorry A b the augmented matrix, I would like to solve the system A and identity here. I want to solve, so this is an n cross n matrix.

This is also an n cross n matrix and I want to solve it, fine. Is that ok? So, when I want to solve it, the idea is to apply the Gauss Jordan method and get the RREF fine. So, what we will do is we will multiply this by; so, what we have seen here is that getting an RREF is same as multiplying by an invertible matrix fine, which is nothing but product of elementary matrices as such fine.

So, I apply elementary matrices and so apply elementary matrices and get here some R here and some matrix say E here alright fine. When I apply the RREF elementary matrices, I will get this fine. What does this mean here? So, when I am applying elementary matrices, they are invertible.

So, I multiplying certain matrices e 1, e 2, e k, is that ok? Now, as I said I would write to write it as some P fine. So, what we are getting here is that P applied to this part is equal to R applied to E. Is that fine. So, from here, what do I get here is that, P times A is R and P is same as E. This is very important for us. Look at here P times identity is E. So, this is very important that you need to understand fine.

So, what we are saying here is since P is E, these two together gives me E times A is R fine. So, be careful about this E times A is R and what is E? E is nothing but, P and P is invertible. Why is P invertible? Invertible because, P is product of elementary matrices fine.

So, again let us understand that I have A and identity, I started with that. I am applying elementary matrices; I have got R comma E fine. So, what I am doing is that I have multiplied elementary matrices say e 1, e 2, e k with us. This I am writing it as P, I wrote it has P fine.

So, P times A of I and A and I n gives me R comma E and look at the two matrices separately, it gives me P times A is R and P is E. Since, P is E, these two together gives me E times A is R fine and P is product of elementary matrices. So, P is invertible, so E is invertible fine. E is invertible, that is very very important alright.

Now, what can I say about R? Let us try to understand that fine. So, let me write it here before going to the next page. So, what we are saying is that E is invertible, now if I look at A, so what we have learnt was that, let me go back here that let F be equal to this; B is equal to this, the first s-columns alright.

B consist of the first s-columns of A, then the RREF of B, this new matrix is nothing but the first s-columns of RREF of A itself alright. It is just the first s-columns here, this is what it is fine.

So, what we are saying is that, if this is the RREF of A and I n, then R is the RREF of; so, this implies by the previous theorem that R is indeed equal to RREF of A fine. So, please go back to the previous one again, let us see. It says that, F is the RREF of A. So, for us in this example, what we have is F is equal to RREF of this fine, which is nothing but, R and E, this is what we wrote it as fine.

So, RREF of A and I n was this and therefore, I am looking at A, the first n columns, the first n columns of this matrix and therefore, its RREF alright. So, the RREF of A has to be this part. Is that ok? This what it says, the first s-column, the first n-columns of F. So, the first n columns of F which is nothing but, R fine.

So, this is what it is important. You get R equal to RREF. Therefore, if R is a identity, then E times A is identity. Look at this part from here, E times A is R. R, I am saying is identity. So, E times A is identity; E is invertible implies E times A is equal to I is equal to A times E and therefore, A is invertible, fine.

So, when I apply RREF to A and this and it turns out that the left hand side this R turns out to be identity fine. If it turns out to be identity, then I know that the right hand side is the inverse of A and if R is not the identity, then I know that A is not invertible alright. So, let me write that part.

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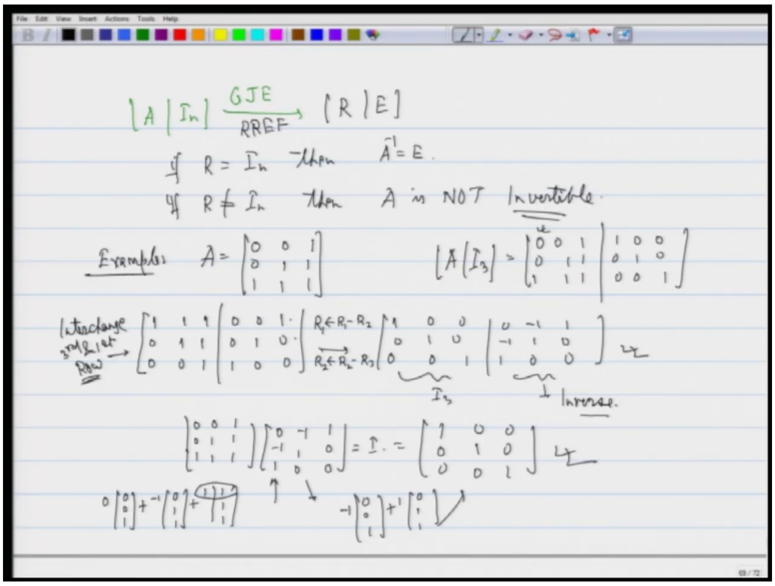

So, what we are done is I have A and identity with me, applied the Gauss Jordan method which gives us alright. If the Gauss Jordan method which gives us the RREF, RREF. This is supposed to be R and E. If R is identity, then A inverse is E; if R is not equal to I n, then A is not invertible. So, we are getting both of them at the same time. Is that ok? Fine. So, let me just explain this with a small example, how to compute this inverse.

So, example. So, I have the matrix A as 0 0 1, 0 1 1, 1 1 1. This is the matrix I have. I want to see whether this matrix is invertible or not. If it is invertible, find its inverse. So, we write A and I 3 which is nothing but, 0 0 1, 0 1 1, 1 1 1, 1 0 0, 0 1 0, 0 0 1 fine; I want this to be the pivot for me.

So, I have to interchange these two, I am not going to write that part. So, let me write one after the other. So, I want to make this as the first row 1 1 1, 0 0 1 fine; 0 1 1 0 1 0, 0 0 1, 1 0 0, I have this.

So, interchange third and first row alright. Important thing is I am allowed only do row operations; no column operations alright fine. At the next stage, I can use this one to make this entry 0, fine. So, I have 0 0 1, 1 0 0. Use this one to make this entry 0 or use this to make this one 0. So, let me do that.

So, I can make 1 0 0 alright. This minus this, so become 0 0 here. It will become 0, 0 minus 1 is minus 1, 1 will remain as it is. I can make this 0 using this. So, I will get this minus this. So, I will get here minus 1, is this correct? I hope 0 here alright; yes fine.

So, what I have done here is R 3, R 1 is being replaced by R 1 minus R 2. So, just look at this, this minus this will give me 0 here, 0 0 minus 1 and 1 and I am also done after that, once I have done this, I am looking at R 2 is being replaced by R 2 minus R 3 this minus this, so we get minus 1 1 0 alright.

So, for me, what I see is that, this is I 3. So, this is the inverse. So, you can check that 0 0 1, 0 1 1, 1 1 1. This matrix times 0 minus 1 1, minus 1 1 0, 1 0 0 is identity alright.

The way to check is look at this, it says that you are looking at 0 times the first column plus minus 1 times the second column plus 1 times the third column. So, here it says one will remain as it is. The first part will remain 1 minus 1 minus 1 will give me 0, 0 here.

If you look at this, it says minus 1 times 0 0 1 plus 1 times 0 1 1. So, this gives me 0; this 0, this one this gives me 1, minus 1 plus 1 again gives me 0 here and this gives me 0 0 1.

So, it is indeed the inverse fine. So, I would like you to learn to compute these things alright and it is very very important that, I start with A and I n. I apply the Gauss Jordan method, if the first part has identity as an RREF, then I get the inverse otherwise A is not invertible.

Thank you.