Linear Algebra

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Lecture – 10
LU Decomposition - Simplest Form

So, let us recall what we had done in the previous class. In the previous class, we had solved a system of equations. And the idea there was that from system of equation, we went to a matrix augmented matrix. We looked at just the coefficient matrix part, and then write try to write A because we took it as a square matrix.

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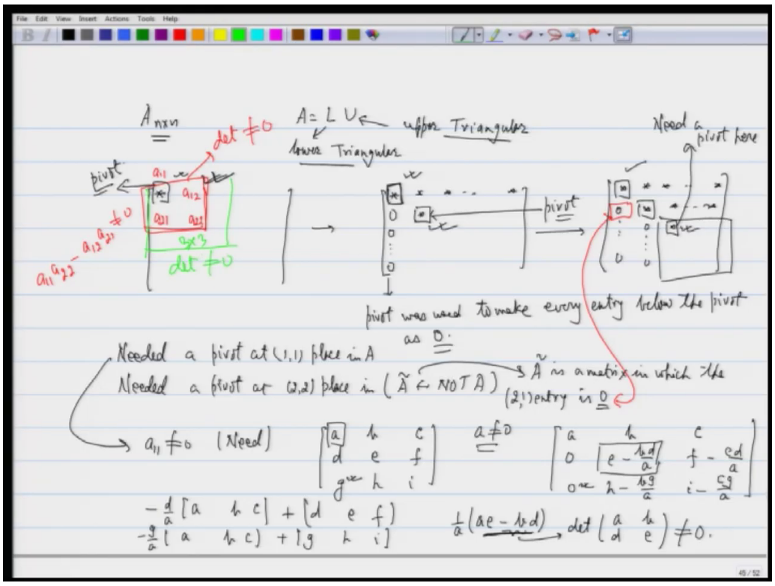

So, what we did was that we had a square matrix A which was n cross n and. For certain examples, we were able to write A as LU, where L was lower triangular, and U was upper triangular fine. This is what the idea was. We wanted to write A as LU; L for lower triangular, U for upper triangular fine. And in the two example that we saw we could do it.

Now, if you recall what exactly the idea was that given any matrix A alright, I needed that this entry should be a pivot alright. I wanted this to be a pivot. Using this pivot, at the next stage I could make, so this is a nonzero entry which is a pivot for me. At the next stage, I could make this entry 0 alright, the two one entry is 0. I can again use this pivot to make this entry 0.

So, every entry of this column other than the 1 1 entry can be made 0 alright. And then I had some things here that I was not bothered about, the idea was that I had a pivot here which was used to make every other entry in that column to be 0 alright. So, pivot was used to make every other entry below the pivot as 0 fine.

Once it was done, we went to the second row alright. And then we looked at a pivot here, again a pivot here. We want another pivot. Once we had this pivot, then you can go to the next stage, So, I have a pivot here I have another pivot here, every entry is 0, again every entry is 0 here. What happens at the top was immaterial for us alright.

So, the idea was that I needed a pivot at 1 1 place. So, needed a pivot at 1 1 place in A; needed a pivot at 2 2 place in A tilde, not A, that is important not A. And what is A tilde? A tilde is a matrix. So, A tilde is a matrix in which the first entry the in which the 2 1 entry is 0 alright. So, what I am saying is that this entry is 0. Is that ok, that is important for me alright.

And the next stage what I will require is that if I want to proceed further with the pivot, I will need a pivot here, alright. So, need a pivot here, pivot here, or we can think of this as a new matrix for me.

And then again I start here alright. So, I have another matrix. I need the pivot at the 1 1 entry of this new matrix as we had at the previous stages. So, the question is what condition do we require as such, fine? So, let us look at this. So, at this stage at the first stage we needed that. So, first stage so this part tells me that I needed a 1 1 to be nonzero this was the need alright.

Now, let us look at if I want to make this as a pivot in A tilde what exactly do I require, alright? So, let us do the simplification for ourselves for 3 by 3 matrix and proceed. So, I start with the matrix with 3 by 3 say a b c d e f alright. And I am not bothered about this part for the time being, or let me write it I think will required e f g h and I alright.

So, I am assuming that a is not 0. So, this is a pivot for me a is not 0 alright. Then what do we do, we leave it the first row as it is, no change. Then I have to make this 0. To make this entry 0, what do I need? I need that I need to multiply the first equation. So, I have to just look at d upon a of a b c, add it to de f. What do I get? Minus d upon a into a is 0 plus this. So, this part is 0.

Now, this part will become d minus b d upon a, this will become f minus c d upon a fine. So, this is the entry that I needed for me. So, I would like you to see that this entry is nothing but I can take 1 upon a common a e minus b d. And what is a e minus b d? Can I say that this is nothing, but the determinant of the matrix a b d e alright. So, look at the determinant of this a e minus b d which is the inside entry here alright.

So, what I need is that this entry determinant of this should be nonzero if I want the pivot at the second place. Is that ok? So, earlier I wanted pivot for a 1 1 to be nonzero, the first entry should be nonzero. Now, I want that the 2 by 2 matrix that I have here 2 cross 2 matrix of a that I am looking at, so it is a 1 1 here a 1 2 a 2 1 a 2 2. So, I want that a 1 1 a 2 2 minus a 1 2 a 2 1 is this should be nonzero or determinant of the first block here determinant of this should be nonzero. Is that ok?

Now, let us go back. If I want the third one what is the requirement that I need, alright? So, let me make this also 0. So, I have got 0 here. Using a similar trick that I am done earlier, I want to get rid of this g, this is what I have done g.

So, this will become h minus o, I have to multiply by minus g upon a to a b c, and then add g h i. I hope I am correct. So, a and a cancels out, I get minus g plus g 0; I get h minus b g upon a, and I get i minus c g upon a. This is the matrix I get fine. So, let us take this matrix that I have here, and write it at the next place.

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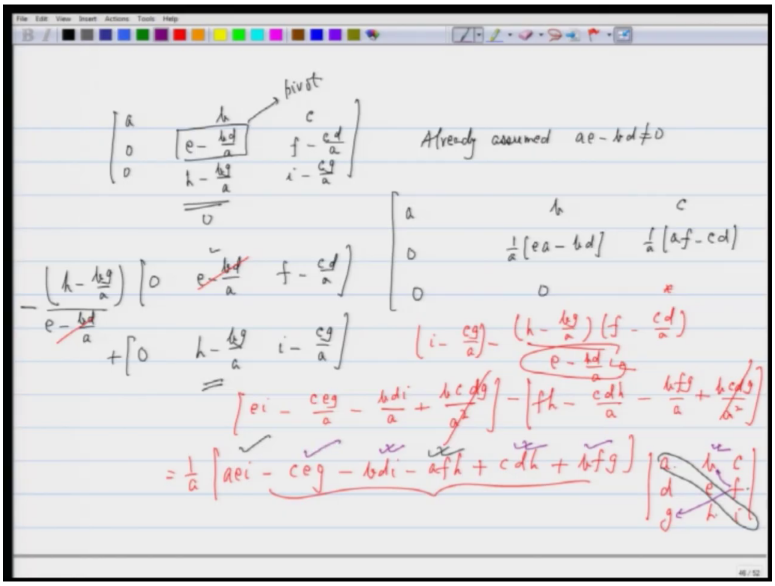

So, is this a is there a. So, what I had was I had let me write it again a b c, 0 here, 0 here. Here I had e minus b d upon a e minus b d upon a, e minus b d upon a alright. Then it was e f, f minus c d upon a I think f minus c d upon a h minus b g upon a h minus b g upon a i minus c g upon a this is the matrix I had. We have already assumed already assumed a e minus b d is not equal to 0, and therefore, this is a pivot for me, and I want to make this entry 0.

So, if I want to make this entry 0, I will have a here, b here, c here, 0 1 upon a times e a minus b d; and this entry is 1 upon a times a f minus c d 0 here, 0 here. So, I need to multiply the this equation the second equation by or the second row by h minus b g upon a to 0, e minus b d upon a f minus c d upon a and divide it by e minus b d upon a, so that this and this cancels out

I get this. So, I have to do this fine, and minus of this I have to do because I have to cancel it out.

So, minus of this plus 0 h minus b g upon a i minus c g upon a this is the matrix I have to look at. So, I get 0 minus 0 as 0. If I look at here it is h minus b g upon a, this and this cancels out, alright.

So, this will cancel out to this one, I will get minus h times b g upon a and h minus b g upon a that will cancel out, I will get 0 fine. The last one entry this entry will be nothing but I already have i minus c g upon a fine minus h minus b g upon a times f minus c d upon a divided by e minus b d upon a alright.

So, let us solve it out. So, I can take out this part common throughout, and I do not have to worry about, just look at the inside part what are the inside part is. So, the inside part is e times i minus c e g upon a fine minus b d i upon a minus minus plus b c d g upon a square this minus f h minus c d h upon a minus b f g upon a minus minus plus b c d g upon a square fine. So, this is the one that is I have taken out this part common and then looking at things.

So, look at here b c d g upon a square and this cancels out, and I take out 1 upon a common here again. What I am left out with a e i fine minus c e g minus b d i minus a f h plus c d h plus b f g. Now, question is what is this inside? So, let us look at the matrix it was a b c d e f g h i. Let us look at the determinant of this.

Determinant of this is nothing but a e i which is here, then minus a f h a f h with a minus sign a f h with a minus sign fine, sorry, then we have with respect to b b d i here, b d i with an minus sign b d i with a minus sign here, then b f g with a plus sign b f g b f g with a plus sign here which is a plus sign. Similarly, c d h with a plus sign c d h with a plus sign here alright, and the last one which is c e g with a minus sign I have got this.

So, what I need is that this part should be 0. So, let us go back to the previous slide, and then what we are saying is in the previous slide this part, the 3 cross 3 determinant should be nonzero alright.

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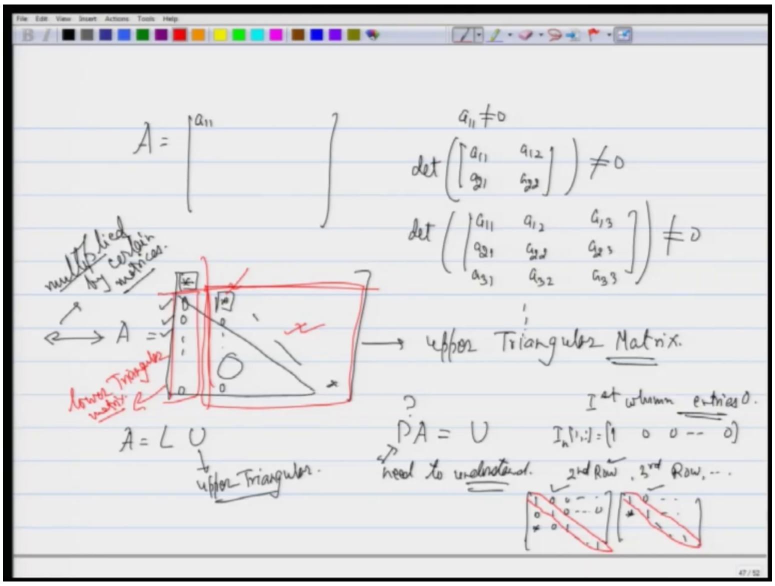

So, what we are saying is that if I have a matrix. So, what we are saying here is finally is that I have a matrix A which says the property that a 1 1 entry is nonzero fine. Then look at this matrix a 1 1 a 1 2 a 2 1 a 2 2 determinant of this has to be nonzero. At the next stage determinant of a 1 1 a 1 2 a 1 3 a 2 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3, determinant of this is not equal to 0 and so on.

If I have such a thing, then that each stage I will have a pivot fine, and this pivot will help us to make this pivot will make every entry 0 here, I will have another pivot here which will make every entry 0 here and so on. So, I will get this matrix as everything here is 0. So, I get this as an upper triangular matrix, already has an upper triangular matrix fine. So, what exactly we have done, so I have A here and on the left multiplied by certain matrices.

Now, what were these certain matrices, alright? So, I have got the upper triangular matrix the idea was to get A equal to LU. I have got U which is upper triangular. So, what at present I have got is I have got some matrix P here. I do not know what this P is, whether it is lower triangular or not, but I have got P times A as U I need to understand what is this P, need to understand this.

So, in the previous examples what we have seen is that if I want to make these entry 0s here, I need to look at the rows of the identity matrix and then do certain manipulation to it. What exactly we are doing is at the first stage if I look at first stage or making the 1st column entry 0 if I want first column entries 0, what I need is I need to use the first row of the identity matrix alright, I want to use this. And then look at the 2nd row of identity, the 3rd row of identity and so on and then make them 0.

So, what exactly we are saying is that the matrix that we are going to have is 1 here, 0 here, there will be something here 1 here this is the first one as per the 2nd row is concerned. At this stage, it will be 1 0 again 0 fine; 2nd rows I am not going to do anything here at this stage because that is all been made 0; at the third stage I would like to make something using this one fine, this again identity for me.

So, when I look at these matrices these are lower triangular matrices alright. So, therefore, to go from first stage to second stage in which all of these entries in which all of these entries have become 0 alright, I just need to multiply by a lower triangular matrix fine.

Once I have done that lower triangular matrix, I have a matrix of this type where the first row is as it is but the every entry in the 1st column is 0 fine. I can use induction or you can use the same idea to say that I can start with this matrix now fine. This entry is nonzero and hence I will get a lower triangular matrix such that this matrix the new matrix that I am looking at alright the lower part has this form which is lower triangular and so on fine.

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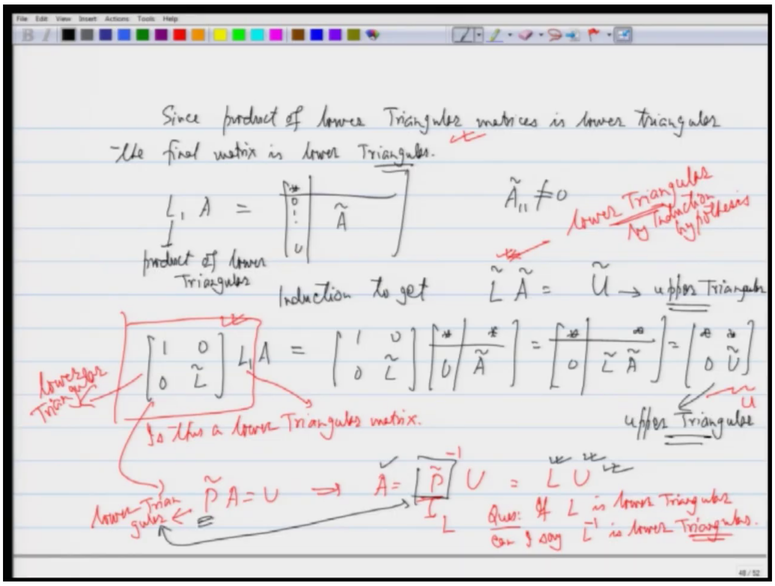

So, what we know is that since product of lower triangular matrices is lower triangular, the final matrix will be lower triangular, the final matrix is lower triangular alright. Or if you want to look at from the induction point of view, what I had was I had A, I had some matrix L 1 say alright.

So, this L 1 was product of lower triangular such that this looked like a star here 0 here 0 and this is my A tilde. And in A tilde of 1 1 is nonzero, the 1 1 entry of A tilde is nonzero fine. So, I can use induction to get say L tilde of A tilde to be is equal to U tilde upper triangular fine.

So, I want to club the two together to understand the final thing. So, let us look at this matrix 1 0 0 L tilde times L 1 of A, I want to compute this matrix product; so this will be equal to 1 0 0 L tilde times star 0 star A tilde.

So, this I am saying is there if you understood matrix product, everything becomes nice. So, this is nothing but just the first row, first row remains as it is no change; 0 times this will be 0, so you get 0 here and this is nothing but L tilde A tilde.

And therefore, you get here star star 0 and U tilde which is an upper triangular, fine. So, you get an upper triangular matrix that you can see you overall, now we need to understand that can I write this in a nice way alright, as it is this a lower triangular matrix alright.

So, what we know is that L 1 is already lower triangular, now this matrix L tilde is lower triangular, L tilde is lower triangular by the induction hypothesis by induction alright. So, since L tilde is lower triangular, this matrix is also lower triangular and hence with the first idea that put it a lower triangular is lower triangular, I get that this is lower triangular.

So, I can write this matrix as say, P tilde which is lower triangular. So, I get that P tilde times A is equal to U, let me write this as U itself here. And therefore, A is equal to P tilde inverse U; I write it as L, so it is L U.

Now, question is if L is lower triangular, can I say L inverse is lower triangular; why am I saying this, because if you note here this matrix P tilde that we wrote here P tilde, P tilde here is lower triangular fine.

Since, I know that P tilde is lower triangular; and inverse of a lower triangular matrix also lower triangular that also we had seen in some of the in the previous lecture or just try to prove yourself that if something is lower triangular, its inverse is also lower triangular; you get that you can write A in terms of a lower triangular and an upper triangular matrix, alright.

Now, what is the use of this that will understand in the next class alright or let me just try to explain it for you for the time being, because I still have some time; so let me do that alright.

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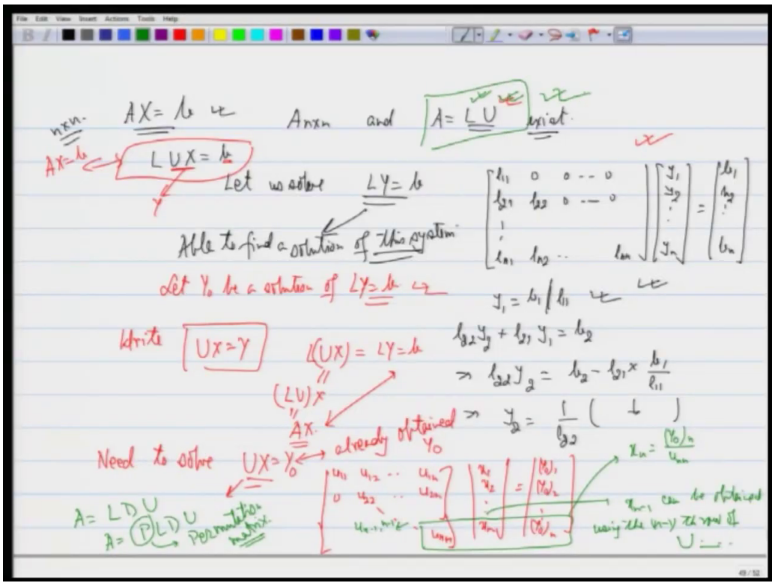

So, suppose I am looking at the system A is equal to now I am looking at the system, A X is equal to b; and I know that A is n cross n, and I am able to get A as L U fine, I may able do that so L U exists. So, I am able to write A as lower and upper triangular fine.

So, I can rewrite this part as L U of X is equal to b fine, so let us solve, so let us solve L Y is equal to b. Now, I know that L is lower triangular, so it means that L looks like some say let me write l 1 1 0 0; l 2 1 l 2 2 0 and so on; l n 1 l n 2 l n n; times y 1 y 2 y n is equal to b 1 b 2 b n, alright.

I started with an n cross n matrix, therefore everything is in this form. So, if you see this L Y equal to b, then I know that y 1 is equal to b 1 upon L 1 1 alright. Similarly, l 2 2 y 2 plus l 2 1 y 1 is equal to b 2 will imply that l 2 2 y 2 is equal to b 2 minus l 2 1 into y 1 y 1 is already known, b 1 upon l 1 1. And this will imply that y 2 is equal to 1 upon l 2 2 times this, alright.

So, what we are saying here is that I am able to find the solution of this, so able to find a solution of this system. Now, why am I going to find it out, so please understand that in the previous slide or wherever we had done this system of equation tried to solve it, the idea was that these matrices had the property that these entries were 1 1 1 1 1, they were never 0; they were the diagonal entries and we did not play with them, fine.

Here also if you look at by induction you can say that the diagonal entries of L tilde are 0, again I have put a 1 here; so again I have something in the diagonal which are all ones, alright. Since, they are all ones this system can always be solved, so I have a solution here for us is that ok, fine. Once I have a solution here, let us look at this one I am assuming that so why so let, y naught be a solution of L y is equal to b.

Once I know and which is very easy to solve, we are just solving it by back substitution or something like that fine we are just solving it here. So, I go back to this equation now which was nothing but A x is equal to b itself. So, I have been able to solve L y is equal to b. So, L y is equal to b means b is already got fine.

And this is the one that I am looking at y is not it. So, if you look at, if I write so write U x is equal to y, then what we see here is that L times U x is equal to L times y which is b; at the same time this is also equal to L U times x which is equal to A x fine. So, indirectly I am solving the system A x is equal to b fine.

Now, if I want to solve now so now, need to solve U x is equal to y naught because already obtained y naught fine. So, again U is upper triangular. So, u have this nice form which is u 1 1 u 1 2 u 1 n 0 u 2 2 u 2 n and so on till u n n times this x 1 x 2 x n is equal to y naught of 1 y naught of 2 y naught of n fine.

From here if you see this part, I get that x n is equal to y naught of n upon u n n. At the next stage this entry at the next stage if I look at I can compute I can use this idea u n minus 1 n minus 1 to get the value of x n minus 1 can be obtained using the n minus oneth row of U and so on. So, I can build up on this idea to solve the whole system.

The important thing here is that it may take time for us to solve this to get the lower and upper triangular form of a writing A as LU, but once you have got L and U, it is very easy to solve it alright. It turns out that the MATLAB package or any package that you are looking at they basically try to write the matrix A in terms of LU, but since there are condition that you need.

So, they do not look at A equal to LU, but they look at what are called writing A is equal to sometimes LDU or sometimes they try to write a as P LDU, where P is a permutation matrix.

So, I have not done permutation matrix, I am not bothered about for the time being. But the idea is that when you solve a system of equation you are allowed to interchange equations, your first equation can become third equation third equation can become second equation, and so on fine.

So, this permutation matrix it changes the see look at this I am writing P on the left, P on the left. And therefore, what we are doing is that we are trying to interchange the equations or the rows of the matrix A, so that the matrix that I have is in nice form what is called a regular form.

And for the new matrix, I can get this decomposition A equal to LU. Is that ok? So, there are some question in the assignment sheet related with LU, just try to solve it. And do not worry about the exam, maybe I will have maximum one question or you need to just have an understanding of the subject that is all for the time being.

Thank you.