Linear Algebra
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Lecture - 01

Alright. So, we start the class on matrices. First we would like to look at notations which are very important as far as our understanding of the subjects are concerned, and this is true of mathematics as such.

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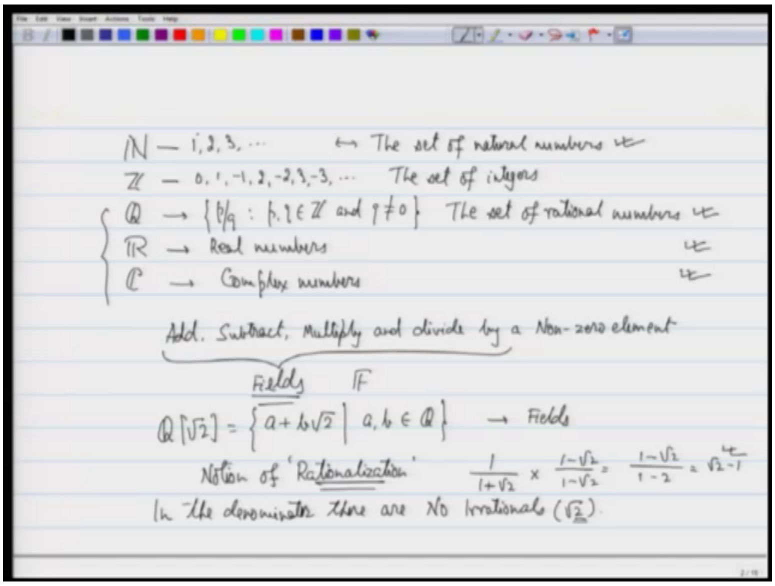

So, for us if I write N, N for the set of natural numbers. So, N is going to look like. So, it has elements of the type 1, 2, 3 and so on and we call it the set of natural numbers. The set of natural numbers.

Similarly, we have the set of integers which have entries 0, 1, minus 1, 2, minus 2, 3, minus 3 and so on. So, we write it as, the set of integers. Similarly, we have the rational numbers Q, which is of the type p upon q p and q are integers, and q is not equal to 0. So, this is called the set of rational numbers.

Then, we have what are called real numbers, C for the set of complex numbers. So, what we know is that if I look at set of natural numbers, then we can add so, if I look at the set of natural numbers we can add two natural numbers, but we cannot subtract two natural numbers. For example, 2 minus 3 does not make sense in the set of natural numbers, but it does make sense in the set of integers fine.

Similarly, if I want to write 2 minus 2 again 0 comes which is not an element of natural numbers, again it is an element of integers. So, in the set of natural numbers you can just add, in the set of integers you can add and subtract. When we want to multiply, natural numbers to natural numbers can be multiplied you are still inside that set. Similarly, set of integers we can multiply, we are still inside that set but, when we want to divide we have problems with natural numbers as well as integers.

So, we go to what are called rational numbers, real numbers and complex numbers. So, if I look at these three sets rational numbers, real numbers, and the complex numbers, they have a very nice property that, we can add two numbers. So we can do add two add two numbers, we can subtract two numbers, multiply two numbers and divide by a non-zero element.

So, what is the non-zero element in all these three sets? So, in all these three sets, 0 is the 0 element. So, I can divide by every number other than 0 and that is a meaningful object fine. So, these three sets where which have the property that you can add, subtract, multiply, and divide by a nonzero element, they are called fields. So, in mathematics a anywhere these things happen we call it as fields and we will write it generally as F here.

So, if you see all the notations what we have done is that, we have integers, natural numbers, rationales, real, complex, F, everything has a extra vertical line. To just to indicate that they are sets and we will be following that notation throughout. We also have sets of the type Q of root 2 which have which are of the type a plus b root 2 and a and b are rational numbers, they are also fields. We had seen in our school days that there was a notion of; so notion of.

So, for example so, let me use the word I do not know whether, you remember it not, rationalisation that is we want to make them rational. So, denominator is to be made as a rational number. So, for example, if I want to write 1 upon 1 plus root 2, I can multiply this by 1 minus root 2 by 1 minus root 2. And therefore, I will get it as 1 minus root 2 divided by, 1 minus 2 which is root 2 minus 1, so the denominator here. So, in the denominator, there are no irrationals. So, here the irrational is root 2. So, in the denominator there is no irrational are the last step alright.

So, we will be looking at such fields we have also have fields what. So, if you look at all the fields that we have seen till, now they have the property that they have infinite number of elements. But, we also have fields which have finite number of elements and it turns out that for every prime P there is a field corresponding to that.

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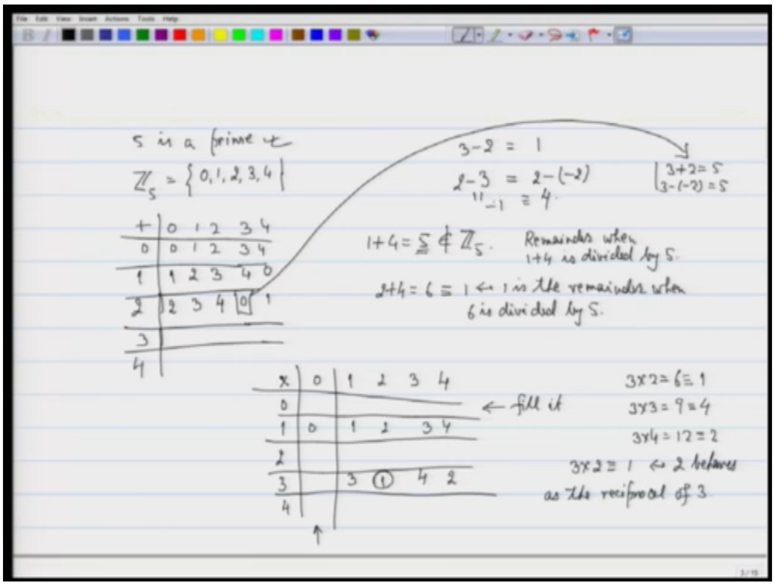

So, for example, let us look at Z 5. So, 5 is a prime; 5 is a prime. So, what is a prime number? A number which is dealt by only 1 or itself so, 5 is a prime number. So, if you want to look at Z 5 what I will do is that its elements are going to look like 0 1 2 3 and 4 the addition here is going to look like.

So, if I want to add 0 which is 0 1 2 3 and 4 alright. So, there is I should have written it slightly differently. So, addition of 0 with 0 will be 0 1 2 3 and 4 with respect to 1 if I want to add, then it is going to be 1 2 3 4, but now note that 1 plus 4, that we have 1 plus 4 is supposed to be 5 as far as natural numbers are concerned. But, 5 is not an element of 5 it does not belong to Z 5.

So, what we do is that we divide this number which is a bigger than 5 by 5. So, we look at the remainder, when 1 plus 4 is divided by 5. So, the remainder is 0 here. So, therefore, the table here will be 0, so 1 plus 4 is 0. Similarly, if I want to look at addition with respect to 2, 2 plus 0 will be 2, 2 plus 1 is 3, 2 plus 2 is 4, 2 plus 3 is 5, which is again 0, 2 plus 4 is 6, which is same as 1. So, 1 is the remainder 1 is the remainder, when 6 is divided by 5.

Similarly, you can complete this table for yourself, I would like you to do it yourself try that out. So, this is as far as the addition is concerned as we said we want this to be a field. So, for a field I need that there should be a multiplication also. So, let us multiply it again let us make a multiplication table for us.

So, I have the numbers 0 1 2 3 4, I also have 0 1 2 3 4. I already know that 0 times any number is 0, so I do not have to worry. One time any number is also the same number, so it will be 0 here, 1 here 2 3 4 so, fill this yourself fill it. Similarly, I would like you to fill this part also fine.

Let us multiply by 3 and C what the table should be. So, 3 times 1 is 3 and 3 is in the set so, 3 remains as it is, but 3 2 times 2 is 6 which is not in the table. So, we have to divide by 5 and if I divide by 5 the remainder is going to be 1. So, as in the case of addition also 6 look like 1 itself 3 into 3 if I look at is 9, again 9 is not an element of the set Z 5. So, I would divide by 5, and the remainder is 4 here.

So, I write 4 here, 3 into 4 will be 12 and which is nothing but 2 and I divided by 5. So, this is what I am going to get. I would like you to complete the this table also. So, what I would like you to observe here that I have a number 1 here. So, what we are seeing is that 3 into 2 is 1. So, what we are saying is that, 2 behaves as the reciprocal, of 3. In the sense that if you remember 3 into 1 upon 3 was 1, similarly here 3 into 2 is 1. And there we are also 1 upon 3 into 3 was 1 similarly 2 into 3 is 1 here.

So, 1 works as a reciprocal identity element of the set. So, 1 is always the identity element of the set. So, Z 5 is a field because, you can add similarly if I want to subtract, then what will be minus 3? So, 2 minus 3 if I want to look at what is 2 minus 3 here. So, 2 minus 3 will be, 2 and I can replace minus 3 by or just right so what should I say. So, you can write 3 as yes anything that you can think of I can write 3 as minus 2 itself, because 3 plus 2 is 5.

So, let us look at the table 3 plus 2 is 5, this is the table that we have. So, from here we see that 3 plus 2 is 5. So, I can replace 3 by minus 2 itself, because 3 minus minus 2 is 5. So, I can replace 3 by minus 2, but it is the additive inverse. So, therefore, this is nothing, but 4. So, is that ok? So, understand again 3 minus 2 is 1 and 2 minus 3. So, 2 minus 3 as such is supposed to be minus 1, but minus 1 is also same as 4 because, minus 1plus 5 is 4 or when I divide minus 1 by 5 the remainder is 4, is that ok.

So, for every prime P we have a field Z P, and sometimes we may like to look at things there also if at all, but I am not sure when I will be able to depending on the syllabus that I can cover, but we have to study things over these fields also. So, whenever we study things I will be writing F in place of rational numbers, real numbers, complex numbers, or any field of this type which has finite number of elements fine.

So, now let us go back to what we are supposed to do, we are supposed to study matrices. So, this is as far as the notations are concerned. So, now, let us go back to the actual syllabus for us. So, we look at what are called matrices. So, what is a matrix?

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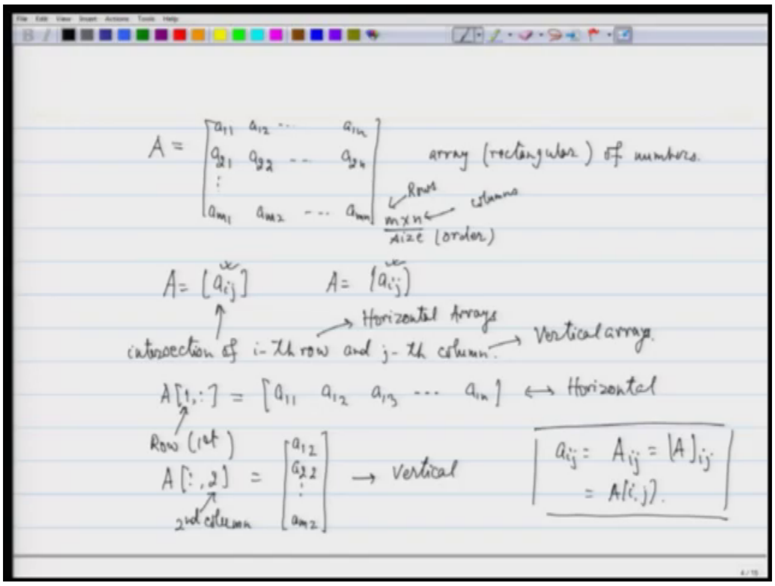

So, matrix is we generally write it as ABCD we will be writing it as capital. And it is nothing but an array. So, it is just an array generally we say it is a rectangular array of numbers. So, this is the way we write we just say it is any matrix A is an array rectangular array rectangle could be via squared also.

So, it is just an array of numbers and the increase of these could be a any of the numbers or it could be functions it could be anything depending on the requirement we will change our ideas, but for the time being our entries are going to be mostly real numbers. Once in a while I look at complex numbers as entries, but mostly it will be real numbers for us.

So, for example, we write it as so, the matrix so if this array this matrix is of size m cross n or order. So, or the order of this matrix is m cross n; then the entries of this matrix generally written as A. So, a small a corresponding to that a 11, a 12 and a n is there, so it is a 1n a 21, a 22, a 2n a m1, a m2 a mn. So, the entries of this matrix as I said they are of the type. So, we generally write A as a ij sometimes you write like this, sometimes we will write it as a ij.

So, depending on the situation these brackets can change, but the idea is that a ij here or here it means that I am looking at an entry which is at the intersection of i-th row and j-th column. So, the first thing that we need to understand is that what is a row and what is a column. So, row corresponds to; so row corresponds to what are called the horizontal arrays and the columns they correspond to vertical arrays.

Fine, so for example, if I want to write the first row. So, this is the corresponding to the row and we are looking at the first row. So, this corresponds to the entry a 11, a 12, a 13 till a 1n. Similarly, if I want to look at the second column, second column the entries are going to look like a 12, a 22 till a m2. So, as I said the rows are nothing, but horizontal and these are vertical. Sometimes you also write a ij as A of ij or maybe a half ij or as a half ij depending on different situations our entries are going to be the to look like this, alright.

So, this is as far as our notations are concerned about the matrices, we will come back to them again. So, as so remind yourself that a matrix A, has what are called m rows here a so, first row is a 11, a 12, a 1n, second row till m rows. So, m the first here corresponds to the rows and n corresponds to the columns. So, number of rows and number of columns that, we are looking at fine.

So, it is an array consisting of rows and columns, a ij corresponds to the entry in the intersection of i-th row and j-th column. When I write a 1 and with a colon it means that, I am looking at that row. So, a 3 comma colon means the third row, similarly colon comma number means I am looking at that corresponding column, fine.

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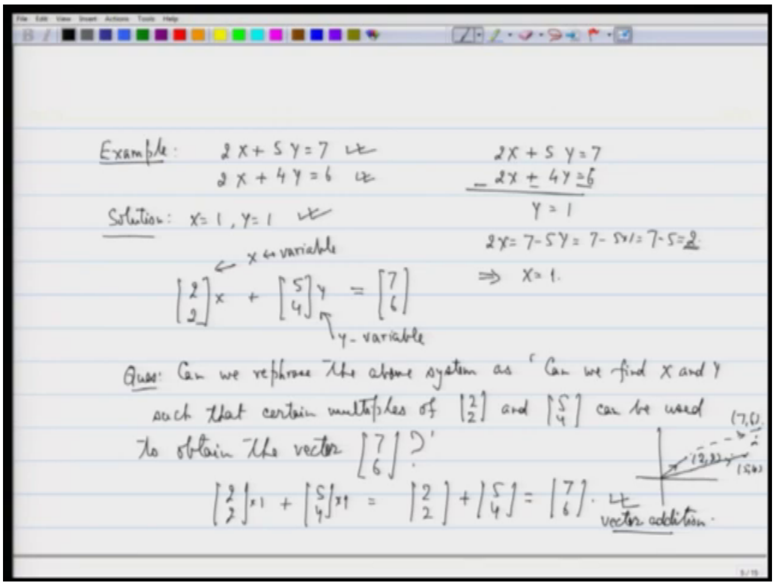

So, now let us look at an example and why do we need to understand these matrices. So, example so, they are going to solve a system of linear equation here, we have already done in our school. So, let us look at the system again. So, we are going to solve 2 x plus 5 y equal to 7 and 2 x plus 4 y is equal to 6.

So, what we do is when we solve the system, we just since it looks very simple what we will do is that we will just subtract it alright. So, write it back minus, minus, minus what we get is y is equal to 1. Now, putting it back what we will get is that 2 x is equal to 7 minus 5 y which is nothing, but 7 minus 5 times 1, which is 7 minus 5 which is 2 alright and therefore, what we get is x is also 1.

So, the solution that we have here is solution x is 1, y is 1, in some sense we are saying that 2 times 1 plus 5 times 1 is 7 and 2 times 1 plus 4 times 1 is 6 and therefore, both the equations are being satisfied. What would like you to understand further after these lectures is that, this vector so there is a vector in r 2 that we are looking at 2 2 this corresponds to the variable x.

So, we have the variable x in the above equations and this the vector 2 comma 2 is associated with x, or let us we write it as it is. Similarly, the vector 5 comma 4 is associated with a vector y, y is again a variable unknown and what we are looking at is that this vector is this equal to this. So, we are can I say that what we are looking at is that.

So, question can we rephrase the above system as, can we find x and y such that certain multiples of 2 2 and 5 4 can be used to obtain the vector 7 comma 6? So, this is the question that, I am asking. Can I find some multiples of 2 2 and 5 4 whenever we add it we do get back the vector 7 6. So, here if you look at our solution was as far as solving the system of equations the solution was x was 1 y was 1. So, here if I look at 2 2 times 1 plus 5 4 times 1 because y was also 1 is nothing but 2 2 plus 5 4 which is nothing but 7 6.

So, we see that we do get back what we were asked for that, yes there is a solution for x and y here and this is nothing, but our vector addition where we are adding the vectors that we have a vector here which is 2 comma 2 and I have a vector 5 comma 4. So, I have a vector here which is say 5 comma 4. So, it will be somewhere here. So, this is the vector 5 comma 4 then we are saying that we can add these 2 vectors here and get the vector 7 comma 6.

What is called the vector addition? So, this is the vector addition that we had learnt in our school. So, we are looking at vector addition. So, I would like you to understand this once more that when we are solving a system of equation, we are not just solving a system of equation, we are also looking at what is called vector addition. So, we are just multiplying vectors by certain size to deflate their size or reduce their size and then add them.

Now, does this also make sense as far as questions in 3 dimensions are concerned, vectors in 3 dimensions are concerned?

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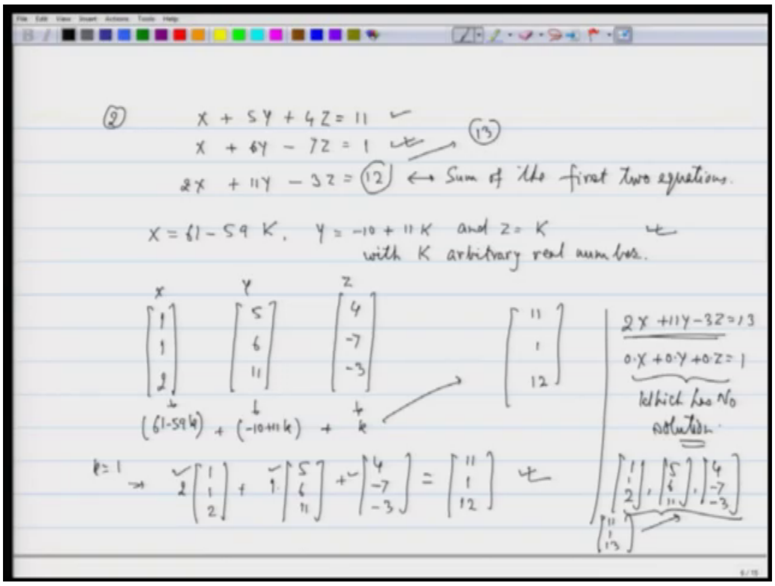

So, let us look at another example, example 2 where we are looking at x plus 5 y plus 4 z is equal to 11, x plus 6 y minus 7 z is 1 and the third equation is just we add it up 2 x plus 11 y minus 3 z is equal to 12. So, if you see this system alright the third equation is nothing, but, some of the first two equations.

So, therefore, the third equation is redundant for us and therefore, what we understand is that the first equation is a plane, second equation is also a plane, and you can see that they are not parallel and therefore, they intersect in a line. So, we have infinite number of solutions for the system, I would like you to verify that the solution here is nothing, but 61 minus 59 K y is minus 1 plus 11 K and z is equal to K with K arbitrary alright.

So, I am not solving it I had done it myself earlier so alright. So, we need to understand that, this you have already solved. So, let us try to understand, what do you mean by the solution here. The solution here means that, if you plug in the values of x y and z you do get by the equation, I would like you to understand which is very important here is that look at the vectors corresponding to the variables x y and z.

So, corresponding to x you have the variable 1 1 2, corresponding to you y you have the variable 5 6 11, corresponding to z you have the variable 4 minus 7, minus 3 and what we are saying here is that if I multiply this vector by 61 minus 59 K. This vector by minus 10 plus 11 K and this vector by K and then add them up what we are going to get is 11 1 12.

For example, alright if I take K to be 1. So, k equal to 1 will give me 1 1 2 times 2 plus minus 1 times 5 6 11 plus 4 minus 7 minus 3 is equal to 5 plus 4 is 9 oh sorry 5 sorry 4 plus 2 is 6 6 minus 5 is 1 have I done some mistake here K is 1. So, K is 1oh sorry it should be plus 1 here; so alright. So, K is 1 here also 5 plus 4 is 9, 9 plus 2 is 11 minus 7 and 6 plus 2 is 8 and this minus 7 will give you 1, minus 3 and then 11 plus 4 is 15, 15 minus 3 is 12.

So, you can see that we do have vector addition again which is coming into play here fine. And the important thing to note is that if I replace this 12 by say 13 then, we then the third equation that we are going to get here is 2 x plus 11 y, minus 3 z is equal to 13. And if I subtract this equation from the first two equations I will get 0 times x, plus 0 times y, plus 0 times z as 1, which has no solution.

So, I would again like you to understand here that this will again tell me that, if I am looking at the vectors 1 1 2, 5 6 11 and 4 minus 7 minus 3, then I cannot write so the vector. So, if I look at this vector. So, let me look at the place if I look at the vector say 11, then 11 1 13 cannot be written as sum or multiple of things like vectors of this type fine. So, I will end the lecture here itself and in the next class we will again look come back to these equations.

Thank you.