Calculus of Several Real Variables
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Lecture – 09
Vector-valued maps and Jacobian matrix

Welcome once again to the running of this course. I hope you are enjoying it I am trying to give it as slowly as possible keeping the fair amount of variation in the audience. So, today we are going to talk about derivatives of many variables. So you may wonder that once we have spoken our partial derivatives.

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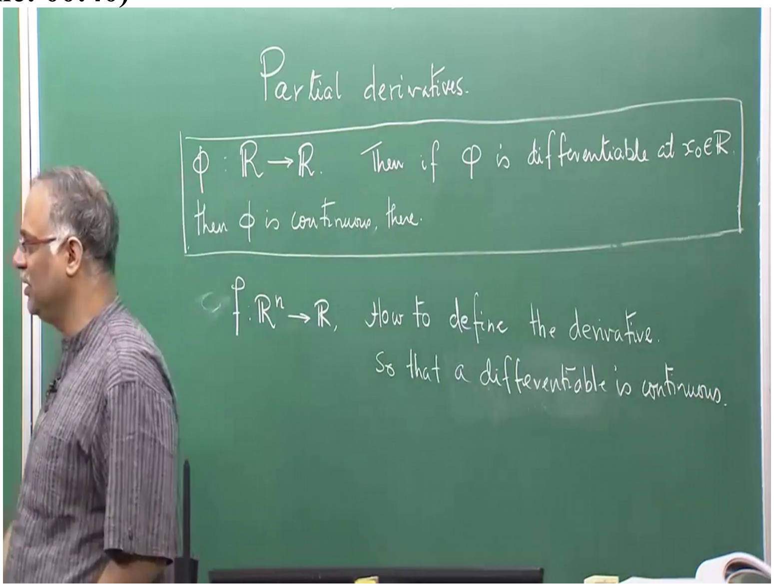

Why do I require to speak about this directives of functions? So, many variables I could say okay. that partial derivative means the derivative that is all. So, I do not care about anything else. See the idea of getting the derivative of functions of two variables. So many variables to mathematicians many years to formalize and make the idea look consistent look all right.

The key idea if you look at the derivatives is that if we just take a function phi from R to R then if phi is differentiate well at a point x naught differentiable and x naught in R then five is continued was there these are very fundamental fact. So, existence of a derivative at a given point guarantees continuity of the function at that point. The reason that we need to talk about functions, derivatives of many variables separately.

And not just consider partial derivatives is precisely the fact that there can be a situation where the function is discontinuous at a given point here the partial derivative exists. So, if you look at this statement what he says differentiable it implies continuity no continuity at a given point implies no differentiate ability.

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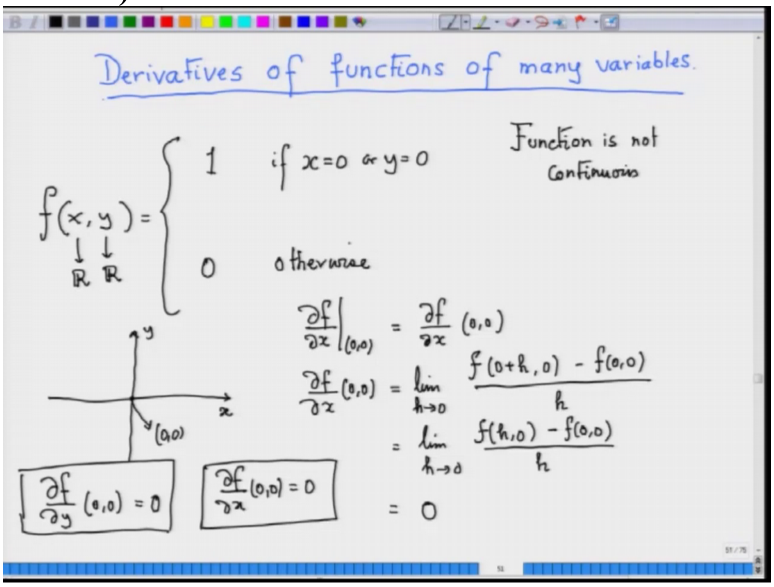

So, it is very important that can we define a kind of derivative Right. And for functions of many variables which will make the function continuous at a given point. So, now look, let me clarify that the partial derivative exists, while the function itself is discontinuous at a given point define this function f of x, y, x is in R, and why is in R function of two variables. And this is defined as follows is equal to this is a definition from the book way this is an example from this book itself.

So, I will just write down the exact example. So, it is one if x is equal to zero, or equal to zero means also the origin it is one and zero otherwise So, if you draw the x axis y axis and this is the point 00. So, along the x axis and y axis is a function as the other one, the graph is like this while is zero otherwise, so it does have a discontinued the axis introduce you.

So, what I want to calculate the Del f of Del x at 00. So, instead of writing like this, I will start writing it like this. It is much easier Del f and Del x at 00. So, how do I calculate the Del f and Del x at 00? Is limit h tends to zero f of zero plus h zero minus f 00 divided by h what is f of zero plus h? So, what are you going to have? Your limit h tends to zero f of h zero minus f of 00 by h.

Now, we have said that if either x or y zero the function is one, so this FH zero is one and f0 this F h zero that you see here is one and F 00 is also one. So the limit becomes 01 minus 100. So the partial derivative at x exists. Similarly, you can compute Del f Del y at 00 is also zero, I leave the competition to you. So, you have this as well as you have this fact.

So, the partial derivative exists a simple example at the point 00 and the values are zero, but the function itself is not continuous the function is not continuous. See mathematics you might say why you need to do some certain thing like that why you need to be worried about this issue when you come to higher dimension. You see in mathematics the progress is done in the following way.

That once you are built a very basic theory like that of calculus or one real variables, then when you build theories in higher dimension, you want to mimic properties of one dimension in higher dimension. And you think that is those properties are carried over to higher dimensions and naturalization in general that is the way of thinking that pervades all of mathematics that okay.

If I am trying to generalize something to higher dimensions, the most important and elegant properties must carry on to higher dimensions. Of course, there will be certain properties which will be very intrinsic to this thing, which will carry over to higher dimensions, but, the most elegant properties which would appear to be free of any additional structures have to have to have that property that it is being carried over to the higher dimension that can be carried over.

So, can what should be the definition of the derivative of functions operating a function. Now, if from R into R my question is, how to define that derivative. How do we find the derivative? So as our differentiable function is not necessarily continuous is continuous which means necessarily continuous sometimes you might find the language of math becoming very.

So, how do I do it? Let me show you how to do it there are two ways one is the geometric way one is the individual way. So we will use first one in 2d way which I liked very much and then we will go to the geometric way.

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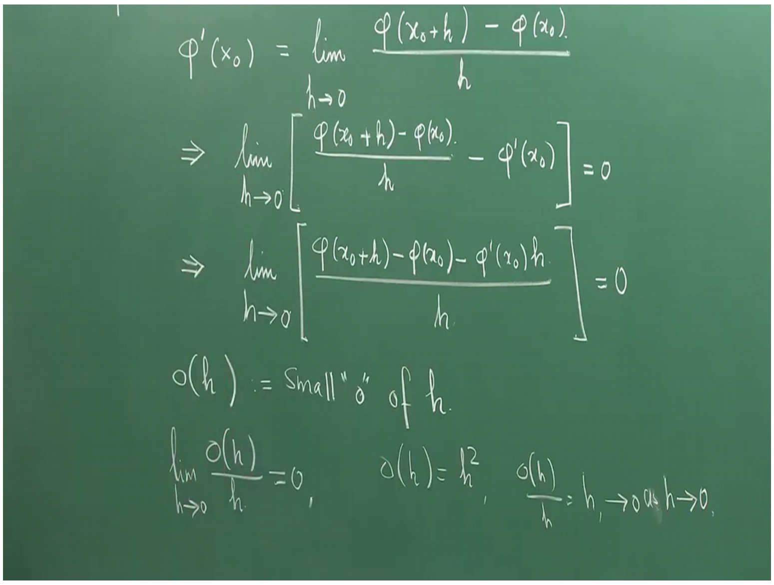

So, if you go to the definition of function is supposed to function phi. From R to R and then I want to talk about the derivative. So, derivative at a given point x naught. That limit of h going to 05 of x naught plus h minus five x naught by age this is zero and this limited existence about this limit exists. Now, what happens, what happens is the following you know that, if I write limited of x tends to zero limit of phi X naught it will become phi X naught because this is just a number. So, I can write this in a slightly different way. So, this would imply that limit h tends to zero.

Phi of x naught Plus h minus five of X naught by h minus phi dash x naught is zero because you would take the limit here this limit to the phi dash x naught and this because it is constantly claiming to be this is all we are also the me to define phi of x naught so they will cancel up and gives you just a rearrangement this also implies that limit h tends to zero phi of x naught plus h minus phi of x naught minus phi dash x naught h divided by h.

That goes to see you Now, what does it Say what does this story tell me? The story tells me something else it tells me which are right on this side now, because I am sure that you have noted down what I have done. So what does this still, this tells us that h is also going to zero, but there is a numerator and the ratio is also going to zero, which means that the numerator must be going to zero faster than h.

Otherwise it will blow up. So the numerator goes to zero faster than h. Then we say that the numerator is a small O of h quantity. So what is this? This is a function of h because excellent is fixed. This is a function of h. And this function divided by h is going to zero so this function goes to zero faster than h. So we define something called all h. O of h is a function small O of h.

So, these are function which satisfies this property O h by h as limit h tends to zero is equal to zero. As the example I am giving you all h. O h is h squared. Right? h square because observed that O h by h is the golden h, and this goes to zero as h goes to view So this is very, very clear that point that it is naught very difficult to construct OH or functions which vanishes go runs to zero faster than H.

See, what happens look at just this function, so h squared is smaller than h when he is very near zero and that is what it is going to do Faster, just look at the graph of y equals x squared. So, now so by the definition of a small O h, what do I get?

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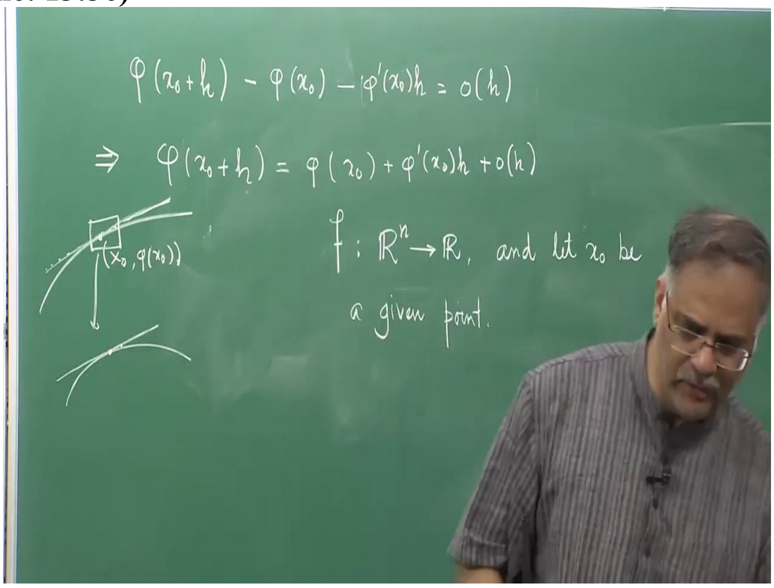

I get that phi of x zero plus h minus phi of x zero minus phi dash zero h is either small of h one d t that is what I get from this definition of Small o h. So, this quantity was very small or which quantity. So, this implies I can write a derivative and we now defined as a number which satisfies this condition what does it say that as the error as h goes to zero the error that you have between this and this these are the kinds of errors.

So, this phi x naught plus h is equal to phi x naught plus phi x naught h this is actually if we forget this, this line actually is the tangent that takes x naught phi x naught. What we are telling that when several he is very small, then functional allowed x naught plus h can be very well approximated by this tangent line basically tells you this fact. So, this is it this is x naught phi x naught then if we take the tangent line So if you are very near magnify the zone, we are very, very near x naught, this this point, then you tangent line value and the value of the function does naught differ much there is less.

So as you move towards this point there becomes smaller and smaller. That is, exactly what it says. So, now, how can I use this idea? This thing that I have written this can I consider this is a definition of the derivative Yes. Now, if I consider this as the definition of the derivative, I need to go ahead and define the definition of the derivative in that fashion or what I will try to do is I will try to define the derivative in this fashion.

Let me see how does it work? Now I will say take a function f from R n to R and let x zero we are given point so this is with this reminds I am trying to define the derivative at the given point so the here I will keep it to write it down so that you have it through out.

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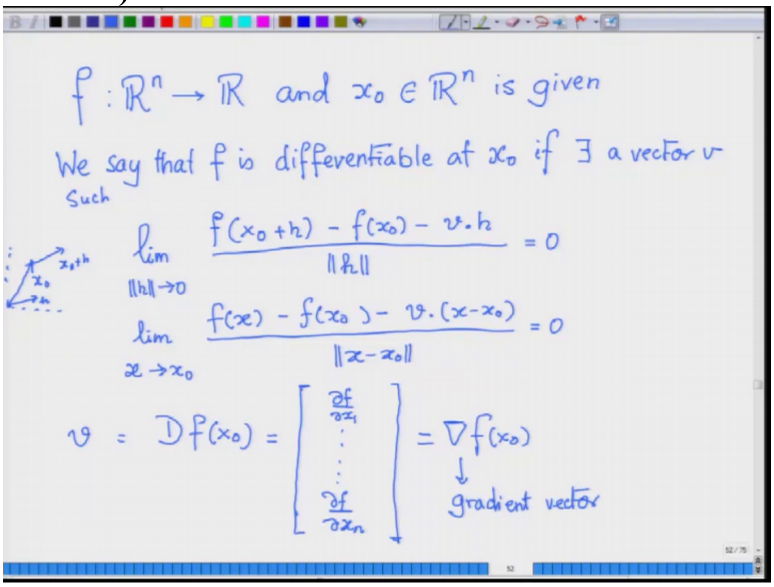

So, f is from R n to R an x naught element Rn is given. We say that if f is differentiable at x naught if differentiate of x naught if the limit of norm h 10 into zero. Now, h is now a vector. It is a vector variation, right? You move from external from one vector and you move along the direction of the vector h. So here, x naught Just looking at in our area of extraordinary move in this direction this is if this is your h.

So, you move in the direction excellent come here at x naught plus h right. So, this is what you have now, f of x naught plus h minus f of x naught. So, what is the derivative? So let me now wright down. If these are some sign of their exists in mathematics if there exists a vector v such that limit of v dot product h you know multiplication here I was multiplying the derivative with h here of the derivative is multiplied with h but now, when I come to higher dimension.

The naught ion of multiplication gets changed to dot product which is the most natural generalization So we put the dot product so it is a kind of intuitive step that we are taking, you know, it is not kind of rigorous way of defining just to we are telling the world we are trying to look at that, that definition that rearrangement, and then we are trying to play it out here these divided by normal of h this limit should be zero.

So, we say that our function is differentiable, if you can find a vector v for which this is true, you can also write it like this vector extending to x naught f of x minus x naught minus v dot x minus x naught h is now x minus x naught physically avoiding x naught plus h It is omics phenomena normal This is norm x minus x.

There is a way of defining a very general way of writing this naught ion of a derivative of symbolically the derivative is written as v is d of f of x naught DFX naught the derivative V is a derivative at x naught. So you can write v x and I am just writing it because you know that v is only for x naught here.

Now, if you know that the derivative exists, you can show that this is nothing but the following vector which has as its components to partial derivatives for them derivatives components of vector. If it exists, if selector exists then it must be the vector of partial derivatives. This victory sometimes also called a gradient vector of f at x naught. Now, f a function from Rn to R.

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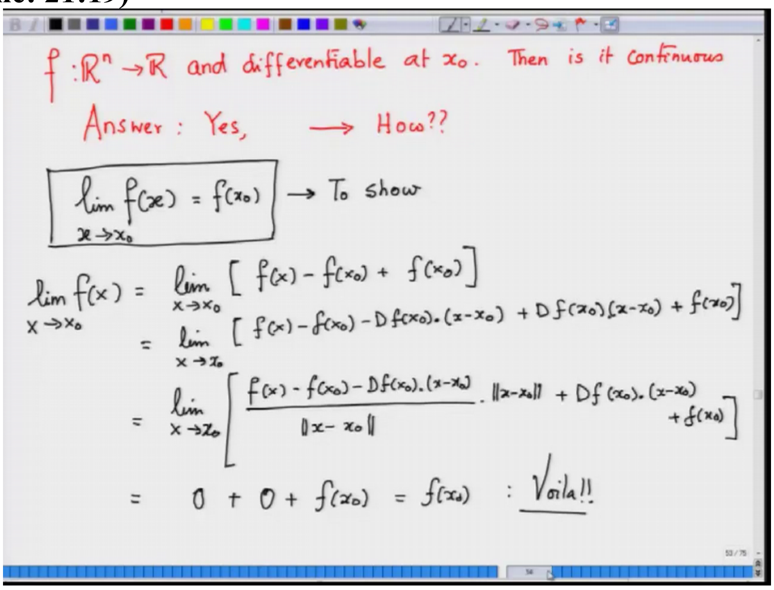

f is a function from Rn to R and differentiable at x naught that is a derivative exists. Then is it continuous that is my question? Then is it continuous the answer Yes. So, you ask how? let me show you that how. So, we have to show that limit of f of x as extends to x naught must be f x zero for x let to be continuous at zero, this is what we have to show. This is to show.

So, let me start with writing computing this limit f of x, x going to x naught limit x tends to x naught a for x minus f of x naught plus f of x naught is equal to limit x tends to x naught a for x minus f of x naught minus d f of x naught dot product x minus x naught plus d. So, you are adding and subtracting this term dot product x minus x naught plus f of x zero. So, then you right limit x tends to x naught f of x minus f of x naught minus d f of x naught dot x minus x naught Divide by norm x minus x naught x is not equal to x naught.

So, multiplying and dividing and multiplying x naught plus d f of x naught dot x minus x naught plus of x naught. Now, you got to understand when you take the limit, this one becomes the derivative which is just a fixed vector, because the definition of the derivative when you run the limit, this limit is zero and this is also the overexposed two x naught so these both the limits to be equal to both the limits, this is zero and this is zero,

So, this whole product is zero plus, here as x goes to x naught this will become zero. And because inner product itself is a continuous function of x, it is actually a linear function if you look at it very carefully, so, this will go to zero. And what we have left with is this constant fx naught so, this is f of x and Voila, we are true as a friend would say, Voila we are done.

So, which means that we have actually defined the derivative in a way which maintains the main question that okay if I have a differential function is a function is a function of revolving differentiable x naught then is it continuous at external The answer is yes so, our definition does make sense.

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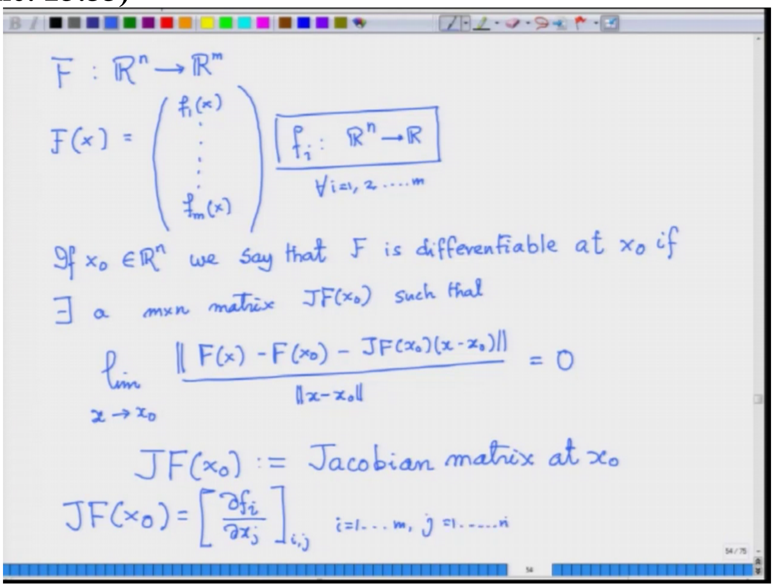

So, to keep into keep in view of so much discussion let us consider a function f from R into R because we were already talking about Vector valued functions of vector valued variables means you are taking a mini variable function which takes a vector and maps it to another vector it does naught map into a scalar. So, in general f can be taught to have n components if I evaluate it as x m components and each of them are viewed as functions.

m functions from Rn to R. So here phi i is a function from Rn to R this is something to remember is true for all i Okay. Now, once I have done that, how do I define Can I do the imitation of the definition we have for Rn to R. The answer is yes. But here, the thing would be slightly different. The derivative would not be a vector, it will be a matrix. If x naught is element of Rn.

We say that if is differentiate at x naught if there exists of m cross x in matrix jf x naught We went in the same sign such that limit x goes to x naught the norm of because now you are been because upper level is see if you look at this we look at this definition when we are used for function from R into R top level is really Lama the Dalai Lama so devoted to real numbers, but there is nothing like dividing two vectors.

So, what we have to define in is in this form that you have to take the norm of upper and lower level. So, capital F of X minus capital F of X naught minus JFX naught this vector this matrix multiplied with a vector x minus x naught normal this divided by the norm of x minus x naught we have already learned about norm in the previous section so, you see how useful those naught ions were in a productive norm and all those things.

This is you this mapping this vector GFX naught is often called the Jacobian mapping the Jacobian mapping or Jacobian matrix at x naught and how does what are the elements of the Jacobean matrix. So, the Jacobian matrix at x naught consists of elements of this form, they will have a Del of fi Del of xj. So, the IJ element is of this form. So, I is running from one to m and so it is on from one to m and j runs from one to m.

That is what we have launched. So, this is the basic understanding of the naught ion of a derivative of five actions of more than one variable. Now for the remaining part, I will talk about the geometry. So, let us see geometrically can we think about functions of when it from the geometrical idea, can we do get the definitions that we have got? If you look at the definition of geometry definition of the derivative of a function of one real very well.

The geometry definition is that it is the slope of the tangent at the point you are where you are competing the derivative. So, if you are competing the derivative at say this is phi x. This is phi x. And suppose you are competing the derivative at these x naught and this is x naught phi naught completing the derivative of x naught and the slope of the tangent.

This one is this this is the slope, this is Theta tan of theta then sorry, here and tan of Theta is phi of x naught that is the interpretation that is the meaning of the rivet. But how do I get put this idea into work when I am in higher dimensions. So, he let us look at a function

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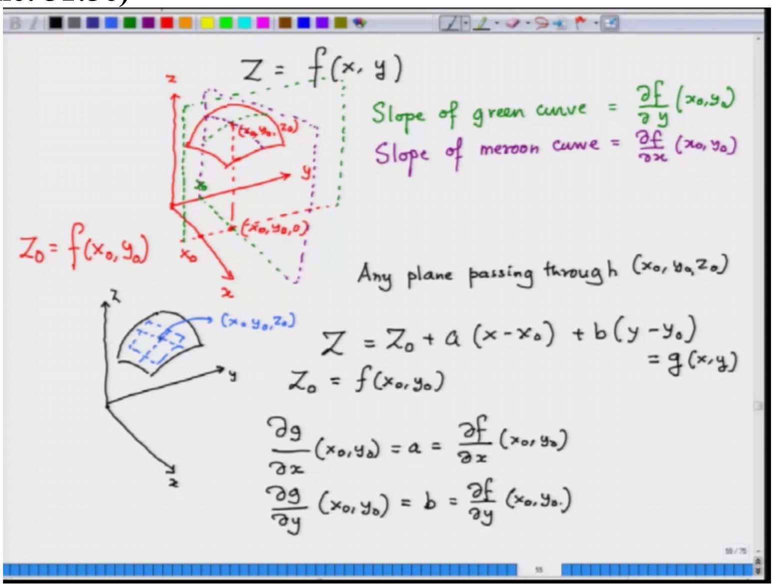

Z equal to f x y and let us try to imagine a car ride try to draw the surface. Try to draw the graph Let this be the surface it let it look like this. This is your z called fxy. Now take a point x naught y naught. They are naught this is naught y naught zero and this is a point x naught y naught and some z naught where is it not is of course, f of x naught y naught Now, you fix your x naught this is the x naught and what do you get?

Basically you are drawing a plane, which is cutting perpendicularly the xy plane and passing through x naught. It is kind of a plane like this. Control the plane and different colour for the plane is passing through is cutting Similarly, I can think of a plane which will make y naught fixed and his passing is cutting the xy plane or in a perpendicular way while passing through the point y naught.

So, basically I can think of our kind of core. So, the slope of the tangent right, the slope of the tangent slope of this line, when x naught is fixed, slope of the green curve, green curve slope of green curve is nothing but the Del f of y evaluated at x naught y naught Because you are extranet escaped fixed slope of the maroon curve slope of Maroon carve is equal to Del f Del x evaluated at x naught y naught.

So, here is a geometrical explanation, of the partial derivatives. Now, how far we can take the geometry explanation, suppose now we draw a tangent plane passing through x naught y naught That is you imagine something like this z axis x axis XSCV think of a tangent plane at this point the point x naught y naught z naught then the tangent lines passing through x naught y naught z naught which are actually tangents to the curve of green curve the maroon curve.

Then the slopes of the green curve and the maroon curve and these tangent lines would coincide this is a very reasonable geometrical factory just you can visualize this. So, now, what is the expression of a tangent plane what is the expression of a plane passing through the points that you have been passing to the point x naught y naught z naught y naught is therefore, it is naught y naught right this is something.

So, any tangent plane be given as follows. So, naught tangent plane the plane. So, any plane is a tangent plane in this case any plane passing through x naught y naught z naught will have an expression z it called to z naught plus a x minus x naught plus b y minus y z naught even nothing but a for x naught y naught that you know they did not is though it is naught y naught they are not disappoint lying on the surface.

What is a and b that is what we have to find out. Let us call this function as g of x y what we said that the slope of these Functions under the slope along x direction and y direction that is a partial derivatives of this function at x and a partial derivative of given function x naught y naught y naught should match that is Del g Del X at naught y naught which is a should match Del f Del x at x naught y naught.

Similarly, Del g Del y at x naught y naught should match Del f Del y at x naught y naught. So, the tangent line can now be expressed.

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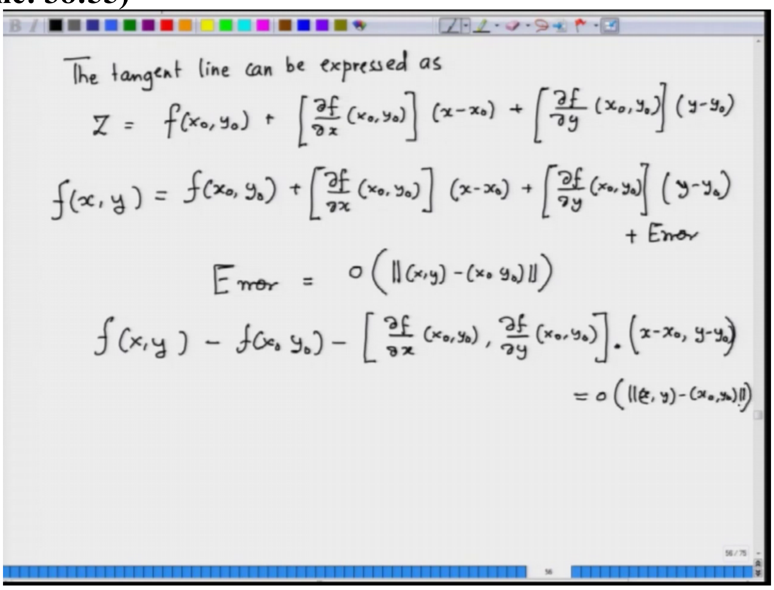

as z is equal to z naught y naught plus Del f x at x naught y naught. So, here that you see here, this what i right here is a dark product that I just wanted to let you know. So dot product with x minus x naught plus Del f Del y dot y minus y naught y naught? Now, you see if you want to define the derivative, so, this is a linear approximation which says so, the value of the tangent.

When the distance between x and x naught and y and y naught there is very less and then x y and x naught why naught is very less than the value of the tangent is very well approximates f of x and f of y. So, in general I can write f of xy is f of x naught y naught plus Del f Del x naught y naught dot x minus x naught plus Del f Del y at x naught y naught see this all evaluated x naught why naught? See all the all the vectors etc that we have done here before everything here is evaluated at x naught y naught.

See, here it is x naught, the vector x to Rn and basically, so we are just doing it for two very will see when I am doing the geometry, for two variables, right. So I am just trying to explain in terms of two variables, this y minus y naught plus error? And how should be that error? That is the question. That error should be in the following form.

That error term should be that when x is x going to x naught and y is going to y naught the norm x y is going to be more x, y is going to x naught y naught? Then this error terms would vanish very fast, right? So it is a kind of term in general should be O small quantity of norm of x y minus x naught y naught that kind of quantity should be there.

So, you see if you look at this, so if what I do if I take f x y minus x naught why naught minus this thing. So, if I look at this thing, sorry is naught about product, this is just x minus x naught making a mistake, because these are exes are just two variables. So, now I make a mistake, please have a look. This is just a real number A into x minus x number being two y minus y naught x, y is the vector x and y are real numbers.

So, these are not caught products, right? These are just y into y naught. This is this into the y x and y. These are not vectors. So, I am confusing. I am just thinking as if I am writing in higher dimension which I am so habituated to, though, so these x minus x naught and y minus y naught is naught a real number because we are in two dimensions. So, x is real number y is a real number forming the coordinate xy.

So, here if you look at this what I have if x and y, if I write like this I can write minus x naught y naught Minus if I look at this thing it is a vector del f del x x naught y naught del f del y which is nothing but the gradient vector x naught y naught inner product with the vector x minus x naught y minus y naught there is exactly the definition and this is nothing but O of norm x y minus x x naught y naught.

So, you see from the basic geometry of the tangent when the tangent line is a very good approximation to the function value for all xy near the top point x all x y values near the point x naught y naught. So, that with that idea, you see, we are bringing we got get back the same definition of the derivative, which we have bought intuitively by doing certain algebra functions or one variable.

So, with this very simple geometrical approach this gives us exactly how our derivative should look like in two dimensions, we get so, bored the cases we get back the same form of the derivative which is a gradient vector and does this gives us a very meaningful way of doing the things and with this idea, now, we can proceed further which will come we will talk about chain rules for partial derivatives in the next class.

As well as we will talk about many other things as we go on. We will gradually get into deeper issues like gradients and sorry directional derivatives and the relation between directional derivatives and the gradients and also about implicit function theorems and all sorts things. So, we are gradually getting deep into calculus of function of many variables, you see.

So, this is a very interesting the whole geometrical thing can come and match with the intuition that you already had from which you define the derivative and you said that the definition of the derivative we what we made was very good because we could prove that we that definition of the derivative of function is differentiable at a given point, it is continuous at that point. So, you see that we have been able to maintain this very important property and hence why what we did is meaningful. Thank you very much, hoping to see you in the next class.