Calculus of Several Real Variables

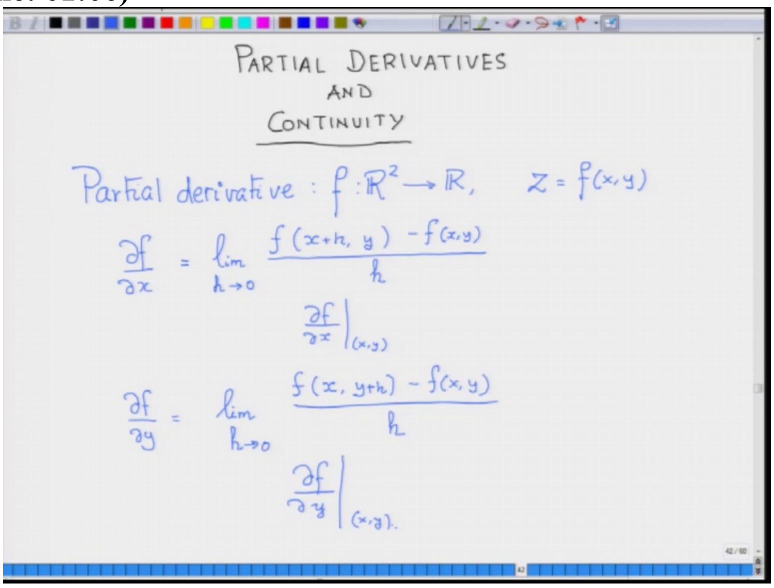
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology – Kanpur

Lecture – 08
Partial Derivatives and Continuity

Good afternoon. Welcome once again to the third talk of the second week. So here we are going to talk about partial derivatives and continuity. You might wonder that when I spoken about functions, as you have learned in the case of one variable calculus, I should talk about continuity of the functions and properties. But here I want to talk about derivatives because derivatives were always spoken about after continuity.

So, why I am speaking about partial derivatives, first means kind of derivative and then I am talking about continuity. The fact is that partial derivatives can be understood from the knowledge of a function of a single variable. For continuity of functions of more than one variable needs the kind of jumping imagination and that can only come later on. Okay, so what a partial derivatives.

(Refer Slide Time: 01:06)



So, let me define a partial derivative. First I will define it for a function of 2 variables. That is all considered a function from R2 to R and I write, this function as Z is equal to F and x y. So, when I am talking about partial derivatives, I am essentially thinking that this variable y is constant we can put some number in place of y. And then you take the derivative, thinking that y is constant to take a derivative with respect to x.

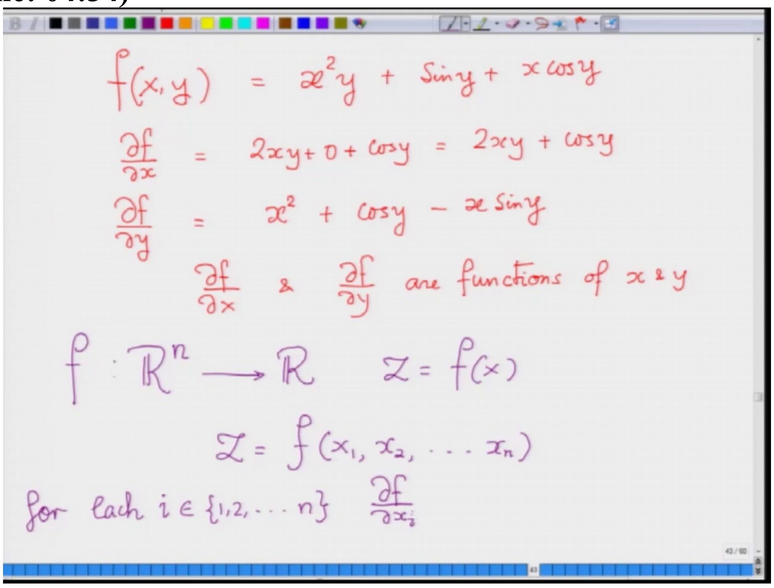
Because once you think that wise constant, if x y becomes a function of y and as a function of x and as a result of which you can easily take the derivative which you know. So, you can do the same thing with x, you can fix up the x and then take a derivative with respect to y. So, when you take a fixed y and take a derivative with respect to x, we call it we say we have taken a partial derivative with respect to x.

And dollar one the partial derivative with respect to y. So, how do I do it? So at a given say point xy the partial derivative of f with respect to x is given as the limit of F, so I am just changing the x and y is fixed x + h y at the point xy. The partial derivative is given as this partial derivative with respect to x as a studio. So I am using my knowledge of one variable calculus.

So, of course, you can say x 0 y 0 and all those things to calculate that at that particular point. So, if you want to be more precise, you can write this as Del F Del x evaluated at the point x y. See on to evaluate the partial derivative with respect to y. So the here we so what we can do, it is a kind of partial differentiation. So, you do not bother what the nature of the full function but bother about the nature only of the part of the function.

So, you fix one variable vary and keep all others fixed. So if you want to talk about n variable now, so you kept say you want to take n derivative with respect to x1 x2 to x and you can get fixed and then do it. So limit age tends to 0. F of x is now fixed y + h - F xy divided by h and often we will write this as evaluated at the point Del f Del y evaluated at the point y. So, this is the way I would like to express just I have learned about function of 2 variables could did opponent for example, if I take

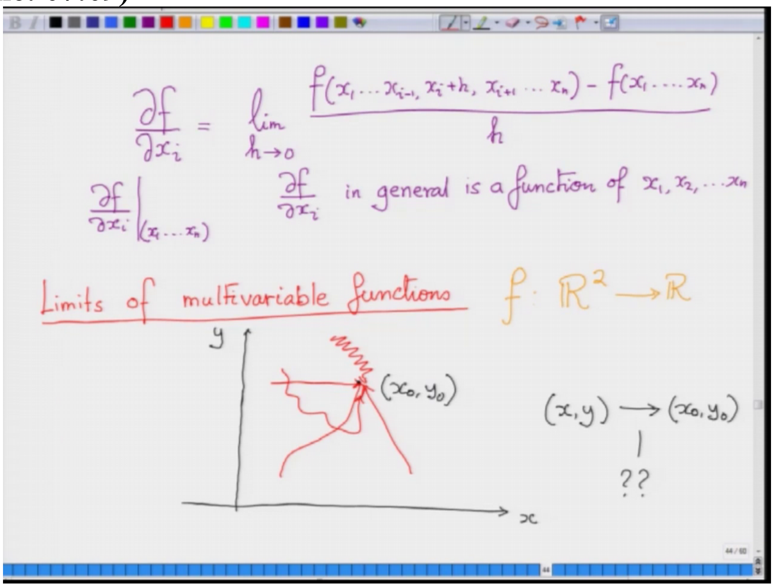
(Refer Slide Time: 04:34)



F x y if x square y + sin y + x cos y. If I do it, then how what is my first derivative So, del f of del x elevated at any point xy, so first I think that y is fixed. So I left to have x into y here. And here there is y is fixed. So this constant so + 0 + he had just had to take a derivative with respect to x, so it is cos y. So, this will give me 2 x y + cosine of y. Now if I take derivatives respect to x. Sorry y, then the first case I will get x square because derivative of y is 1 + sin wise derivative is cos y. And here I will get - x sin cos y derivatives is - sin y keeping x fixed. So, this is exactly 2 derivatives you see both F xy that Del Fx and Del Fy. Our functions of x and y now I am looking at a function of more than one variable.

So, now I am looking at F from Rn to R. So, we write this as z is equal to f of x. Or if you want to be more precise, you can write this z is equal to f of x1 + x2 x1 x2 xn not + sorry x1 x2 xn now, basically, for partial derivatives for each i, each i from 1 to n, you can define a partial derivative Del f, Del xi. So how do you do it? You will just define it in the same way.

(Refer Slide Time: 07:09)

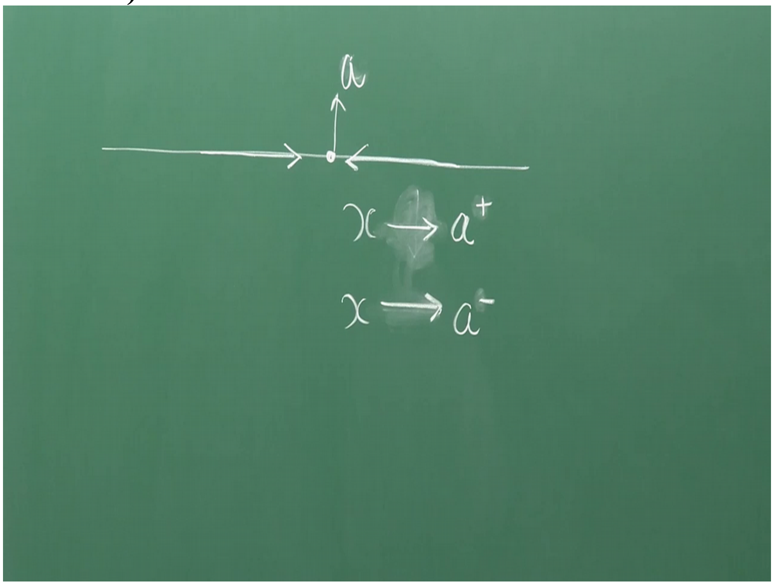


That Del F of Del xi is equal to limit of h tends to 0, f of x1, xi - 1, xi + h. I am just moving the xi as others have kept fixed xi + 1 xn - f of x1 xn divided by h. So of course, you have to understand that Del f Del xi is evaluated at the given point x1 to xn and Del F Del xi in general does in general is a function of x1 x2 and xn. So, this is basic idea. So, that is already we have studied this kind of things.

Now, we are going to talk about limits for function of n variables, limits of multi variable functions. What do you mean? This is a very key issue and I spoken when I give introduction to the subject I told you that this is the key fact which will differentiate our game in the one dimension as compared to all game in the higher dimension the games that differ on this issue. Limits of multi variable multivariate functions, multi variable function maybe I should write that. So, when I am talking about limits of multi variable function, let me first look at a function from R2 to R that f is from R2 to R and then let us see what is the meaning of limit in that space. Consider in R2 there is a point given x 0 y 0?

What do I mean by the expression? xy tending to x 0 y 0? So, my question is what does this mean? This is a very key question and this is this needs to be understood, because without understanding this idea, nothing else can be achieved, you have to understand on the plane on the board that suppose I have a point here on the plane and I have a point on the real line.

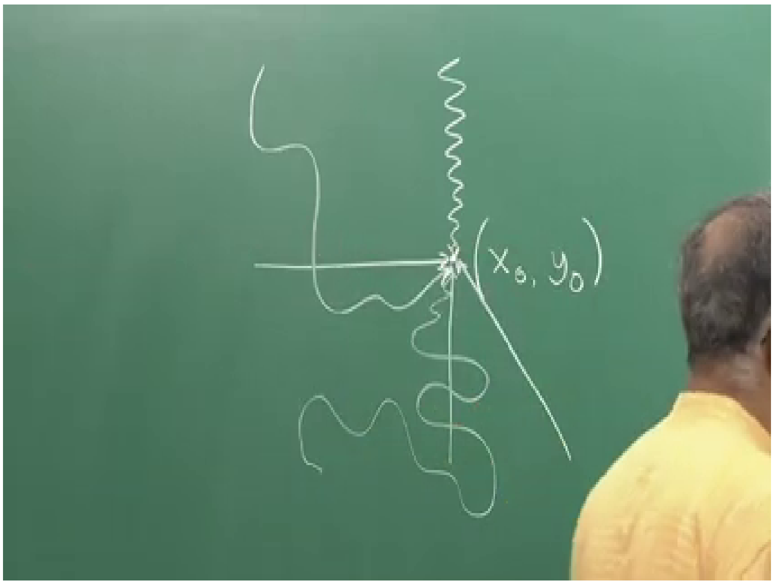
(Refer Slide Time: 10: 45)



Now, when you are on the real line, your degree of freedom is only 2 because either you can approach the problem from the right. What you can approach the problem from the left approach the sorry, problem; I am sorry, approach the point. So this is my point x or for a point A, then x either approaches it from the right, A + or x approaches it from the left, except process. So either approaching it from the left or it approaches from the right, we do not have I do not like the symbolism. One can say x approaches from the right, when is goes down.

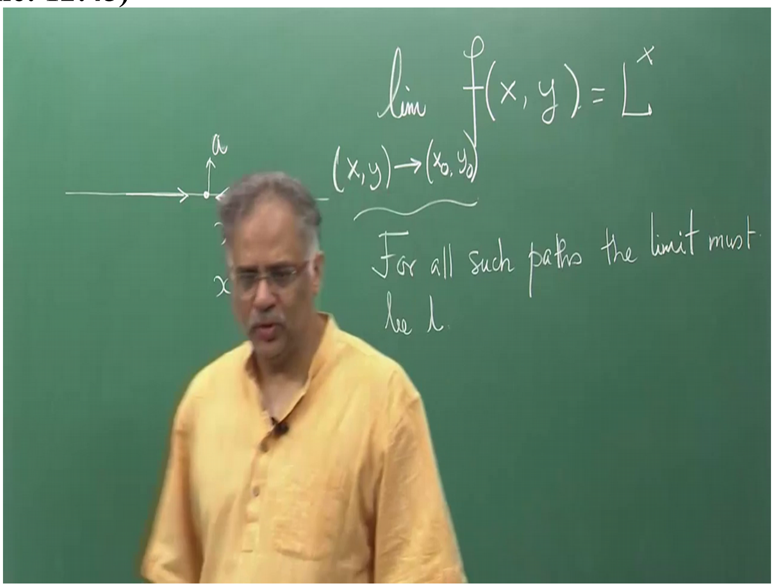
Or every that is a better symbol xa+ I sometimes get confused with symbols. I do not bother and not got too much used to standardizations.

(Refer Slide Time: 12:04)



But when you come to a point on the plane, just like what I thrown in the board x 0 y 0 your freedom of movement towards x 0 y 0 is infinite not only that, it is countably infinite in the sense again come like this again come like this again come like this. I can also come like this I can also come approach it like this again come approach it like this.

(Refer Slide Time: 12:45)

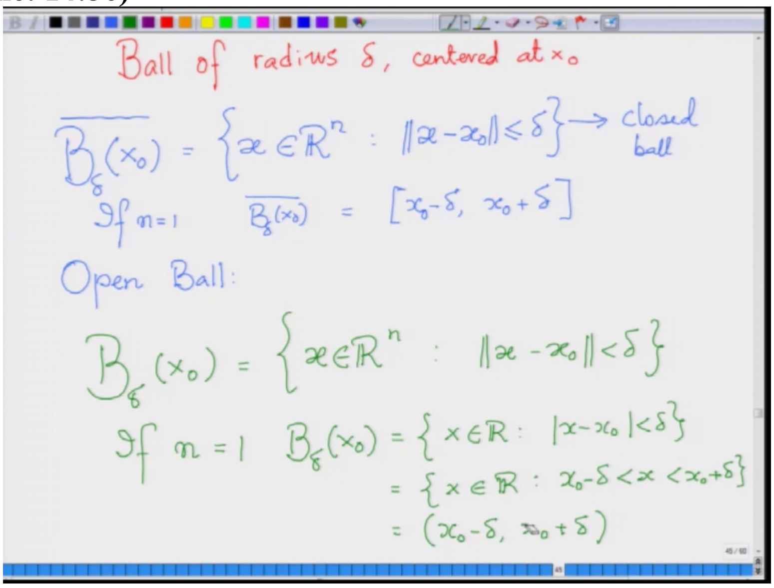


So, when I write the term function of xy limit when xy tends to 00 we need not bother with it the function is defined that x 0 y 0 but the function value tends to some place then you have to understand. If I said this is equal to L, then it means whichever path I approach, x 0 y 0 for all such path the limit must be L. If there is one path for which the limit is not L and we say that the limit does not exist. So, for all such paths, so, remember this means for all such paths limit must be L.

So, here as I would like to redraw this thing again, so many ways of approaching this point this way straight in this way when a zigzag way whichever way or maybe this way whatever.

Now, we would formally define just like the epsilon delta definition that you know for functions of one variable. Similarly, we can see epsilon delta definitions of functions of more than one variable and for that we need to know how to make a generalization of the delta neighbourhood that we continuously have used when we spoke about function of one real very well you have a look at the one real variable course. So, then let me just go ahead and define what is called a ball of delta radius.

(Refer Slide Time: 14:56)



So, ball of radius Delta centred at x 0. What does it mean? This way right as ball of radius delta so, will symbolize like this is given us the set of all x in R n we are not talking in terms of RN such that the norm the distance between x and x 0 must be less than or equal to delta. This is called a closed ball is this sphere with everything is stuffed inside the different 3 dimensions, closed ball.

So, we are basically writing everything algebraically but bringing in the imagination, what we have called till the third dimension we bring it up. So, that is leaving imagination. So, if you just have one variable here, so, mod of x - x 0 is less than delta x is line between and x 0 + delta x 0 - delta. So, if n is 1 and B delta x 0 is nothing but x - x 0 - delta to x 0 + delta.

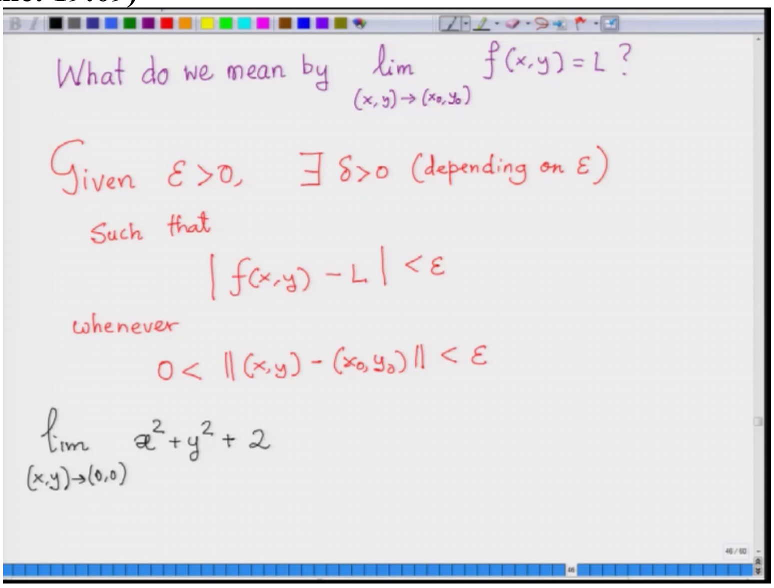
So, that is kind of important that terrorists you understood that these are 2 generalization. So, what we need largely is the open ball and the open ball is something like this. Just you do not have the boundary of the ball. So, it is all x in R n says that the norm of x - x 0 is strictly less than delta. So, if n = 1, then it is very important to understand then what would happen.

Then this B delta x 0 would be nothing but the set of x in R and normal becomes modulus when x - x 0 is less than delta or it is nothing but the set of all x in R. such that x is line between in x 0 - delta and x 0 + delta. So, this another way of writing is this is x 0 open interval x 0 - delta 2x 0 + delta. So, you see this open ball and closed balls Rx exactly generalizations.

Of the Delta closed interval and delta open interval right. This is the Delta closed interval and then this is a Delta open interval. So, this is what we have summarize now, this idea of the ball because it because a Delta ball was required to define the meaning of a limit these acts as a kind of neighbourhoods to a given point, the same idea is open balls and closed ball will act as a kind of neighbourhood in the higher dimension.

So, let me go and write the definition and the general level let me just not bother about 2 dimensional thing. You can do text wherever they will just bother about 2 dimensional things that I will first bother I write it in R n and then we will see what we mean by the 2 dimensional stuff.

(Refer Slide Time: 19:09)

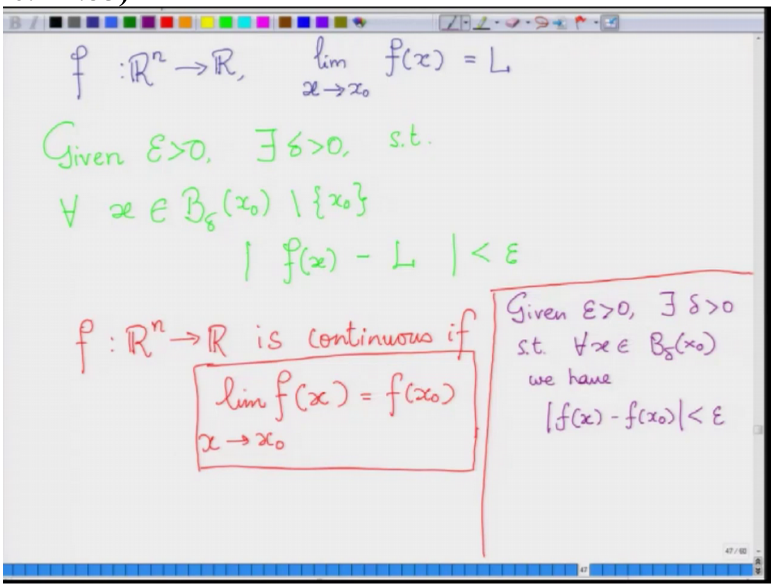


So, what do you mean by this statement limit xy going to x 0 y 0. F xy is equal to L. What do you mean by this? This means, given if epsilon greater than 0. So, this is what I am writing in R2 first and I will read in Rn. So, given epsilon greater than 0, there exists a delta greater than 0 Depending on epsilon such that the mod of Fxy - L is less than epsilon. Whenever xy is not same as x 0, y 0 and lies in the open ball. So, for this I have to use the norm because we are in 2 dimension.

We do not have just mod here we are dealing with the real number F xy is real number right. So, how do I do this? So, given this is what happened this is the definition right. So, if you for example try to find a limit of find the limit as xy tends to 0. x square + y squared + 2 of course with inside you know, it does not matter whichever road I come to 0, these x square +

y squared dollars be 0. So this answer will be 2. So, this is the kind of ideas that you can use. Now let us go to higher dimensions. Now when I come to higher dimensions like

(Refer Slide Time: 21:53)



F is from R n to R. How do I speak about continuity sorry, how do you find talk about limit? So, what do I mean by limit x going to x 0 y is a vector in Rn. x 0 is a vector in Rn. And that limit is L. These second means the following given if epsilon greater than 0. No matter how small in read my undergraduate teachers solve use this term let epsilon be given if epsilon greater than 0 be given and no matter how small that sounds still you know a booms in my ear he had a very booming voice and booms in my ear, so I have got also got this I also started using this thing that led epsilon greater than 0 be given no matter how small.

Once epsilon greater than 0 is given, we can find an existence of a delta greater than 0 such that for all x element of open neighbourhood B delta x 0 we accept the point x 0 this absolute value f of x - L for all such points. I have this less than if delta of course, different zone of epsilon, I have this fact when I am talking about the limit x 0 may 0 be defined that f itself, if so, if f did not have a definition of export.

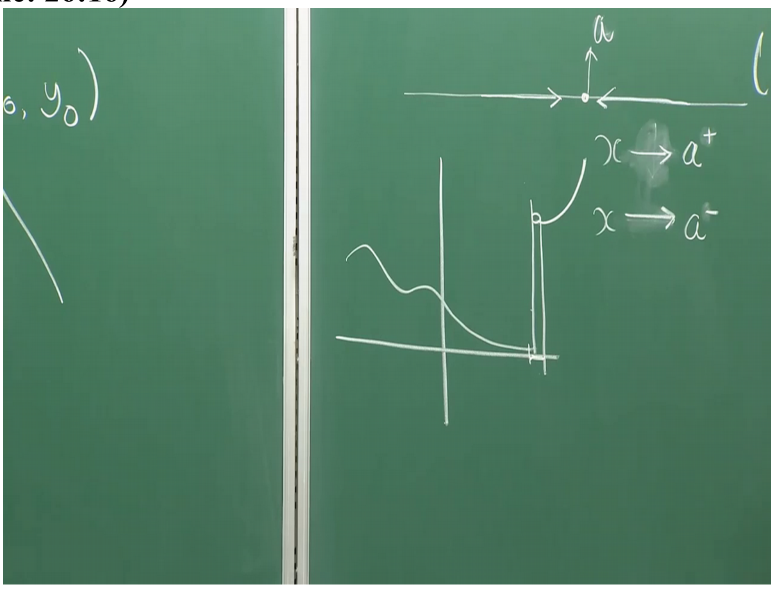
So, this is something we have to keep in mind if we are the definition of x 0 and you have that the limit itself is equal to Fx not then the function is continuous and that is exactly what we are going to talk about. So, function say f from R n to R is continuous if limit f of x, x goes to x 0 is f of x 0. In this case I am expecting it to be defined as x 0. If this is true and we say that the function is continuous, of course.

We can have an epsilon delta definition. Continuity simply means that if there is a small change in x there must be a small change in y if for a very small change in x as a huge large such and in y. I then the function is really 0 continuous continuity crudely means, a small change in x must give me a small change in y and small change in f. So, if you can, if you want to write down the epsilon delta definition of this.

We, just try out the traded epsilon Delta definition, I do not know why if you will read the epsilon delta definition, it simply says that if you give me a very tiny amount and say okay. If f changes by this amount, tell me the corresponding change of x that is required to affect this change in f. So, it is another water way of telling that the small change in f corresponding the very small change in f there must be a corresponding very small change in x.

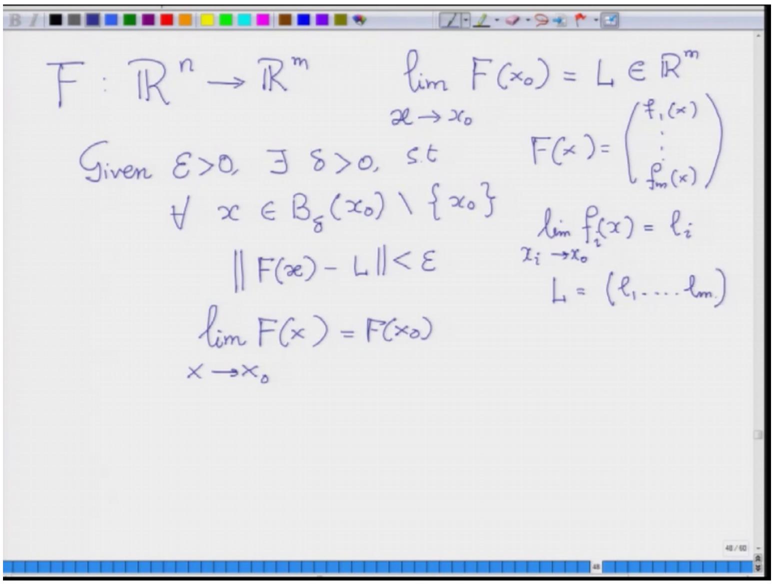
If the function is a non constant function so, this is something already given money already inbuilt in my thinking. So, that is the idea continued that you can really move without taking your pencil of the pen of the paper. So, you have a small changes made in x, the small change in made in y. there is these changes corresponding to some small change made in x it cannot be corresponding to some money big change except constant function. So, we are very small change in x is that hugely huge change in y so that for example will function like this.

(Refer Slide Time: 26:16)



So, for a very small change in x here in this neighbourhood there is small change within this neighbourhood function values have changed very large heavily. So, such things such as things present this continuity. So, this is something very important to understand. So, here if I write it down epsilon delta definition will symbol mean given. Epsilon greater than 0 there exists delta greater than 0 such that for all x element of p delta x 0 here x 0 x could be x 0 also because the function is defined as x 0, we f x - f x 0 less than epsilon know suppose instead of this if I consider a function f, which is a function from R n to R m capital F, then how do I go about talking about the limit.

(Refer Slide Time: 27:56)

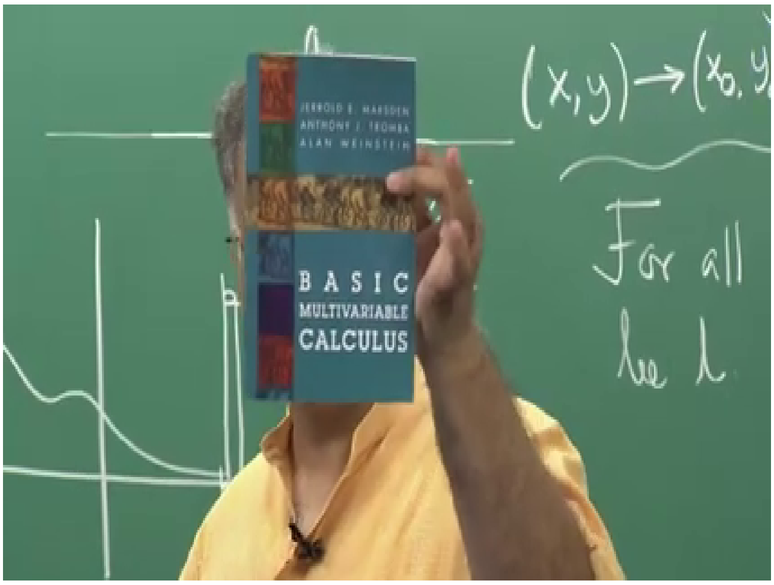


So, now, let me talk about capital F, which is a function From R n to R m and I say that the limit x going to x 0 F of x 0 is vector L in R m. What does it mean? It the vector function goes to some vector it will mean in a similar way a given epsilon greater than 0 there exists Delta rather than 0 such that for all x element of the B delta sorry x going to x 0 sorry, there is mistake please check it x 0 B delta x 0 and without the one x 0 for them the norm of F x - L the norm this is a vector.

So, the link the distance between F x and L must be less than epsilon. Similarly, when we talk about the continuity of this class of function which i am not writing down a thing bigger you can write it down for a continuity you also that same story will come limit of F of x, x goes to x 0 must be F x 0 that if must be defined at x 0. This is so you are seeking for a component. So when I am talking about continued at x 0 I am showing telly proving and whenever x is approaching x 0 from whichever path F of x values must move towards F x 0. The, Fx of vector must approach F x 0. Of course, you can look at everything from a component wise point of view. Instead of writing this you can look at F of x itself as made up of say imparts.

And if limit of F x sorry limit of Fi goes to x 0 is some Li. And, this vector L that you have formed here is nothing but L 1, L 2, and Lm. So, you can do things even for a vector function, but let us know for symbol 2 dimensions or to do all let us look at a few examples which must be using an epsilon delta type thing, a very unlikely example. So, here is an example which I am giving from this book.

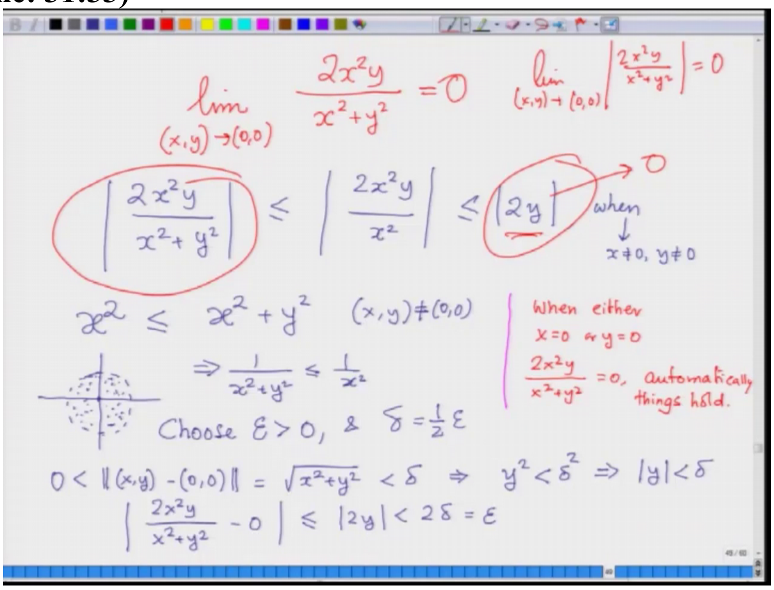
(Refer Slide Time: 30:59)



Which I can want to show you by margin Trumbo in Weinstein basic multi variable calculus, I again tell you that this book is available in Indian print. It is a very, good book and it should be in the library of maths teachers, as well as all those who are interested in calculus, they will really learn the subject is very well from this book and those who use multi variable calculus in the research scientists and engineers economics where specifically they should actually use this book.

So for example, I would ask if those who have access to MATLAB and all those things can actually draw this graph. So I am looking for this function of 2 variables.

(Refer Slide Time: 31:33)



So I am asking you to find the limit of x y going to 00. And, 2 x squared y, by x squared + y squared, it is a kind of very strange situation because here the upper part will go to 0 lower part will also go to 0. This kind of 0 situation, but I claimed that this limit is 0. So, you have to show that this limit is 0 I really have to use the silent technique. So, let us look at this expression 2 x square y by x square y square.

This I can write is less than 2 x square y by x square, this is the key fact. And then this is less than equal to mod 2y is the key fact because you know that x squared is less than equal to x squared + y squared. So, we are approaching 0 so, we are approaching 0 but we are not doing it in such a way that excess couple of y squared both are not 0 suppose. I am approaching x y equals to 00 but through all through the points where x and y are not 0 Of course, you can approach it to a point where x is 0 and wise on 0 by the y access. So in that case, I have to use y square and instead of access squared.

So whatever, so both are not 0 when I am approaching 00 from all the sides. So, this is valid or y squared less than equal to x square is valid whatever. So whenever I am very near, basically I am, here is my 00 point. And I have drawn a small ball of some radius around it is a delta. And when I am here and whenever x and y are non 0 that is x y is not equal to 00. Then I can always write this and this would imply, one by x square + y square is less than equal to one by x square, and I get this.

So now choose if silence to be greater than 0 and Delta to be half of epsilon. So, basically, what you have is that now look at all such x y is which are lying in this ball that is around 00 line this ball around 00 but not equal to 00 Of course, you can say what would happen 2 points which are lying on the y axis right. What would happen to points which are lying on the y axis?

For example, x is 0 and y is on 0 the function value is 0 anywhere is 0000. So, if you approach it along the x axis that is y axis towards 00 the and your y is some fixed value and your x anywhere 0 uniform values anywhere 0. So, if you approach along this y axis right, if you approach along this y axis, then the function values are anywhere 0. So, the limit will go to 0 if you approach along the y axis, now.

We have to investigate what is remaining; how we have to approach it along other at other parts, and in that case x is not 0 and that is why what we have written is true. So, now, let this be less than delta. And so, this is what if I do this, this is nothing but normal x y which is route over x squared + y square and that has to be less than delta. So, this implies that y square is strictly less than delta square.

Which implies this x squared + y squares and delta square which implies that mod y is less than delta. And then does that means that 2 have x Square y divided by x square + y squared -

0 is less than equal to 2 Ry, which is less than equal to 2 of delta and delta is half of epsilon. So, this is nothing but half of this epsilon. So what I have that 2 have x squared + y squared by x squared + y squared - 0 is strictly less and epsilon.

Whenever x and y is within this neighbourhood. Whenever is it is within our within this delta neighbourhood. I am having this. I am having this fact for all x y. Now this is true for all x y in the neighbourhood, can be very careful here all this is true for all x y in this neighbourhood but not for points for where suppose you have x squared 0. So, you will have yet 0. So, you cannot have one by y square less than one by 0 square and have that so far wall.

So here I am looking at all the points for which. So, you know for all the points which are lying on the y axis be very carefully this value is always 0. So, these differences anyway equal to 0 less than epsilon for whenever x and y xy is strictly less than delta and y is lying x is xy is lying on the y axis. Now for remaining points we have this inequality. So, for remaining points we have this equality.

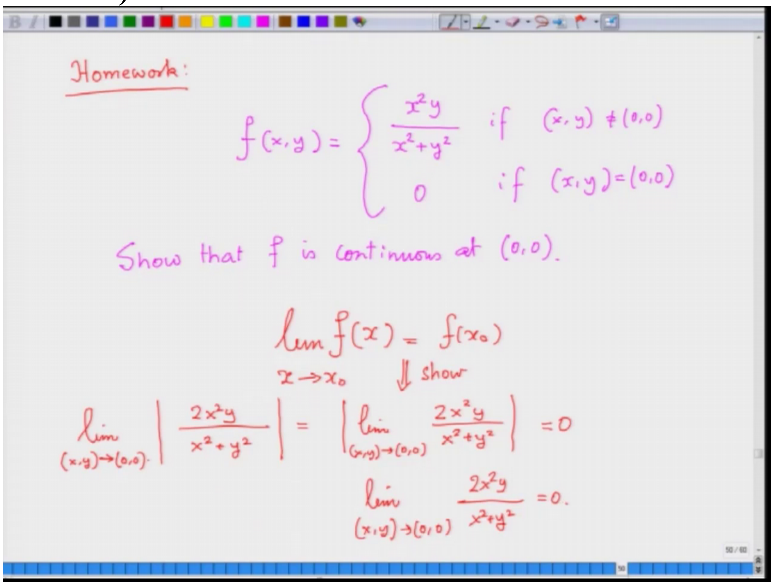
Right means in this case basically I want to say that x is not equal to 0 and y is not equal to 0. So, and for remaining points you have to run this show that we have just shown. So, far remaining other the xy that is also strictly less than epsilon. So, for the one which is lying in the y axis, this is anyway strictly less than whatever epsilon you choose for whatever However, for any point that is lying on the y axis this is 0.

So, does not matter whatever xy you choose it when it is within that delta neighbourhood that you have got your whatever Delta you have chosen for that also it will be 00 is anyway less then epsilon is greater than 0, but for all other points, where x and y both are non 0, right where x is at least non 0, the story will go through and hence, this is what we will get, right.

So, even if y is 0 for example, see this on the x axis, then also these will be 0. So, x axis and y axis basically you do not have to bother because in both cases the function value while he whether he x is 0 y 0 this is 0 it will be anyway less than epsilon for whichever be your delta whichever be the Delta in a neighbourhood, but for all other points, you have to choose the delta neighbourhood in this way.

And show for all such wants also this distance is an excellent. So, and for every xy in this neighbourhood, so, which now takes points here as well as the points on the axis, this is less than epsilon, that is what it means, and that shows that the limit is 0. So, here we have given you quite a bit of understanding of how things are working. I make you try as a homework as a homework maybe I will ask you to do.

(Refer Slide Time: 39:48)



A problem which I take from the book an example problem which you can try the homework. So, as a homework given to f of x y is x square y by x square + y square similar sort of structure you see xy is not equal to 00 is going to 0 if x y is equal to 0. So it means that this function is defined segmented way. Now I am showing that money you have to show that f is continuous at 00. One must be very cautious here again, when you look into this case, be very cautious again.

So if either x and y or 0, I am repeating these values anyways 0 always. So these distance this - 0 would always even 0 when our x and y any one of them are 0, when both are non 0, then all this missionary that we have done will work. So this is only when both are non 0 so when both are 0. We should write on the side when either x equal to 0, or y equal to 0 to x square y by x squared + y squared is equal to 0, and automatically things holds. That is it. So you have to be careful when you make arguments. You have to be extremely careful and you have arguments you have to see. First you check what happens on the x axis y axis whether to that x axis y axis.

Your arguments are all right, then you check out what are the other points. So here, this delta is actually for the points where xy both are non 0, because for the x axis y axis have short things are 0. So I am repeating it, because you need to be very careful when you are looking into this kind of things. I was just talking about y axis is when i was explaining it, but it was also true for x axis.

I just missed that point that I should have spoken out both x and y axis. So anyway, you have the thing right in front of you, you go through this example very carefully. So, very simple application of the epsilon delta thing. So if I am just to look at this thing, how do I decide? The only way I decide is in this way, so what I do I take the limit. Here only this is a function of one variable limit y goes to 0.

So what happens? Suppose I want to just calculate the limit. So I take this one and I know this is less than this. So, when I take the limit of this not be less than limit of this, but limit here there is no x there is only y. So, this because a single variable limit to this y, limit goes to 0. And so, limit of this. So, what would happen limit of xy goes to 00 norm of 2 x square y by x square + y square.

That would actually go to 00 and that would be exactly 00. And a very important result about limits is that this is a trick that you should always learn, because this is a fact about limits that if the mod itself is a continuous function, so, the limit and the mod can be changed. It, limit and the mod can be changed. So, you have put the limit inside. So limit of this mod of that will be 0.

So, limit of that would also be sorry these 0. I am a mistake not 00 is 0. So, it is numeric number 0. So, if you have a continuous function a very important criteria about continuous function is this. That limit F x right. extends to F 0 is F x0 the modulus of a function of a number of a real number is a real number is a continuous function and you taking the limit so, modulus of x, F could be the modulus of x, and it is a real number, right? It is a vector, right? x y is a vector, we are taking the modulus of that the real number, you are taking the modulus of that going to 00 this actually means the same as this is something important to remember.

So, basically, you can change the modules and the limits by the from the definition, because you can actually choose the definition of the continuous function. You can using this definition you can show that you can make this change. From here you can show I am not getting the details, but you can show that this change has been made. So if this is equal to 0, you have limit. So when you want to come to the limit by hand, not by a certain delta.

So, this is a real number which is equal to 0 moves modulus is 0, so it must be 0. With this, I end my talk here, and hope that all the things that I am trying to tell you here is folding much in easy terms of sequences, in our tour sequences in origin, we will have a chance to speak a bit about them. I do not know whether I really have a time to but then that takes me more into the direction of analysis. Rather into keeping you grounded in calculus. But let us see but just take it from me for the moment that you can really use this idea to show this thing and hence, you can compute the limit in this way. Thank you very much.