Calculus of Several Real Variables
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Lecture – 07
Function of more than one variable

Welcome to this lecture and here as I want to tell you that we are getting into you might say that the real analysis the real higher dimension analysis, we are going to study today function of more than one variables

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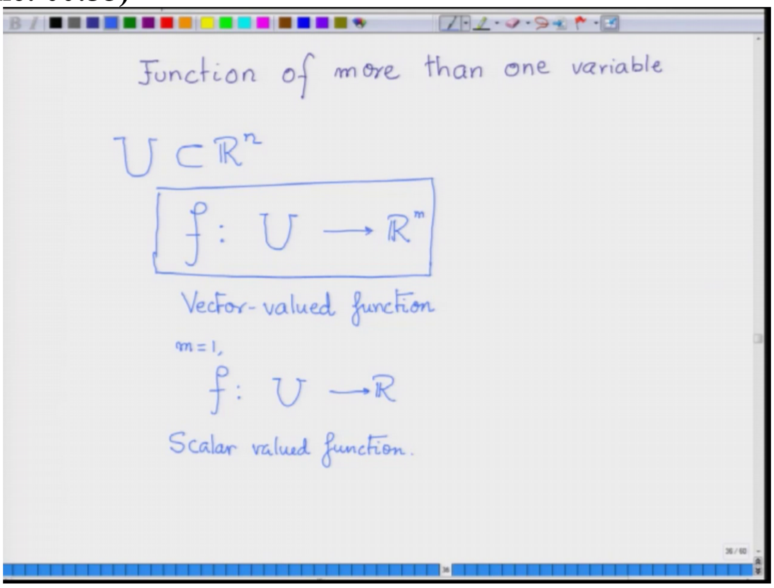

So, how do I do it, let us without getting into too many details? Let us start by actually very formally defining it is sometimes very good to be more formal, then start telling stories only to you or sometimes other ways better. So, you consider as a set U subset of R n and consider f a function which carries every element of U to some element in R m.

So, this is the most natural form of a function higher dimension and this is what is called a vector valued function Okay, and if m = 1 that is f is from U to R then we call scalar valued functions. I will give you 2 examples physical examples to tell you that what is what. For example, if I consider a 3 dimensional situation not 3 dimensional space in 3 dimensional space take any point xyz

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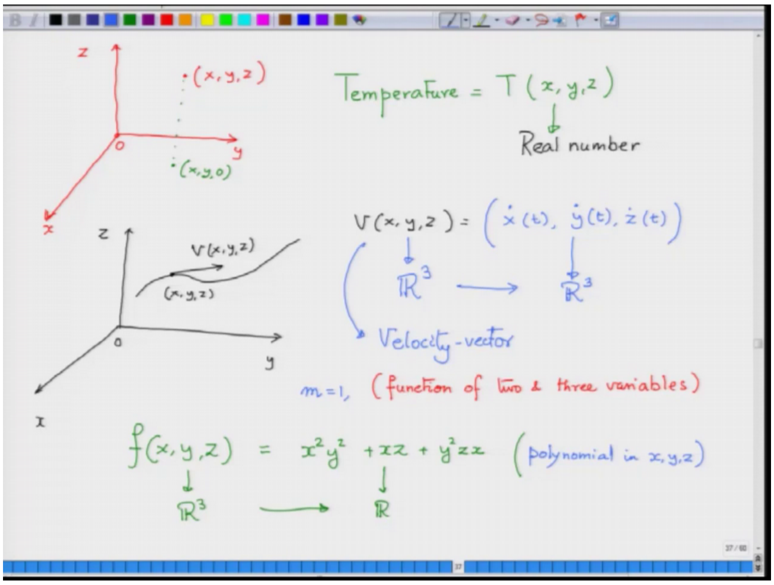

And then you try to see the temperature associate with it temperature T is actually a function T of the position. so if xyz is on the plane, say you take a kind of projection here, if it is the point xyz laid on the plane, the temperature is higher and if it is going up, the temperature is lower if suppose you are climbing the mountain. So this is a scalar. This is a real number. So this is a real number. So this is an example the temperature at any given point in space is an example of a scalar valued function.

So given xyz did not measure how coordinate point is not measured the temperature what about a vector valued function. We have already had an idea of a director world function when we started force when we started velocity, just to remind you let me go ahead drawing this. Okay let a particle move along carve then at any point xyz on the particle the velocity vector is tangential to it.

Of course, all these points are changing with time. And you know we know that the velocity vector V xyz is usually given as x dot t derivative with respect to t plus derivative of a function of one variable, y dot t and z dot t here is an example of a function which takes a point in R 3.And maps it to another point in R 3 so, this is the velocity vector. And so, if you have not written it down less written down this is a velocity vector is called the velocity vector represented as a function from R 3 to R 3.

So, here you have these two things very well said and very well done right. So, things are very clear now, so, let us take a very simple example, like if you want to talk about a function of 2 variables. So, let us just first concern ourselves with a case where m = 1 so, let us concerned with this function of we want to scalar valued function of 2 variables and 3 variables. Let us see that. So let us just take an example to see, okay, whether I can talk about something.

So when we see so let us take up functions f of xyz is x square y square plus xz plus y square zx. This for example, is a function from R 3.This is these are actually R 3 and this is a function from R 3 to R and these kind of functions or something names, these are not sometimes they are always called I am just skip of these are called polynomial functions polynomial in xyz and forms a very important class of functions.

Let us go over a bit and try to understand how do we represent such functions? How do we look at such function. Because if you have a function from R to R you know how to draw the graph of the function. So, consider a function Phi from real line to real line.

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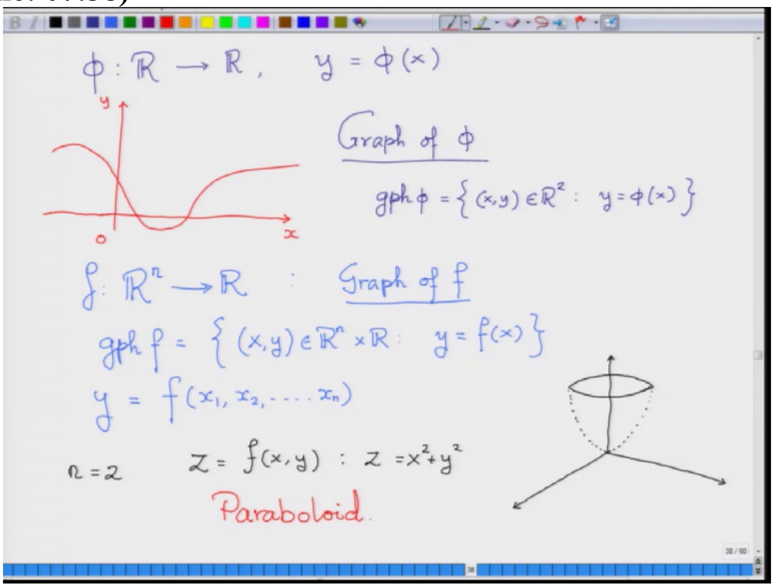

So, basically what how do you write y is Phi of x this is something known to you. So, what do you draw, you draw a graph of that function. So, you draw the x axis and the y axis and then each element, you know you are just floating the value of y it could be say like this for example. So, basically what you have done so what you have done for the graph of Phi for the graph of Phi which we indicate as gph Phi this will be the standard integration for the graph.

So, it collects all elements xy in R 2 such that, y is a function of Phi x this is the meaning of the graph, this is all the graph is written in a more algebraic form. So, in that on all analytical forms you want to say so, now, how do I get come and talk about graphs in higher dimensions. So, for example, if I have a function f from R n to R and what is the meaning of graph Phi. So, how do I speak about graph of Phi sorry graph of f.

So, here graph of f is equal to collection of elements x and y, where x belongs to R n and y belongs to R that is xy belongs to R n plus one such that, y is a function of x like you can visualise the graph or sometimes you know some people who write to write winning a function of in variables.They want to make it explicit by telling that okay the n variables of it this x actually represents a vector in R n so, which has n coordinate some people who try to like it like this, but I want to write it in a very compact way that x means in R n.

Once I say x is in R n, it is clear that it has n components which you can write as x 1 x 2 and x n whatever way you want to write it. So, that fact that x is R 1 means is a vector has to be clear and this way of writing is much easier because this compact representation will also help you to understand more advanced mathematics. Now, you have to understand that when you are looking at a function from R to R it is very easy to visualize it.

But why you are talking about function R n to R it is not possible to visualize it unless n = 2. So, let us see we allow to go into a particular situation n = 2, right. For example, if I say that z as a function of x and y that z and later let us define this function as there was given take an example z x square plus y square. So, which means what I will have, so, x and y will lie on the plane while its value has to be promoted in a 3 dimensional setting.

So, let me see how they would look like. So, what I am trying to draw is draw the graph. Okay, so what I am going to do so this graph looks like this it is called a paraboloid. What you can generate nowadays by the standard mathematical packages. So it is everything in the surface. Nothing is in the interior is hollow it is like a kind of not a pizza but calzone derivative you find go to pizza hut all the shops will find calzone.

It will be something like this or pita bread or right maybe better idea to talk order a pita bread. So pita bread hollow which you can put in some vegetables or whatever. So this is a ship or kind of a fully a pita bread slightly round the pita bread. So this is not ellipse that you see this is actually a circle, but I am looking at it from a moral from my own perspective on the plane and that is why I have drawn it like this. I am not drawn it like this.

So it is slightly more giving in view of the perspective right. So, this is called a paraboloid. So when you put x = 0 going to be 000 and then you put anything that is it gives you a non negative number. Because these are all squares to this. So this represent a paraboloid. Now, once you know something about paraboloid you can talk about other shapes. So, what about other forms of functions. Can I visualize them?

It is here this is an easy example, I will show you something more. So, for example, if I talk about an example like this z again f of xy.

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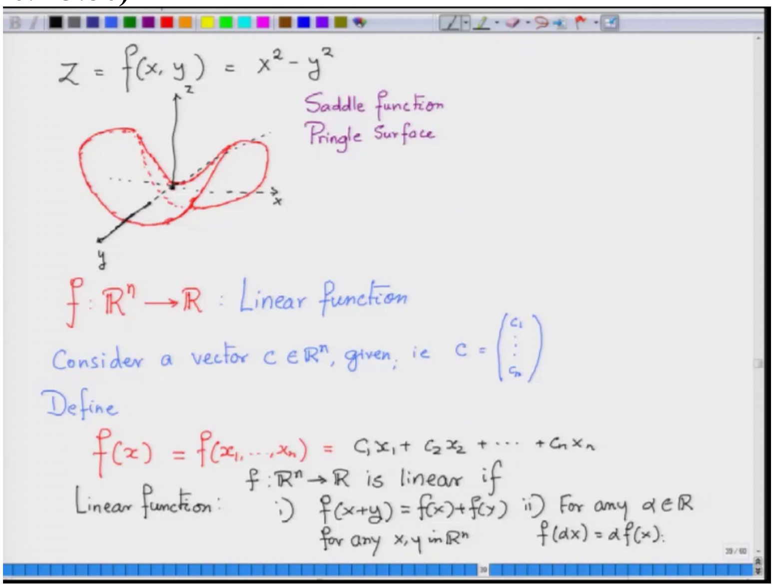

And this is equal to x square minus y square. So here again, if I have the three coordinates. The shape of this curve looks like this. So this looks like something like a saddle it may an outline it properly. It takes time to draw what it is so it looks like a saddle of a horse. It is also called a saddle. It is also called a saddle function. Or it is called hyperbolic parabola sometimes, but we will just call it a saddle function. So what happens is that if I fix my x, and let us look at y the function of y eit gts the function of y it gets maximized.

And if I fix the y is a function of x as a function of sorry, it should be a function of x. So I think it should be in this direction, functional y. So it should not be it x should be here and y should be here. So, this is this. So, these are the interesting fact that if I fix the x then it is as a function of y it gets adjusted the functional y to maximize that, y this actually this y axis is cutting through this part.

Actually 2 axis coming out of this part, this from this point it is cutting through it. And if I the y as a function of x it gets minimized at 00 as a function of y even as a function once I fix x it gets maximized at 00. So this is the idea of the saddle is also called those who take Pringle nowadays Pringle. Potato Chips will also called a Pringle surface it bigger it looks like the surface of the Potato Chips also called Pringle surface.

Another interesting class of functions that I want to talk about is the following. That is the class of linear functions now you might be wondering that I am only talking about functions from R n to R, R 3 to R, R 2 to R and just keeping everything very simple but what I want to giving a general example of function from R n to R a very simple example of function f from R n to R. R is a function called linear function.

It does not matter what is the n, how does the linear function constructed on consider a vector C element in R n given that is, C is equal to the column vector c 1 c 2 c n and then define f of x as if you want to be more precise f of x 1 x 2 x n if you feel that you are still uncomfortable with just writing a compact x x n is c 1 into x 1 c 2 into x 2 plus c n into x n what does such a function represent?

You are seeing this set of function with p 1 p 2 p n and when we are talking about the diet problem, these kinds of functions are called linear functions. So, Water what are the linear functions and what is the definition. So, function f from R n to R so R n is the domain and R is the range. So, when we are discussing in the general form you as the domain so, R n to R and not R n domain, but the core domain,

So, not the range but the core domain range is of course, the image of R n under the function. So, now, what is a linear function we have defined what is the linear vector function, one in the matrix are the we have shown that matrix is a vector function which is linear but here for scalar function the linear function is given in the following way is linear. If number one, if f this is additive f of x + y is f of x plus f of y.

For any xy and for any scalar means any real number for any alpha in the real line. f of alpha x is alpha of fx this is what you have. This is called homogeneity. This property is called additivity property. The next property so, let us write down if you are confused. So, the property one.

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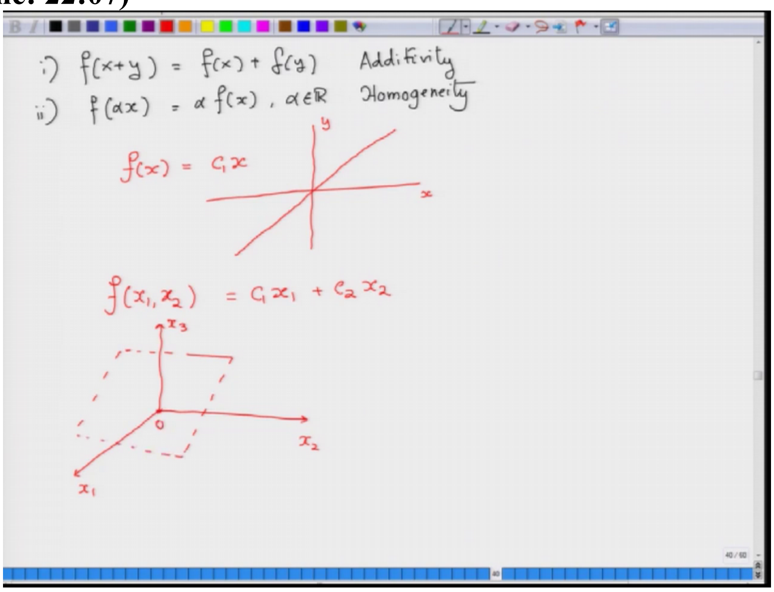

That is f of x plus y is equal to f of x plus f of y this is called additivity property, additivity okay and then this is called positive homogeneity. So, if you scale the vector x scale also the function by the same amount that is exactly the meaning. So, this is called homogeneity not positive or negative because alpha is R sorry for the word positive is called homogeneity. Now, let us see what does the real function do or what are the real function look like.

So, if you put say one variable x 1 then that is f of x is equal to this x is in R is c 1 of x then this represents a straight line passing through the origin. But if you have two variables f of x 1 x 2 is c 1 x 1 + c 2 x 2. What does that represent graphically. If this is your y c 1 x or z or whatever, is f of x 3 is equal to c 1 x 1 + c 2 x2. Then what does this represent, this would represent a plane passing through the origin so it would present the plane in 3 dimensions are passing through the Origin because 00 would satisfy because if I would x 1 equal to x 2 both of them 0.

Then I will get 0 when away x 1 and x 2 then if it is a valid also 0. So, if I put x 1, x 2, x 3, this is x, this is y. So then this represents a plane, for plane passing through the origin plane, of course, is infinitely expanded in all directions. It is sometimes become difficult for now, for example, if I look at a function like this, say a function phi x, y, and z like all these W and I write it as x square plus y square plus z square.

Can I see the graph can I visualize the graph? The answer is no. know unless you have some kind of a special way of viewing because the graph is in 4 dimensions the 4 dimension graph is in the 4 dimension. How do I tackled, how do I even visualize such an object? Is there any way to visualize such an object? What do I do in such situations? That situation tells us brings us into the four of idea called level surface level surface becomes very important in subjects like optimization.

And we will talk about maximum minimum. But let us see, so we could not see the graph in 4 dimensions, but can we see some kind of its image, some kind of its shadow, not image other a shadow in 3 dimensions and red wine and I can have an idea of what it is. So that gives rise to what is called a level surface. So, if I am in for this function, so level surface level five level surface of five is a set of all level surface or a given level surface or a given level say alpha.

Alpha the real number is a set of all xyz in R 3 such that Phi of xyz is equal to alpha. Now, let us come to this specific example where my Phi xyz. I have taken this example as x square plus y square plus z square. Okay, what is a level surface of this? Now for any alpha in R to do a draw level surface so as you have changed your alphas, this level surfaces change you can start with 0 and then 1 and 2 or 3 or anything in between and keep on increasing it.

If I had function of 2 variables, and we will call it level curve, we will just write down on the side. If we have z = fxy. Then we have level curves so, whatever is in 3 dimension, I can see it in 2 dimension I can get a feel of that in 2 dimensions level so we will have a level curve instead these idea can be define even n dimensions, please understand that this is can be really extended.

Just to have R n here x is R n of Phi x is alpha, but that is that we could not even visualize so, we are just keeping ourselves to the situation where we can visualize things. So level curve is nothing but the same thing. We just level f, the set of all x and y such that f of xy is equal to alpha. So if you draw the level surface of this, what does it mean x axis, y axis and z axis. So, what do I mean by this? So if I put x square plus y square plus z square equal to alpha, this shows that I am talking about all the points which satisfy these must lie on a sphere.

Which is of radius root alpha, and has a centre at origin. So basically then you are drawing us sphere so basically what happens if I keep on changing my alpha I will mix I keep on increasing my alpha I will get bigger and bigger and bigger spheres so similar let us try to draw the level curve of paraboloid.

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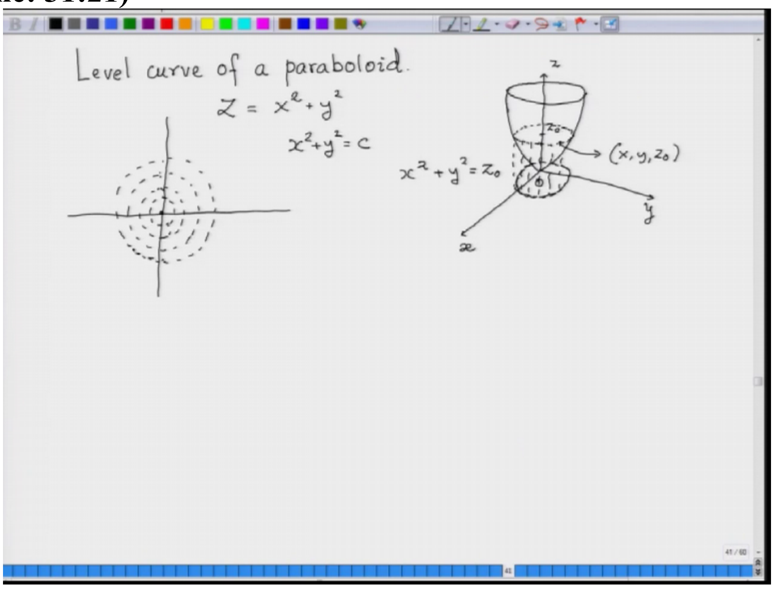

This is really important level curve of paraboloid so how do I look at level curve of paraboloid. Now we have already drawn the paraboloid say it was z equal to x square plus y square so now you look into the fact that so we look into the family of level curve so we will look into a scenario where we put x square plus y square equals to some C. So basically we will be drawing it with radius root C and centre 0. So basically the level curves associated with the paraboloid forms concentric circles with centre 0.

These are the levels of see if you just draw the paraboloid again. This time in 3 dimensions. What I am doing is, I am fixing, so I am drawing paraboloid. I am drawing it as a 3 dimensional quantity. And since I am drawing it as a 3 dimension quantity, Let us fix this particular value of z say z 0. And then basically take a plane which is perpendicular to the xy plane and cut that and pass it to that point. So, basically what will remain the intersection would be a circle.

And in that circle, what will be the scenario basically, they will satisfy at every point of that circle, the z value is z 0 so these any point to take here has a coordinate of the form xy and z 0 and this so, so which means it will satisfy the equation x square plus y square is equal to z 0 because that is a form of the function z value is z 0, so hence basically you then, what do you do, you keep on traveling perpendicular on the xy plane and so on.

You see that circular level surface been level curves been formed on the xy plane. That is the story. I guess we have now got a fairly good idea as water surfaces how we can visualize now, what are the functions of n variables you see, I will just reversed back that you know why we call it linear programming. Our last lecture, because we were actually minimizing a linear function programming term came that the history is different, but we were minimizing a linear function subject to linear constants.

And that is why we called it linear programming, the objective that we wanted to minimize the function was a linear function. So, some examples of functions how to define functions of more than one variable, we have pretty good idea about how to draw their graphs and how to visualize them at least for functions, in functions defined on 2 dimensions and 3 dimensions. And some functions about n dimensions.

Which can be easily defined in and dimensions like the linear functions, the many, many ways to do it. For example, the x square plus y square z is equal to x square plus y square often called the quadratic function. So the many forms while I am not getting into details or some exercise can be given. So we stop here today stock and tomorrow.

We are going to speak about the continuation of limit continuity of a function of 2 variables and then talk about the notion of derivatives, the notion of partial derivatives that is what will play a very important role in tomorrow's discussion. Okay. That is why the next chapter the lecture is called partial derivatives and continuity.

Thank you very much. Thank you for giving good amount of time to listen to me. And I hope you are enjoying the course if there is anything that you want to tell me there is anything that you want to share with us in the course. Please do write on the forum will be back or my tears would be answering as soon as possible. Thank you very much.