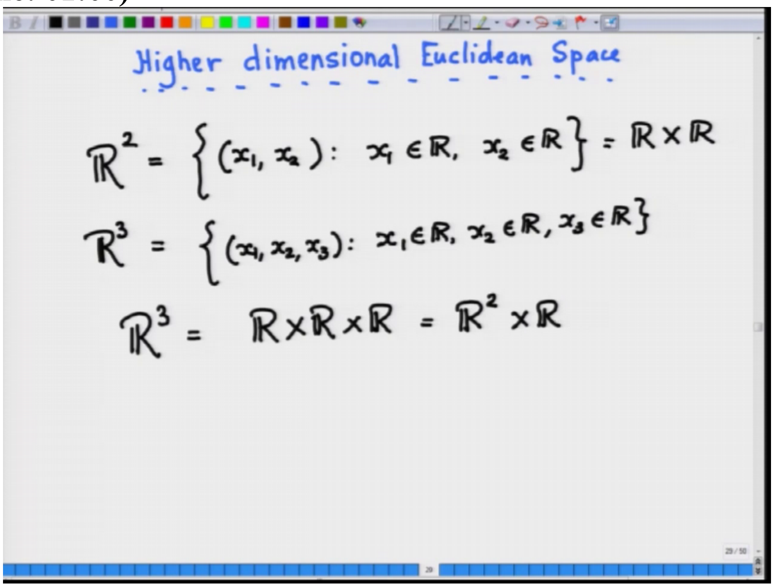
Calculus of Several Real Variables

Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology – Kanpur

Lecture – 06
Higher Dimensional Euclidean Space

I do not know what for the time that you are watching the video, but for my side is good afternoon it is almost going to be five year at IIT Kanpur. When I start this talk, the sixth talk of the series of the program and this first half of the second week and today we are going to speak on higher dimensional Euclidean spaces. Higher dimensional Euclidean space we have been speaking on the spaces of two dimensional three dimensional. These are spaces of our own experience. So, how can I construct a space?

(Refer Slide Time: 01:00)

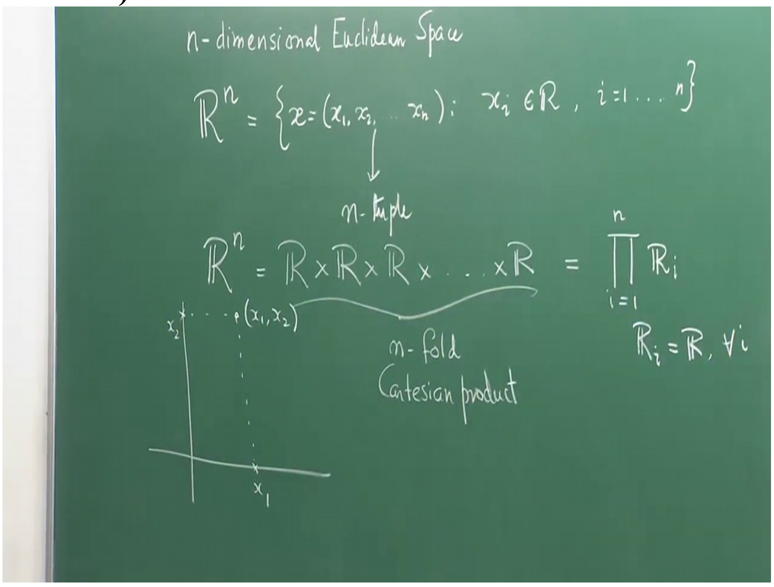


Which has a higher dimension? How do I construct a space which has a higher dimension period we tell you that a lot of things in mathematics can be gauged through intuition? For example, we have studied the spaces r 2 and r 3. And not to say everything is actually built up from real set of real numbers x 1 x 2 x 1 is in r and x 2 is in r.

This is what we have studied and this can also be written as a Cartesian product. This is called a Cartesian product r with r. Now, DoCoMo dot three so now he has three coordinates to represent the point. That is our space, our own dimension own experiences. So, here x1 is elemental r x2 is elemental r x 3 is elemental r. So, these are my standard spaces r 1 r 2 r 3 r 1 is r so r 3 is the Cartesian product r with r and with r another sense.

It is a Cartesian product r 2 with r. how can I improve upon this and make a higher dimensional space to make a higher dimensional space. I just a tool to take a little leap in imagination and consider in dimensional space RN.

(Refer Slide Time: 02:55)



So, what I want to write in dimensional Euclidean space so, that is denoted as RN whose consists of vectors x which I call vectors because anything r 2 r 3 everything consists of vectors. As we are already studied x 1 x 2 and it has n real numbers as its components it is represented by toppled in real numbers in toppled. So, this is sometimes called as in toppled and where each of these exercise R element in RI runs from one to end. So, what is RI terms of a Cartesian product it is N fold Cartesian product of R in Can we written us R cross R. This called N fold Cartesian product.

And this also symbolically sometimes written as the final product is this product of sets. Cartesian product of sets I is equal to 1 2 end RI each RI is equal to R are all I. Get it like this although this is also all right. So, it is just a Cartesian product so, what is r 2 physical you take any x 1 here and any x 2 here. You know, just drop perpendicular and this x 1 x 2 is you know, so this kind of formation of toppled is called audition product.

So, now you might ask that okay. When I when we are in our existence spatial existence, that is when we are in our familiar world of link breath and height, we understand when we are we are standing, we can touch this board, we can touch the ground, I can understand we can touch the side of the room. But what about having such an abstract space? Does it really fit in?

Does it really help us in our understanding of the physical world or anything relevant to do with humankind? The answer surprisingly turns out to be yes. Because one of the first examples that I show you where we go beyond the third dimension is Special relativity Albert Einstein. In special relativity was developed because Einstein was trying to see any special relativity how good are Newton mechanics and electrode.

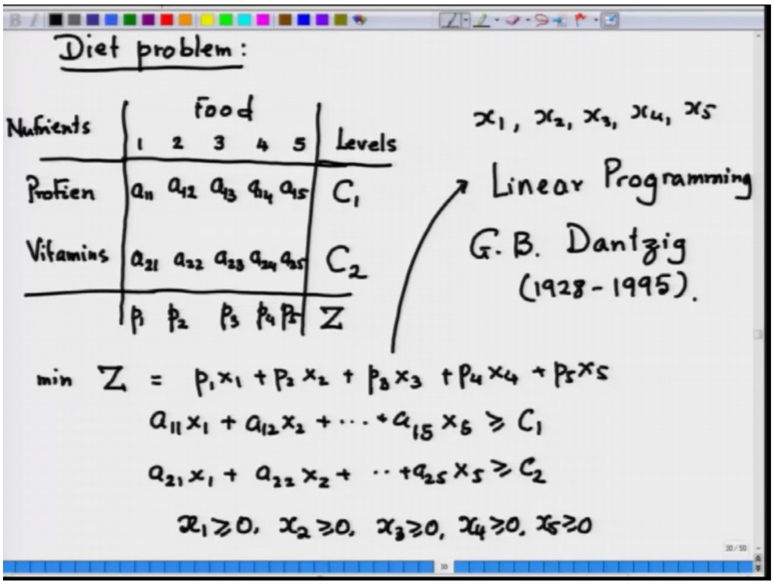
And theory will have 2 dynamics of Maxwell. Maxwell equation of electromagnetic waves, how are they compatible and he found that everyone are compatible and in order to make a system of dynamics, where the electromagnetic theory is compatible, he invented what is called special relativity, in which he had these kind of assumptions that we will talk about initial frames.

Which we will not get into details and nothing in this universe moves at a speed higher than the speed of light in vacuum. Which you many of us know denoted see Nothing special relativity idea of time and space is actually still separate entities which what we learned in mechanic. Because, we are also detailed with applications to mechanics of these ideas of vectors. So, those in especially ready to the idea of time and space been separate entities is washed away and a single entity called space time is studied.

And in order to study the geometry of the space time, you need a coordinate system. Where, every point is denoted by 4 coordinates or time coordinate and 3 special coordinates. So, here you see you immediately in understanding better understanding of a physical theory, you need 4 dimensions. Another application where you might need 20 dimension or 20 or 50 or millions of variables millions of such x 1 x 2 x 3 is a field called linear programming.

And that has got nothing to do with the physical world or more with the human world that we live in it is an area of optimization. And I will give you an example through which you will see why this structure of all in this abstract space becomes interesting and important in our day today scenario, we will introduce to you what is called a diet problem.

(Refer Slide Time: 08:24)



So, this was a problem used for US Army how what amount of diet I should give to a soldier, suppose, I should give them good amount of proteins and good amount of vitamins, right. So, let me see, so, I have 3 nutritional things that I want to get protein sensitive protein, and vitamins, these are the 2 things I want to be very careful about protein and say vitamins. And at every meal, I say at lunch.

I am giving them 5 different kinds of food or in the whole day to give them 5 different kinds of food. So food one, food two, food three, food four, food five. And this is the proteins and vitamins is called the nutrients and these are the levels. So, suppose for the protein, the first one has a 11 amount of protein per unit. The second one is a 12 level of protein per unit without food as a 13 level of protein for food a14 the fifth one a15.

Similarly, a 21 a 22 a 23 a 24 a 25 shows the various levels of vitamins in various to nutrients per unit quantity. Now, some the minimum level of protein I required an army say I am soldier required is C 1 and the minimum level of vitamin C 2 required. So, I have to do now, I have to determine how much quantity of the food I should give to this to a soldier so that my cost is minimized but at the same time maintain his protein and vitamin levels.

So, let P 1 be the price of the food one per unit P 2 is the price for food two per unit P 3 P 4 and P 5 these are greater than 0. Now x 1 x 2 x 3 x 4 x 5 is the amount of food the amount of the first food amount of the second food amount of the third food amount of the fourth food amount of the fifth food that I need to give to the soldiers. Z is a total cost which I need to minimize. So if this is the levels I should give my cost is P 1 into x 1.

Total cost is P 2 into x 2 plus P 3 x 3 plus P 4 x 4 plus P 5 x 5. So, I need to minimize this cost. But, I have to make sure that the soldiers get their level of proteins and level of vitamins, minimum level of proteins and vitamins. So, that the amount of protein I if I give me x amount of food the amount of protein I give which is a 11 x 1. Similarly, a 12 x 2 plus a 15 x 5 and this total protein has to be minimum see one similarly, this is the amount of vitamins in various total vitamins that have to give a 25 x 5.

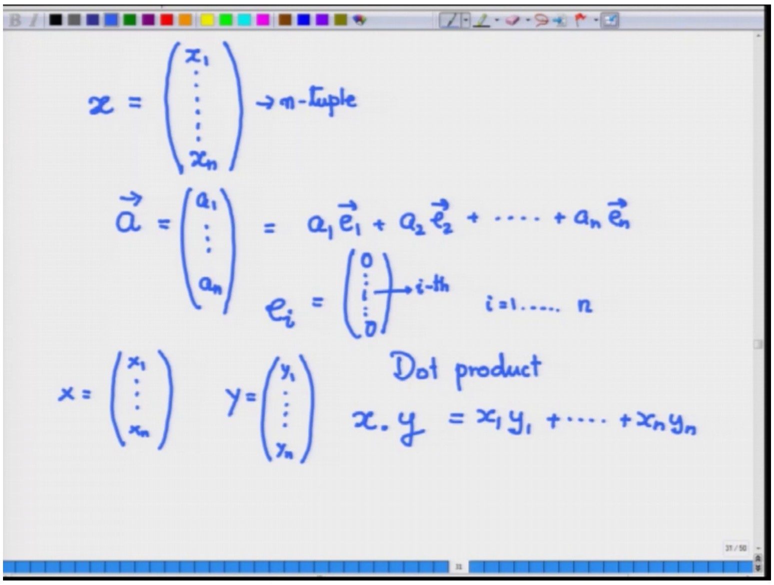
But this has to be greater than C 2 the minimum requirement. And of course, the amount of food that you will give the way amount of quantities of various foods that you would give has to be non negative integers x for greater than equal to 0 all of them are non-negative. So, this what I have written down is called a linear programming problem and it has a huge amount of applicability in business and economics and actual army operations.

So, this thing as a whole the senior program in this the interesting part of that this problem would be solved algebraically and it was first solve a beautiful approach to with some cumbersome algebraic manipulations was given in 1947 by George Danzig. Sometimes called the father of modern optimization. So, George Danzig had solved this problem. So you see that I got a 5 food. So it could be 5 food, it could be 10 food, it could be 15 food.

That you have used throughout the day, it could be 20 food it could be 30 food. I do not know how much I want the food that you give. So, per unit of food, this is the amount of proteins one gram two gram three gram whatever. So this problem is a very important problem called the diet problem that is actually used by many hospitals. In the world, it is part of health economics also.

And this diet problem has recently been used by economics to talk about poverty line. So that is the kind of thing that this interestingly small problem can do. Now, then have come and see how we look at vectors array.

(Refer Slide Time: 15:03)



In what I mean my vectors in array So, given any vector x just write it without the arrow can we represent as an element which is n coordinates right it is called the n tuple now you can also talk about a vector in origin. And write in a vector and if this a vector is coordinates x 1 x 2 x n and I suppose I it has this coordinates x 1 x 2 x n and also a1 into a n if you want to write electronically.

See, we get we will all write vectors column vectors but sometimes we can use it as O vector because you see here we have written them as row vectors because for convenience of writing nothing else. So, you can write them as this form once you know that these are coordinates it can be written as a 1e 1 vector plus a 2 e 2 vector plus a n e n vector.

So, when I write x not put the ix mean just an element of ordinance to direct dial understand is a vector. So, you want to write it just like a vector in physics then you write it like this and we each e i is a vector 0 1 diet position and 0 everywhere is called canonical this even into e n these set of vectors home the canonical basis or the standard basis of the space or in so basically they are replacement of the 1 0 0 on 1 0 0 0 1 0 and 0 0 1 vectors that you know a1 a2 a3.

Given 2 vectors say x and y x is x 12 x n and y is y 1 to y n then the dot product so when I am writing x and y, these sort of things, I will not put the vector sign on the top but when I am reading AB does standard vector illustrations I will put the vector sign on the top. So if I am talking about so this is the distinction the style of writing I will adopt, but when I am writing x is x 1 x 2 x n vector x is actually a vector.

So, that how do you define the dot product of x and y that is x dot y, x dot y is same as what we learned in r 2, r 3, but now you just extend it to r n. So, once a dot product is defined can the idea of the Cauchy Schwartz inequality very far. So, let me write down this. So, given a vector x what is the normal length of the vector x.

So, given the vector x, what is the normal length of the vector x length of x or also called the norm of x that is given us norm x is square root this call is sometimes called the Euclidean norm. So, we are we bother at this point only with the Euclidean known at this level you do not need to bother about anything else. This is exactly an extension of the Pythagorean Theorem to end dimensions. And this is exactly what you have learned in 2 dimensions when you put n equal to 2 or 3 dimensions.

When you put n equal to 3 any of us take extended the geometry of two dimensional spaces? When you look at 3 former geometry absolutely pure geometry point of view is very different in 2 and 3 dimensions because for a 2 dimensional space these 2 lines of parallel only if there is a sheet of paper in these 2 lines a parallel, but in a 3 dimensional space these 2 lines all the parallel is called skew parallel.

So, from your genetic point of view, there are difference between 2 and 3 dimensional space or do you just look at from the algebraic point of view then it is exactly that kind of you just do the algebra so it is whatever you do for 2 can be repeated for ends and that is exactly the way the definition extended so the Cauchy Schwartz inequality So the dot product of x dot y, or any vector x and y is less than equal to norm x into norm y.

This is the Cauchy Schwartz inequality. Now if x is equal to 0 or y equal to 0, it automatically holds both sides will become 0. This is automatic holding. While so how do I prove this? So if I take the length of the vector, you know that length of this is norm for all properties of the length. So, I am not writing the properties of the norm which you can easily figure out.

I would like you to check out the properties on check the Wikipedia never mind check out the properties of norm exactly the one you learned in you have seen 2 dimensional 3 dimensional. Of course wrong kind of various definition. So here, but here will be usable is this definition the Euclidean definition this is what we are set to prove. So, you know that norm of x y norm x x is a vector multiplied by 1 by norm x so now I am taking x and y both not equal to 0 vectors.

So, now that has linked 0 so I can divide them by so I can one by norm x is a number which I can multiply scalar multiply with a vector x norm square of this is equal to greater than equal to 0. so obvious and now you will do exactly what you do in the case, you know in 2 dimension and 3 dimension, it will be the same kind of expansion norm x square by norm x square minus 2x dot y norm x norm y plus norm y square by norm y square.

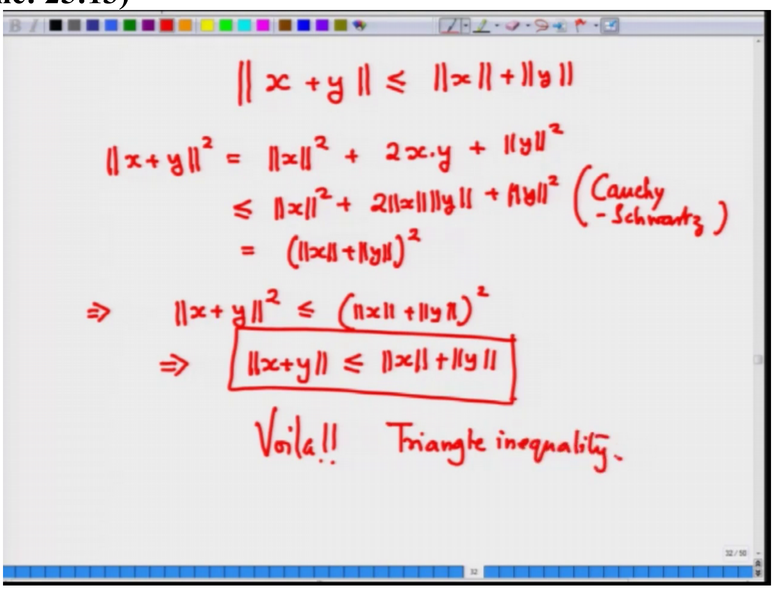
So, this is one this is one and I would like you to check the calculation. So, the norm of this whole square so, then one by norm whole square go out and this will be norm x square and they will cancel and you have 1 - 2 x dot y by norm x norm y plus one is greater than equal to 0, which will simply tell me that if I collect the two and give it here and take it on the other side,

So I will have 2 >= 2 of x dot y by norm x into norm y. So, if I cancel what I finally get is x dot y is less than norm x. norm y similarly, let us take we put plus here it is x y, norm x minus y by norm y whole square now this insert a minus I put a plus and put greater than equal to 0 and doing the same kind of operation here minus would become a plus you would be able to prove that minus x dot y is less than equal to norm x norm y.

So, what do you have final if I combine these two it will become x dot y is less than equal to norm x norm y and is greater than equal to from this minus norm x minus norm y they simply means that the modulus of this is less than norm x into norm y number lies between minus and plus a modulus of the number is typically less than a So, start xy x dot y this is what is a Cauchy Schwartz inequality so he approved the Cauchy Schwartz inequality.

Once Cauchy Schwartz inequality is proved the triangle of inequality cannot be far behind. So, we know proportional to the triangle inequality on this board and this electronic board. And so, the triangle inequality again asks us to find.

(Refer Slide Time: 25:13)



Given a vector x and y so x plus y in dimensions is norm x plus norm y now, let us look at this expression norm x plus y whole square. So, this is equal to norm x square plus 2 norm x dot y plus norm y square in applying the Cauchy Schwartz inequality, I have norm x square plus 2 norm x norm y this is less than or equal to norm x norm y plus norm y square.

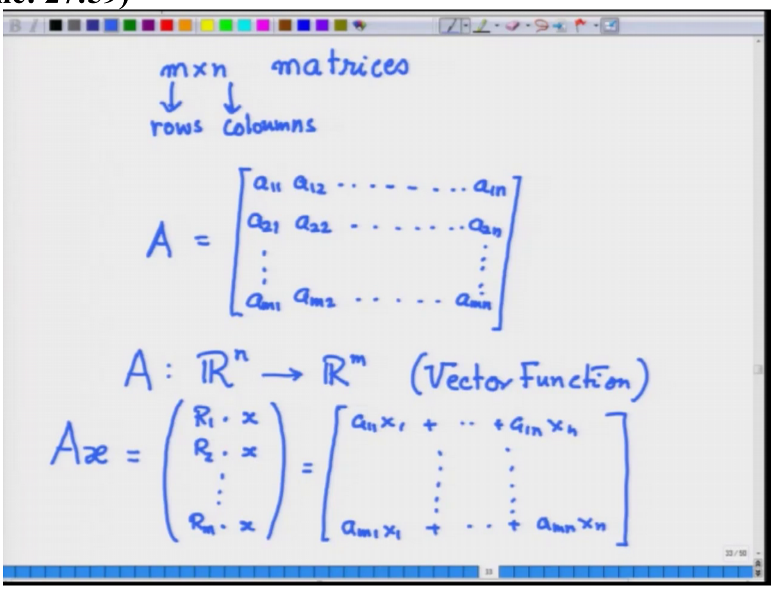
Here, applied a Cauchy Schwartz policy Russians obviously call it the Cauchy Schwartz inequality as I told you earlier. Now, here what I will get this is nothing but equal to norm x plus norm y whole square. So, this implies that norm x plus y whole square is less than equal to norm x plus norm y whole square.

Now, if interrupt to positive quantities on a non negative quantities of squares or while squares are one is greater than square or the other, it implies one does while the one square is greater than the number 2 square is lesser. So, a square is less than equal to be square and a and b are both non negative than as less than b.

So, here it implies that x plus y is less than norm x plus norm y and as a French would say, we write it here as Voila done. And that is what we have as a triangle inequality that is it that solves a problem. Now, we are going to talk about matrices. Now, once we have spoken about things in in dimension, we are going to speak about matrices. So what are matrices?

And maybe I should add it on this board. And can we talk about a matrix of a given number of rows and given number of columns and row and column need not the same. We are not talking about square metrics for spoken about square matrices and square matrices and in when we are talking about 2 and 3 dimensional spaces, we of course, to focus our 2 cross 2 plus 3 cross 3 matrices which we are now going to talk about m cross n matrices.

(Refer Slide Time: 27:59)



So, m is a number of rows and n is a number of columns so where m and n they could be different and could be done other. So, how does the matrix should look like? So, we have such a matrix already when we wrote a linear programming problem. So, we had basically 2 rows because there are 2 nutrients and 5 columns there because there are 5 foodstuff.

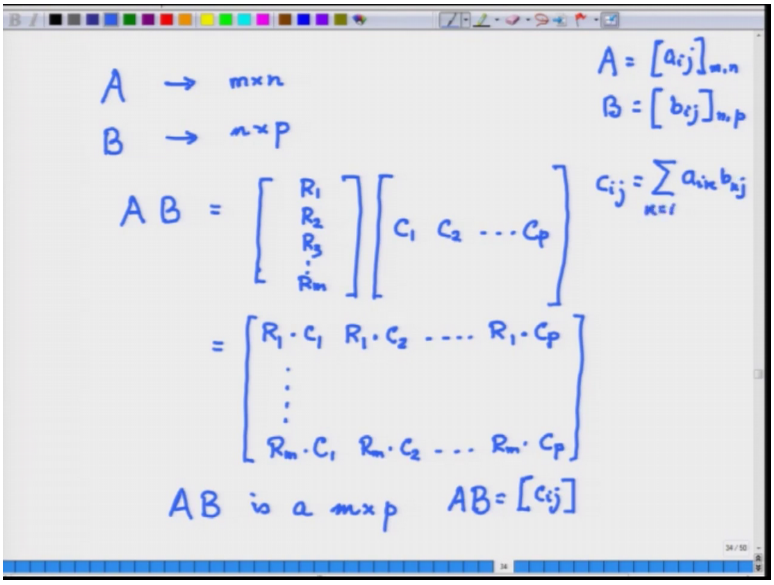
So, here the generally writing it as a 11 a 12 a 1n a 21 a 22 a 2n a m1 a m2 a mn. So, this is an m cross n matrix okay. What happens I can view that m cross n matrix A as a function from R n to R m. So, I can view it as a so called is the first step to view it view as a vector function. So, which carries one vector to another vector. Vector of one dimension to a vector of another dimension.

So, basically I am looking at by laws of matrix multiplication what would be A of x A of x if I call each of these row vectors as R 1 R 2 and R 3 and then this A x is nothing but R 1 dot x R 2 dot x R n dot x So, how do I do it? This comes out of the policy of matrix multiplication, you have learned about matrix multiplication 2 2 and 2 3 same type of thing can be done for example, here what will happen this is A x would be. So, it will be a 11 x 1 plus a1nxn.

And the last would be last component will be am 1 x1 plus a mn x n and so, this is nothing but the product of R 1 into the x 1 x 2 x n vector. If you look at it, you will immediately get some sense of what we are doing some sense of matrix multiplication. So, I am trying to inform matrix multiplication to this case, I am going from here to matrix multiplication. So basically, if you just have a vector, you take every row and take the inner product with that vector.

So, now instead of just a vector I have a matrix which has some columns. So basically what we have to do, we have to take the take the first column, and do the same thing, we have to take the second column and we have to do the same thing. Let us see, All right, I will try to write it in a very simple way.

(Refer Slide Time: 31:28)



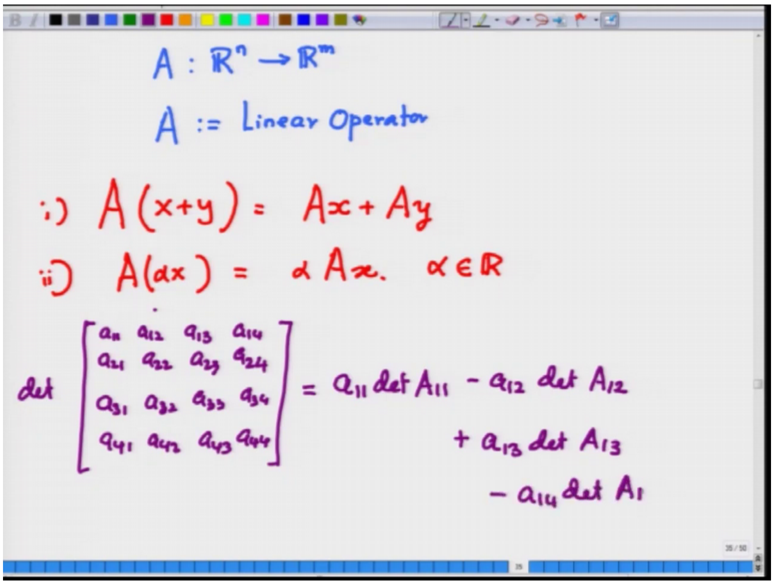
So, let A be m cross n vector see number of rows, number of columns in A number of columns in B a number rows in B must match. If I have to write down the product AB n cross p. So I want to write AB so AB what is AB? Or how I write down so A has m rows and n columns right. So, just bother about the m rows are in R 1 R 2 R 3 Rm and these are in columns B So, column is written as C 1 C 2 C P now R 1 is a vector in R n and C 1 is a vector Rn and there So, what I right so, what are I what is my first new matrix.

So, I will R 1 into C 1 or R 1 the next component is R 2 into C 2 R 1 into CP the dot product and the last row would be R m dot C1 the same story all right Rm the C will be changing Rm dot Cp. So, now, what is the order of this matrix it is m cross p, so, AB is a m cross p matrix. So any element of AB, if I take an element of AB and I write it as a C ij, I call every element of AB as C ij.

Instead of a ij or b ij I ready it as a c ij where a ij is an element of AB and C ij is a element of AB b ij element of B So, I can write this as a ij as mn matrix and B again write as b ij as mp matrix. Now what is AB it is, so what is C ij? So, how does C ij look? So, every C ij let me write it down here C ij is nothing about the inner product of the first row right. a ik b kj where k is varying from 1 to n.

Because, k denotes a column of A and k denotes the row of B they must be n here. So, that is one way of writing and but the simplest way of looking at is this one through the dot product. That is exactly what happens. Now, A has a functional vector for R into Rm.

(Refer Slide Time: 34:55)



The standard notion of a function as input is Rn output is Rm. So, then A of x. A becomes what is called a linear operator A becomes what is me is like what is called a linear operator. So, let us see what is the property of the linear operator? So, A of x + y is a linear operator has 2 properties number A of x + y is A x + A y number 2 A of alpha x is equal to alpha A x.

But alpha is energy or lumber there are certain other issues out matrices which I do not want to get into too much in detail right Now, but you might be wondering how to compute determinant of square matrix if I had one determined is only computed for square matrices please understand.

So if I have a matrix save of 4 a 11 a 12 a 13 a 14. So a 21 a 22 a 23 a 24 a 31 a 32 a 33 a 34 see a 41 a 42 a 43 a 44 how do I come to the determinant of this matrix is the same way giving the same plus minus or learning sign on. So, if I go by the first row a11 into the determinant of this matrix. So this matrix, which consists starts with a 22 and ends with a 44 in the diagonal and that is the matrix obtain by taking all the first row, the first column that is called into the determinant of a 11.

For a 11 is a matrix obtain from the main matrix by taking all first row in the first column. So it is called a kind of a joint matrix. So this is the first term is a 11 into a 11 minus a 12 into and a 12 is a matrix obtained from the original matrix by taking away the first row and the second column plus in the same thing, this is our determinant is calculated. There is several other things which is associated with the matrix, which I do not want to speak at this moment. Let me just come to the end of our discussion today.

So, the you have seen from the beginning to summarize that, why I need to bother about in dimensions and dimensions It is not that mathematicians are just playing a game or let us just put in dimensions and let us play with it. It has applicability the idea when dimensions make sense when we saw the diet problem in linear programming, and hence all these type of things, which these extensions from 2 and 3 dimensions to n dimensions makes sense, because all this can be used.

Now, of course, if I asked you what is the angle between 2 vectors in n dimension 2 vectors x and y it is nothing but the cos inverse of x dot y y inverse into norm y so, that is so, it is the same thing that repeats itself. Now, here you also saw that in our physical theories, we are talking about space time all modern physics uses special relativity and so, we have to talk in terms of the 4 vector.

This is what physics called the 4 vector so you have some idea what matrices now of the type m cross n and how to do their operation of multiplication addition on I am not telling you, you should be able to add just they are adding the corresponding stuff means when you add two matrices, the number of elements must match to add. Basically, you should be able to add the first row elements with the first row elements, second row element with the second row elements. Right? So but when you go to multiplication.

Right, so when you come to, so you have two m cross n matrices, you can add them right because you are same number of rows, same number of columns. What do you have different rows and different columns then Okay fine. Then that becomes very difficult that then addition is not defined. So there suddenly which have left out edition multiplication which you can really figure out for yourself or for books because certain things you also have to think about.

Because why suddenly just spoken about matrix multiplication will mention one level issue will become slightly important as we go on. So, what all you we are now in higher dimensional space. So, here we are going to talk about many are examples will consist consists of function of 2 variables 3 variables maximum 4 variable, but we are our theory would be within variables always.

So, theory would be done in the most abstract form the most general form, while examples because they need a little much less time if you put ask me to do an example with 50 million variables I can do by hand or with even 50 variables I can do by hand. So, to do something by hand we need to do it with 2 or 3 variables. So, examples would do 2 or 3 variables, while our theory would be absolutely general and that is exactly the point I want to stress before I end this lecture.

That we are getting into absolutely proper multi variable calculus will start about function of 2 variables. So it is very functional n variables. And it is associated issues like partial derivatives, higher derivatives, trailer series, maximum, anyone, and all these things would gradually come up. So thank you once again for listening to me. And I hope I am being able to give you some idea of what good analysis means. Thank you very much.