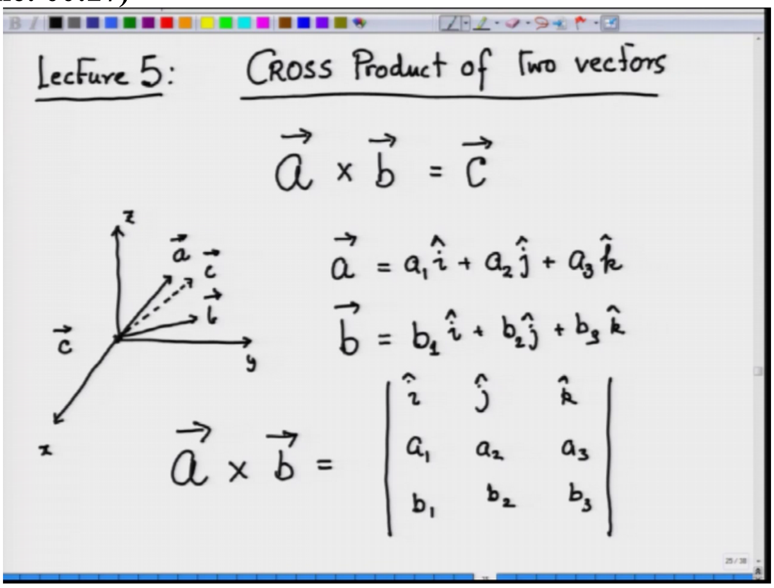
Calculus of Several Real Variables
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology – Kanpur

Lecture – 05
Cross Product of Two Vectors

So, today, we are going to start our last lecture of the fifth lecture of the first week. And here we are going to introduce what is called a cross product of two vectors.

(Refer Slide Time: 00:27)



That is, I define a new kind of multiplication of two vectors. And I will call this as a c vectors. It is a new up so this product of two vector does not give me a number just like the dot product, it gives me a vector. So let us see what do we mean by this? And let us from practical perspective also, you will see what happens. So here is the x y z plane. As always, space. So if a is the one vector here, as b is the another vector in space.

Now C is the cross product of two vectors of vector which is probably if you will write a cross b and c Vector when you add a cross b you move from a towards b and then in the plane formed by the vectors a, and b to c vector dot product or the cross product is perpendicular to the plane in which the vectors a and b are lying. So, why did I draw seen this direction and not the other direction.

So, it is like you know, moving screw if you take a screw and put it in front of your nose straight and you try to rotate it in the counter clockwise direction counter clockwise from your point of view. Now if I am rotating it front of me is counter clockwise from my point of view, but it is clockwise from your point of view. So from my point of view from moving in a counter clockwise direction, you will see that the nose of the, or the tip of the nail, tip of the nail, or the screw will just pointing towards my nose.

So that is exactly what is happening. So if I were from here a to b, when I am moving it from a to b, this tip is moving towards this direction. So I am just going to demonstrate to you to kind of experiment here. So it is kind of here like this. And so I was moving it like this. You see what happened is that so this is our school is moved. So this is crew and I am with us today we are moving it from so there are two vectors, which are defining the plane in which this face of the story is lying.

And I am moving it like this. Then are actually putting the screw and moving it like this, so you see the It is the bottle is going if it is a pocket, it is clockwise, right? So this is a school, which you have got, and you are moving it. So this is screw that you want to put in. Right and you are moving in this direction, clockwise from your direction and see what is happening. The school is getting inside. So basically clockwise.

So if I was standing there then for me, it would appear that I was putting in counter clockwise, and it was going inside like this. So in this case, maybe I am little mistaken. I think I have to this case it might be if I am going from A to B, then I would see from my side it is coming to the counter is going on the clockwise direction. So it seems to see should be something like this. Okay, so how do you come define this vector? So, there is a way of defining this vector. So, we will have a way so we will take a vector as a one I vector.

Where do you want it to be a coordinates of a, the so, we are just in three dimensional because beyond three dimension, you cannot define the cross product. But three dimensional ideas would actually be helpful in the sciences in natural sciences like physics and you will very soon see the applications are towards the end of the course, when we talk about ruins theorem and goals divergence theorem stroke theorem.

See how important this concept of a cross b and a large amount of electromagnetic theory which combines the forces of electricity and magnetism into one framework, which Maxwell did also needs this kind of operation. This is what it is, and b Vector. So the coordinates of beyond b2 b3. So I can represent it as the one i Vector b to j Vector b to k Vector, those are no linear algebra.

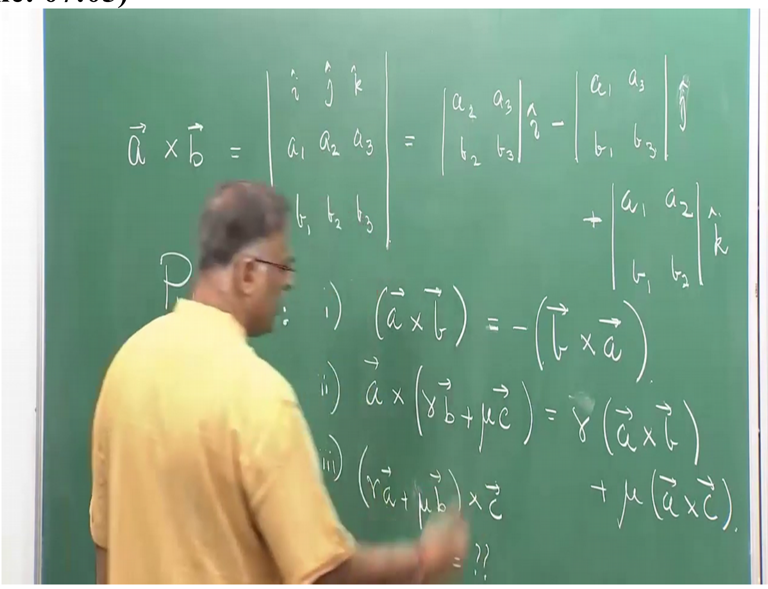
You know, there is nothing but expansion to the standard basis, which is i and k, which is 100001. And so at 1000101001. So I am not getting into all these details, you know these things very well pretty well, we have been writing this for quite some time. Now. Now, let me tell you, how do I define it? Now I how do I define it? How what is a cross be? This is a very strange kind of definition where you will see that we are talking about a determinant.

I am talking about a kind of determinant, I want you to come to the determinant where one of the rules consists of this idea characters IJK. You might say, well, how can you do it with a determinant you always put in some numbers how you suddenly telling you, oh, my God, this has to be something like this. This has to be Kind of this is a metric and how many did these are all vectors.

How can you compute the determinant? Of course I can say okay, you can write down this whole thing in terms of the three vectors and then do the determinant. Right? Then you also get the same answer. But okay, we are not getting the details. So how can I write this? So what I am writing what are the components of a a b with i vector a1 b1 here a2 b2 with a j Victor and the k vector.

So this is the definition. So you might ask, how these definitions meaningful? Because ijk vectors are not numbers, but I can say I can call right down them as numbers as 100010 and 001. And then are you direct to write the expanded in the form or determine and then also, you will get back the same kind of stuff. So that we just expand the determinant and let me just see. How can we write it so what did we have here

(Refer Slide Time: 07:05)



Is that two vectors a cross b. So computing determinant is the same, it is not that I am doing something, it will look like exactly the same. So it will be the same thing here, but I will write it in a slightly different way. So, this will be a real number who take the determinant of this, which will be scalar multiply biggest kilometre liberation with this vector. So it will be a one, a three because if I take I and I take all this first one first column.

So sorry, a2 a3, b2 b3, this multiplied by a vector now minus nine Congress like in order to determine it, now I will have a1 a3 b1 b3 plus this which is deployed to the gym vector and a1, a2 and b1 b2 multiplied k vector. So that is what is how you define the cross product. Okay, and what let us now look into again as I showed you through this that, okay if this is a and this is b, and if I am rotating from a to b in a Popeye's fashion, the c vector is in this direction.

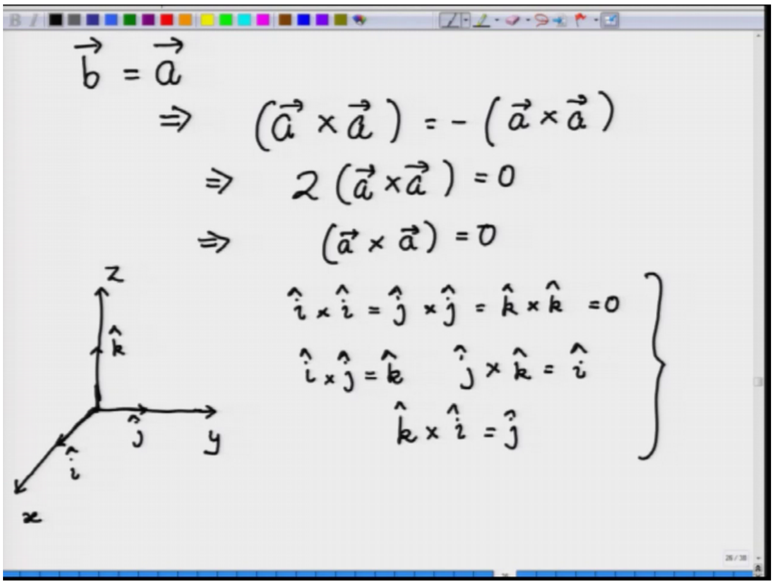
If I wrote it in an anti-clockwise direction, the c vector as I told you, just before the school I am showing this example, you can only begin anti-clockwise towards me and you see that this most of the water is coming towards me. So, this is this is what it does, there is a very important kind of stuff that I really you really need to know before you do anything with this, we need to put certain properties of these vectors into perspective.

So let us put some properties. So number one properties that require A cross b these two, this is the negative of b cross A because the directions will change because here it is a vector it has both magnitude and direction.So we are not computed the magnitude of the magnitude of A cross b will soon do that. And you have so was now I distributed is like it follow the kind of similar laws of a adding that if I distribute this cause for two vector what the addition of two vectors that is a cross gamma b plus mu mean these are real numbers of C vectors scalar multiply. So, this is nothing but gamma of A cross B plus new of shown on camera a cross C.

So, this is this is what you will have this is a property for distribution of multiplication or addition So, again distribution kind of the reverse kind of thing means, basically you have multiplied from this side either multiplied from the other side also. So, I would not do it suppose you have gamma a vector plus mu b vector, and this year multiplied with c vector and easily write down what will be the answer.

I leave this to you, but what you get from this is a following. That if you know this property, then let us look at the first property what does it say if I would b is equal to a?

(Refer Slide Time: 11:51)



So, if for if I put b is equal to the a vector it tells me immediately that a cross a vector is same as the negative of a cross a vector which similarly shows that twice of a cross a vector is 0 which immediately again shows. See this symbol any vector which is equal to is negative must be the zero victory it cannot be otherwise. Okay. So, this is interesting property once you have known this property, you can play around with i and j.

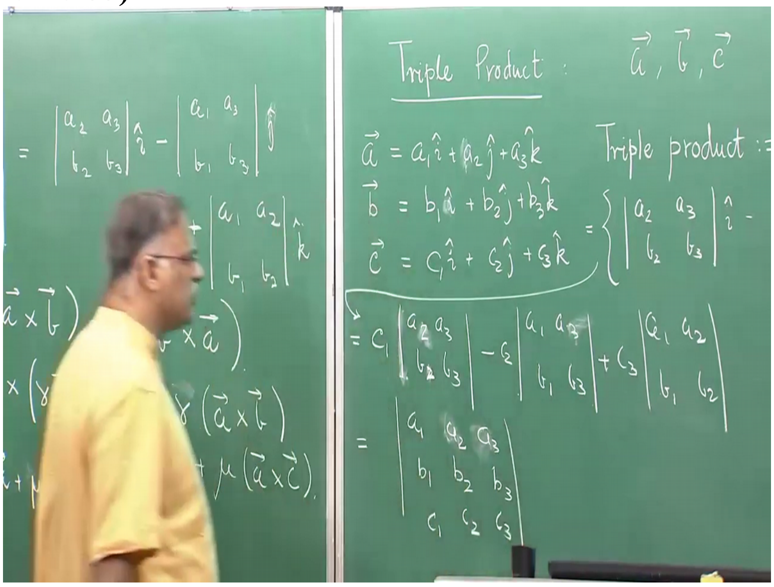
So, let us write down what you will have now let us look at the three dimensional picture again with i a vectors in tow do not mind my drawing Do you the x axis y axis and Z axis and this is the i vector the coordinates 001 is a 100 and these j vector with coordinate 010 and this is k vector with coordinate 001. Now, one thing is clear from the very beginning that i cross i is equal to j cross j is equal to k cross k it must be 0 from what we had just discussed. Now, what is i cross j that if I move i towards i from j so, basically if I am looking from this side.

If I am moving i to j in the xy plane. If I am looking from this side and I am moving i to j right so from my side, I am looking at it some if the observer is standing along the z axis x axis i along the z axis, then he observes that the moves towards j in a counter clockwise fashion. So that the screw will move towards him along the z axis that is x axis on the z axis is it is a k vector which will move towards him. So, i cross j is the k vector.

Similarly, if IJ cross k, so, again if i on the x axis and if I move the vector from j to K if I wrote it from j to j towards k site that I am rotating the whole system in an anti-clockwise direction, so, the i vector will move towards me. So, I will have j cross k and i Victor and I will have k cross i as j Victor. This is a beautiful, simple setup, I think this is something you should always remember because this can be used in many, many, many places.

Now I will talk about something called a triple product okay triple product and I will see what a triple product is a triple product of two vectors is that you take the dot cross product of two vectors and take the dot with the third vector. So, this is called the triple product.

(Refer Slide Time: 15:33)



Triple product is a very useful on as you will see very soon triple product. So, you have three vectors given to you a, b and c and where a vector as before as given us a one I vector a2 j vector and a3 k Vector. While b is a given the similar fashion I think which I should not repeat b is given as b1 i vector. What still I am doing it for the sake of clarity I guess and c is c1 i Vector c2 j vector and c3 k vector.

So Triple product is this a triple product is defined as follows Triple product is a cross be intersections is a dot product c. so I take the cross product of the two vectors and take a dot product with c if I expand it.So what I will get, I will get the following. So I will get to build product is a2 a3 b2 b3 i vector minus a1 a2 b1 b3 j vector, plus a1 a2 b1 b2 as k vector. These dot product into C1 i vector c2 j vector and c3 k vector if you recall, the definition of the dot product which we had already given.

So then this is nothing but so this whole festival for us I am writing it here so that you can see it clearly. So, this is nothing but see one into c1 into the determinant of a1 a3, b1, b3 minus c2 into the determinant of A one A to B one, sorry, A one A three B one B three, plus c3 into the determinant of A one A two B one B two C it can be written either as a determinant whose first choice he wants you to C one C two C three this just this has to be a story it is A, this is a sequence it is a two a three and b two b three.

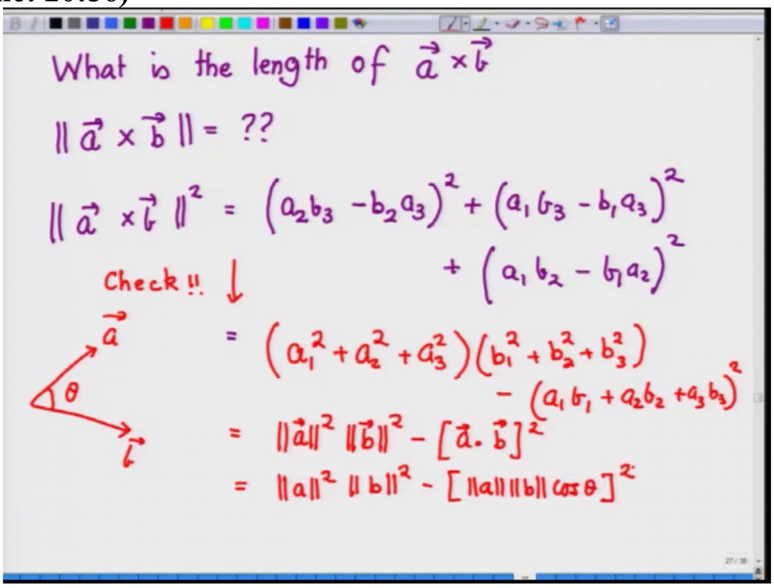
So, if you look at the So, what I did dot product, the first component of this first corner of this with the first quarter of this and second and second coordinate with second quarter, third quarter, third quarter and summer. Now, if you look at this thing, this can be either written as a determinant with the first row c1 c2 c3, or it can be written as a determinant as a third row a c1 c2 c3.

Because a sign convention of the first row and Third row is same. So, actually the determine and if we just said to remind you read their determinant if I if I have nine positions. So, you can start expanding from any of the rows. So, then if you go by the rules in plus, minus, plus, minus, plus, minus plus minus plus this is the way the convention is carried out. So, just keep it keep this in mind. And so if you do that for our convenience just for writing convenience.

I am writing it as a determinant which is expanded with C one c two c three in the third row sorry, you want A one A two A three, B one, B two B three, c one c two c three. So, what do you get? So, what I get is that naturally because A dot B a cross C the vector and see the vectors of the dot product to two vectors is a scalar.

But the scalar can be expressed as a determinant with the inputs. Which are the coordinates of the vectors of A one and A two to anything and this will become pretty important as we go along this concept.

(Refer Slide Time: 20:56)



Now you want to find the area. I want to find the magnitude and length so my question is what is the length of a Cross b magnitude? What is the length or the magnitude or just light length which is much colloquially easier linked of a cross b. In some sense I am asking to do mine the norm a cross b.

And so what is the norm of a cross b that is essentially that is exactly what I am trying to ask you Okay. Now, how do I do it? I have to first note that a cross b is nothing but a vector whose coordinates are this, this and this, which are determinant. So if I write it as a square, norm of the square norms square is then square of the value of these determined square of the value.

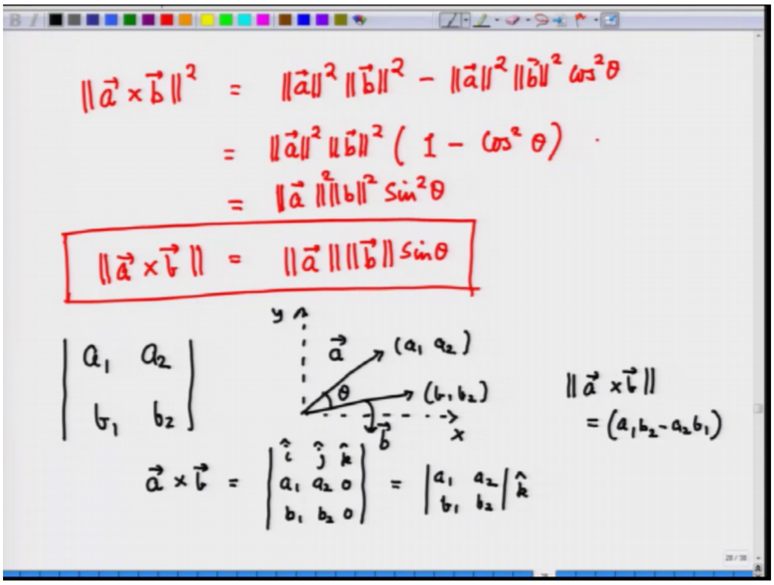
Which means if I go back and write it down the first determinant is a two b three minus B two A three whole square plus a one, B three minus beyond a three whole square plus a one, B two, B one a two, a one, B two minus b one, a two, whole square. This is the length the square of the length. And now I will write it in slightly modified way.

I will just I will square up these as mega square and then rearrange the terms and then I will get the following expression which I am just writing down, I asked you to check it from here to here, your job is to take a cup of tea and sit down and do the check. So, the checking if that will give me a one square, a two square a three square into b one square plus b two square plus b three square minus a one B one plus A two B two plus a three B three square.

So, this is a very quiet cumbersome computation and just I have been lazy and just because of the time constraint, I am leaving out this computation and asking you to check this computation. And then what happens? If I go back please understand that is a vector with a one a two a three s coordinates and v is a vector with B one B two B three as coordinates and you can understand this term a one B one A to B to A three B three is just a dot product.

So, I can write this whole thing as norm square normal norm b square minus a dot b Of course, I should have the arrows on the top whole squared. So, this can be written as norm A square. So, if I have the two vectors A and b and Theta be the angle between them. And I can write this as this square minus norm a norm b cosine of Theta is a angle Between two of them. So, this will give me

(Refer Slide Time: 25:03)



That norm A cross be whole square is norm A square into norm be square minus normal square into norm b square cosine square of Theta and as a result of the second right norm a square into normal b square which we can take out from the expression from the bracket. The secondary distributive law, multiplication over addition standard distributive law with matrix.

So, every time we are actually doing this sort of competition which looks very common and very obvious to us, we are actually doing a very deep operation we are actually writing down certain kind of laws were of arithmetic which was initially written down because they Well figured out to be true with numbers with natural numbers.

And then it was written known as a law that a number which has to a real number it has to follow all these things. So they do follow, right. So, so this is not just on the top Oh, it looks like a just an obvious operation, but it is not. So, very simple as we think this is a kind of axiom kind of thing which actually happened. So we put it as a kind of law, which will, we always have to remember and follow that this is nothing but distribution of multiplication over addition.

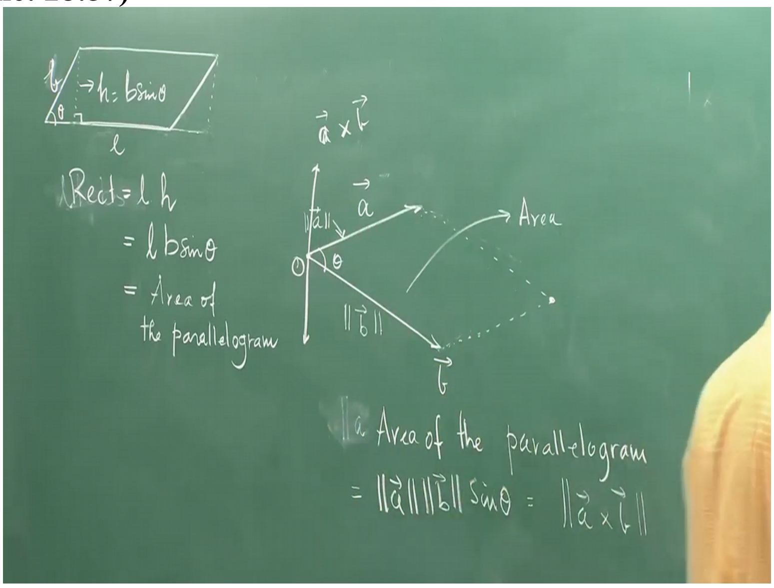
The kind of laws I had written down there. So, this is the thing if you just think that these are real numbers and is just multiplication on this tradition, that is what that is exactly what it is here. So, those who know trigonometry would immediately realize that science courses have lost costs, Court Theatre is one and so as a result of which Sorry I have to put a and B here the arrows I always seem to forget the arrows.

No so I will have here sign square of Theta. So noting that length is always positive assembly have this equal to a b norm a norm b sign of Theta. There is a deeper geometric meaning of this and that is what I want to work out here on the board that what does this thing actually means what does cross product mean geometrically, and you will be surprised to know that the geometrically the cross product simply means Area.

And that is what we are going to Do will show that the area of the rectangle content by the or the parallelogram sorry, not rectangle parallelogram content by the vectors A and B is nothing more the cross product A Crosby and that is what we are going to demonstrate but I am looking for a duster, okay. So with a duster here, let me try to demonstrate this some time is usually lost when I am rubbing the board.

But that is how life is that is how things are done in the class. I sit here before an imaginary audience and I am speaking to you as if I am giving a lecture before you in reality. So let me know write down what that stuff is. So suppose in the three dimensional space,

(Refer Slide Time: 28:57)



Here is my A, this origin right? Here is be maybe I should not make it. So here is your main angle I should make it. By immediately in the space. It might look as if I brought something perpendicular I mean actually not. I should write it like this. So this is my A victor is origin and is a b Victor. So let us make a parallelogram out of it. So maybe something similar to this and something which is power to this. So this is a parallelogram. Now, what I am trying to claim is that a cross b the presence.

The norm of a cross b that you see here actually represents the area of the parallelogram contained Made by these vectors A cross b the plane that contains a vector A cross b because how do you compute the area of a parallelogram in a plane? So, here is the area of the parallelogram plane is this is one. So, what do you do area of the parallelogram how it is computed How does one compute the other parallelogram?

So, this is the height right. So, you basically compute this side and so was this is a length l and this has length b deserves a link b and this is linked L and this is b sign theta and the parallel area of the parallelogram if you define the area, so what is the area? So what do you have constructed this rectangle like this and because everything the same parallel lines the area's same, this is what you learned in your school geometry.

But though it is not so, simple, we can we are doing things very simply, very not so simple. So, this he called to be sine theta is the height. Basically, we take the height, basically, this part and this part because he is also feet and did this part. So, that would, so this height and this, this whole part, this part would make up which is same as this. So this length and this length is same.

So this whole thing is actually L. So if you look at the area of the rectangle that is formed There you have the rectangle is LHB, sorry, LL into that area of the rectangle is L into H the rectangular area rectangle wrecked is L into H length into the breadth, but that is what is L into what is h it is b sine theta. So you observed that this length this line, the side of the rectangle must match this side of the rectangle.

But this line is same as l because these are parallelogram, only sides are equal. So this length is also l this full length, and so l into h to age is area of the rectangle and the area of the rectangle is equal to the l h into b sine theta. And this area of the rectangle is same as the area of the parallelogram. So if you consider this part length as b Vector and this is Theta and this part the length is a victor of this.

And you know, then what is the answer? So, the area of the parallelogram this part of this follows from this one, this this one. So, I am trying to find the area of this area of the parallelogram is nothing but norm a norm b the length into triangle between them sign of Theta and what is it? It is nothing but the norm of a cross b. So a cross b this vector in some way presents the area.

So, the vector equals we could be either in this direction or it could be in this direction perpendicular to the plane, it depends on which way you are rotating the a and b towards a cross b towards a, but this vector say a cross b on this one, whatever you want to say, whichever is across this vector is actually representing the area I enhance area itself can be thought of as a vector.

The area of vector and this would be very helpful when we try to study Stokes theorem and Gaussian which are extremely important in engineering and physics. So please understand that cross product gives the whole goal cross product it is to tell you that we can view the notion of a as of vector, we all saw a very as a scalar a number, but area can also be viewed as a vector.

So, area vector of an object is actually perpendicular to the object to the area, you are trying to calculate length. So for example, the plane for example, rectangular, so to cross product of two vectors is actually the area of an A rectangle. So area where rectangle can be also viewed as a kind as a vector. And that is where the beauty lies in this whole thing. So, let us now do two things before we finish our talk is let us look at the geometry of two by two determinants.

It means basically, I am trying to look at what is this time to time to look at as the determinant of this form? Now, we will soon see something pretty interesting. I suppose this is a plane xy plane and you have two vectors which is A one. And with coordinates you wanted to and another is the vector be with coordinates B one B two. And these angle theta between them.

In a three dimensional setup, their coordinates are A one to zero beyond B to zero. Now, if I look at the cross product of these two vectors, A cross b then A cross product is nothing What I sorry. So comma j and k a one a two zero b one B two zero. So, this is nothing but the determinant of A one A two B one B two k Vector. So, the parallelogram in the plane the magnitude of the parallelogram what is the magnitude of this vector.

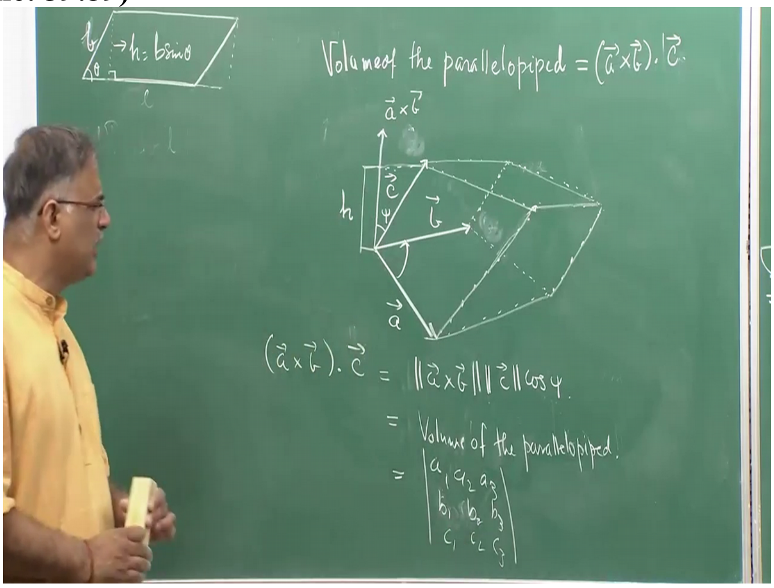
So, magnitude of a cross b in this particular case is nothing but you take the magnitude on both side norm of the case one right these are the K Vector. So if you take the magnitude of this vector and so now we are here we are looking at equals b as if it is a three dimensional vector, so I am putting you on it to zero. And then what I am getting the magnitude of this is nothing but a one, B two minus a two b one.

So, these determine and what does it give it gives them the area of the rectangle A of the parallelogram around formed by the vectors a and b in the two dimensional plane XY. So two dimensional determinant, determinant of with two entries, is actually representing the area of a parallelogram. Similarly, I will show the determinant with three. When I when I am talking about a determinant with three and driven and nine entries.

Basically three entries in row each row, three dimensional determinant, and that determined actually would then speak about the volume of parallel people because the notion of a rectangle, so the notion of a parallelogram not a rectangle, notion of a parallelogram in three dimension becomes what is called a parallelepiped. And that is what I am going to show that A determine the area or the value of the area of a parallelepiped.

Width which is made up of three sides developed by the victors A, B and C are actually is nothing but the determinant of over form this, which is the cross product of three vectors. So, when I draw a parallelepiped, right I so I am two dimension so this is my origin

(Refer Slide Time: 39:59)



Let me see which will go trying a vector b vectors so a vector b Vector and these my c Vector so the parallelogram you can form like this I am not I am trying to draw this but something like this. So this parallel will be it is called a parallelepiped. So area of the parallelepiped actually.

So how does so we are showing how you can read your magical interpret the two notions that we had developed today so let us cover so here is A cross B so these are A vector and this is your B Vector and this was your see Victor so a cross b has some kind of is a perpendicular to the plane assuming that I am moving like this A cross B.

This is my motion. So it now what is the norm of what is what is we reading the dot product with c what is that so a cross b into c is nothing but norm a. So this angle some i so norm a Sorry, it is a norm of a plus b into norm of c into cosine of psi. That that is what you have here. Right? That's that is how that is the definition of the dot product. Right?

But again, what is c cos five what are the c cos five giving me? The c cos five Why is actually this is this height this is actually the height if I replace this parallelepiped with a rectangular parallelepiped standing up between the same two parallel basis. So, see this is the height. So, again if I take the base the idea of the base rectangle and multiplied by the height, then I get the area of the area of the rectangle parallelepiped.

So, basically and that rectangular parallelepiped, but has Should I buy up geometry the same though is three dimensional you can just take your vision for the you can just take a leap in the imagination and know that the rectangular parallelepiped which is standing between the same to basis and this parallelepiped must have the same area. So, this thing is nothing but the area of the parallelepiped.

Because this is this is nothing but the height. So the base the what is the area of the rectangle parallelepiped but the base area into the not area of the parallelepiped whenever you make a mistake sorry I should be corrected volume so emptied adventure there is no area talking or say it is over. So, volume of the panel we please forgive me it should be volume of the parallelepiped.

So, the volume of the parallelepiped is that this is nothing but the base the base area into the height. So if you do not have covered it rectangle parallelepiped is what the base area into it so, but that rectangular parallelepiped we were at the same area as this parallelepiped. So the standing parallelepiped so volume of the parallelepiped. So this is nothing but the volume of the parallelepiped.

And so what is happening? What is the volume on that parallelepiped? I know what is it called a dot across b dot c, it is a one B one c sorry a one a two a three B one B two B C and C one c two c three, which I already know. So, because a triple product is nothing but this determinate, so, this did three dimensional determinant is nothing but he presents the volume parallelepiped.

But, two dimensional determine and represent the volume of our Area of our parallelogram. So, here geometry both three and two dimensions are linked to the notion of determinant. So, algebra and geometry are linked and that is exactly what all mathematics is all about that algebra is somehow geometry and geometry is algebra. Once you want to get a feeling about this, then you will start enjoying mathematics. Okay.

So, I call it off for today and we move forward for the next day's one which will come along which will be will be going now toward higher dimension. So, we will no longer be in three dimension we will be in three to four, five, six operand dimension, and will show by some examples why this in dimensions matter why you have to go beyond three dimensions. One comes from physics, one special relativity.

Another comes at very good example comes from linear programming. And that is where we are going to show that we really need to talk about dimensions more than three, we really need to talk about something called area and variant could be five, ten, twenty anything when we are million, right. So, thank you for your patience.

And, if there is any problem, you can always write it down on the portal of the course. And please go back and listen to this lecture once again. That the co idea is that algebraic definitions and manipulations are actually inspired from geometry. Math is all about visualization. Thank you very much.