**Calculus of Several Real Variables**
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# Lecture – 04
# Matrices & Determinates

Welcome to the fourth lecture, we do have a long way to go five lectures per week, eight weeks 40 lectures, you are in the fourth third is more. Wow, I think that is quite intimidating. But you know, to learn something, you need to be patient and you need to go on for a while and gradually get a hang of things or what is going on. Because what I present to you is very organized knowledge.

But I will also like to tell you beforehand that no knowledge that you have seen such as presented to you in such an organized manner has ever been developed in an organized way. It comes in bits and pieces done by various people in various times. Found In many years after the death of the person who did, the true discover is often forgotten, somebody else takes the name. So it is filled, the stories of such things are filled in science.

So But finally, a good amount of knowledge which is really useful, really applicable, really needed by humanity has been organized into and taught to people so that they can also use them in their profession, in their activity and whatever they want to do. So as a result, when you learn organized knowledge, you really need to be patient, to you know, sit down and listen as to what the hell is going on.

And then only you will gradually have a little understanding of what you are doing. I try to be more moody have more different time I try to be more conversational with you and try to be more give you more motivation than anything else in the sense that I would like to talk to you. And I know, nobody is sitting right here in front of me. But I know there is a huge gamut of people who would be final observing this.

And in keeping in mind that imaginary audience, I would keep on doing this conversations with you. Now, what I am going to learn today is matrices and determinants. And this is something you must have already heard in your high school. So it is lecture 4, I have already written in that electronics lab also matrices and determinants of this, you might suddenly think why I am getting up of this idea of vectors, functions into suddenly into such strange terms.

Matrices and determinants and those who know all matrices and determinants, they would not be finding it surprising, but let me tell you that these are technical things that we have to learn, because matrices and determinants will play on a very fundamental role as we go on matrices would start appearing as derivatives, derivatives of form vector functions. Where one vector space is carried to another vector space in the sense or in is carried to rm.

So, when you talk about such functions, for example, we have seen the force function r2 is carried to r3 just in the in the last lecture. So, when you talk about derivatives of such functions, then you are really going to express such derivatives in terms of matrices. So, what are matrices are see here in this lecture will be just considering 2 by 2 and 3 by 3 matrices. So, matrices are arrays of numbers which are treated now in forms of rows and columns.

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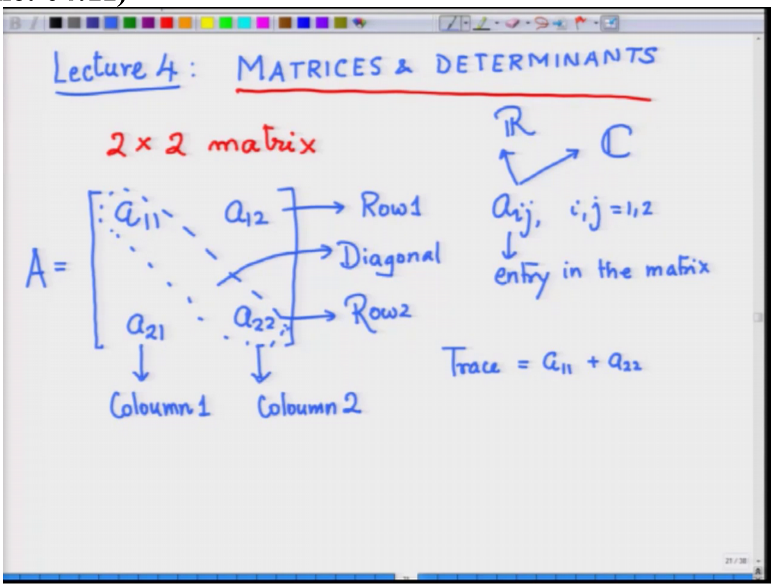


So, that we just write down a 2 by 2 matrix. So, 2 by 2 matrix which I can write as A if you want is given us follows. Where, so this a 11 and a12 this vector not to consists of the first row, row1 and then these are all numbers they could be real number or complex numbers also in physics for example, matrices with complex entries called poly matrices are used very much this row2 while this is column1, overriding the correct spelling, and this is column2. So, this is called a 2 by 2 matrix because there are two rows and two columns.

Of course, you can say that can there be one cross two matrix of course, there can be one cross two matrix that is, for example, this one if I just take a 11 and a 12 then one row and two columns, so, such a matrix will be obviously a row vector. So, this is a matrix, you can obviously if I talk about 3 by 3 matrices, I can so, I have to add one more row and one more column So, write them down. So, let me write down what is a 11.

a 11 means the entry in the first row and first column a 12 means the entry in the first row, but second column, a 21 means entry in the second row and first column a 22 means entering the second row second column. So, this a ij where i and j values are 1 and 2 while are taking the other one and to this a ij is it is called the entry in the matrix and this entry, either is a real number or a complex number.

So that is what I, there is a very basic thing this a 11 a 12 this thing this stuff many of you possibly would know is called a diagonal of the matrix and really some the row2 elements of the diagonal if you some them that is called trace, which is here same as a 21 plus a 22 Okay, you can read or higher dimensional matrices with more rows, more columns, we will, but not now, later on.

So let us get habituated with simpler things first, and then we will talk about much more difficult things. So, I would expect you to know how to write 3 way through matrices. Now, just an extension several I call this matrix A Tilda and this is a 3 by 3 matrix. So, first row first entry, first row, and second entry now, another additional entry first third entry. Similarly, second row first entry second row second entry, and second or third entry.

And similarly we can write down for the third row, for the row one has increased and as a result one column is increased. So, such matrices which has same number of rows and columns, these are called square matrices. These are called square matrices associated with every square matrix is a number which we call determinant.

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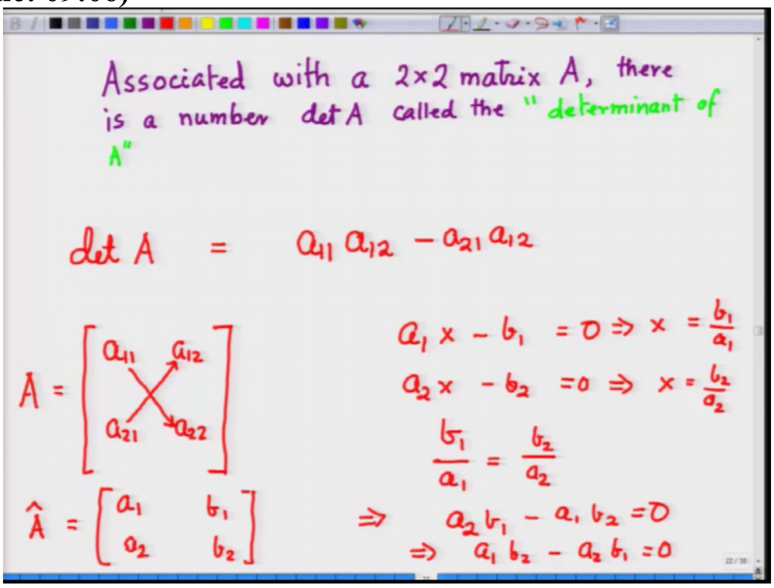


So, associated say with a 2 by 2 matrix with a 2 cross 2 matrix A, there is a number which we call determinant of it there is a number det A, denoted by det A called the determinant of A.

Call the determinant of A now what is this determinant of A, what is this determinant of A. So let me write down the determinant of A.

So we are going to consider the 2 by 2 matrix, this one. So if you have forgotten the 2 by 2 matrix, I just write it down. If you want to, if you do not want to go back and check it, I write it here. Determinant of A is you multiply a 11 with a 12 then you subtract from it the product of a 21 in a 22. So it is like this, multiply this and then subtract this product of this into this. So, it is a11 into a 12 minus a 21 into a 22 that is called a determinant of A.

So, determinant of A is a function, which is on x the set of all 2 cross 2 matrices and it gives me a number. And this is the way I calculate that number. You can why you calculate such a thing and what is the requirement? Okay. Maybe I should try to explain to you why write down these things. So, let me just give a try to give you an explanation. Suppose I have an x which satisfies both these equations and it also satisfies this equation.

So, when can I satisfy both these equations? So let us see. So observed that from the first one here, I get x equal to a 21 by a 11 and from here if a 11 is not 0, if you a a11 is 0, I could not say anything and it will be zero concepts. I assuming that these are not 0, then I will get x equal to a 12 by a 22. So suppose you have a scenario like this. So what is this I am just writing in arbitrarily let me be more precise on this matter.
Let me just write it down in a very simple way. Suppose you have two equations a 1 x minus b 1 equal to 0, then a 2 x minus b 2 equal to 0. So from this I will get x as you may want to not equal to zero. I will get b 1 by a 1 and from here I will get x equal to be to buy a two of what these beyond and it would be 0 than 0. Okay. Otherwise what is going on?

So, then what happen then that does mean that b 1 by a 1 is equal to b 2 by a 2 which implies a 2 b 1 minus a 1 b 2. Now what is this? So this is equal to 0, but what is this quantity? So whether you take a matrix A hat, which is made up of is also you can take it you can write it like this a 1b 2 minus a 2 b 1 equal to 0 just in signs. Let us take into the other side.

So, I can write this as a 1 b 2 if it is A hat matrix is this a 1 b 2 and a 2 b 1, if I have a 2 by 2 matrix like this, then this is nothing but the determinant of A. So, what it says that if there is an X, which satisfies these 2 equation, when you want any two is not equal to zero and says I am that b 1 and b 2 are not equal to 0, then and x which will satisfy these both these equations simultaneously.

When A x will satisfy both discussions Hamilton's you only if the determinant of this matrix is 0. So, this is sometimes also called an element of the simultaneous equation. So, is some link between the question solving and the determinants, we will come to you there will be some more things about determinants, you talk about Kramer's rule in exercises and okay you know how to solve a system of 3 linear equations.

And hence, you know, you will know you need the use of determinants, we will talk about it later on. So now how do I derive the determinant of a 3 by 3 matrix? When I have a 3 by 3 matrix, how do I write down the determinant? For example, this one so, how do I write down the determinant of a 2 by 2 matrix is a building block. So, if you know if you determine that 2 by two matrix wanted to do a determination of 2 by 2 matrix.

Then you can also write down determinant of a 3 by 3 matrix. So I will start in this way. So a11 so what do we take the element in the first row and first column, basically then we basically take of everything that there is in the first row and first column. So what is the remaining is this part, so we write this in determinant of the matrix a 22 a 23 a 32 a 33 minus so the, when I come to a 1 a 12, what I do is I take off this first row and the second column.

But I will put a minus sign that is the definition of the determine I am not getting the detail, but please accept this as a definition. So when I am in 3 dimension I am define defining the determinant in this, I am a defining the determined as you that this is definition, right? So what I am doing is I am trying to use what I know about 2 by 2 matrices or 2 by 2 determinants into this thing.

So it is a12 the signs would be actually plus before these things minus and plus, and they keep an alternative you can have a larger matrix of a 12 into the determinant of the matrix this so it is a 21, a 23, a 31 and a 33, then add a 13 into any multiplier the determinant of now you take the first row and third column out so you are left with this. So it is a 21, a 22, a 31 a 32. So this is the definition of the determinant of A Tilda right.

So here we are using the knowledge of the determinant of a 2 dimensional to do 2 by vector, a 2 by 2 matrix. Now there is some facts about determinants that possibly you should know, which is, well just give some examples of determinant.

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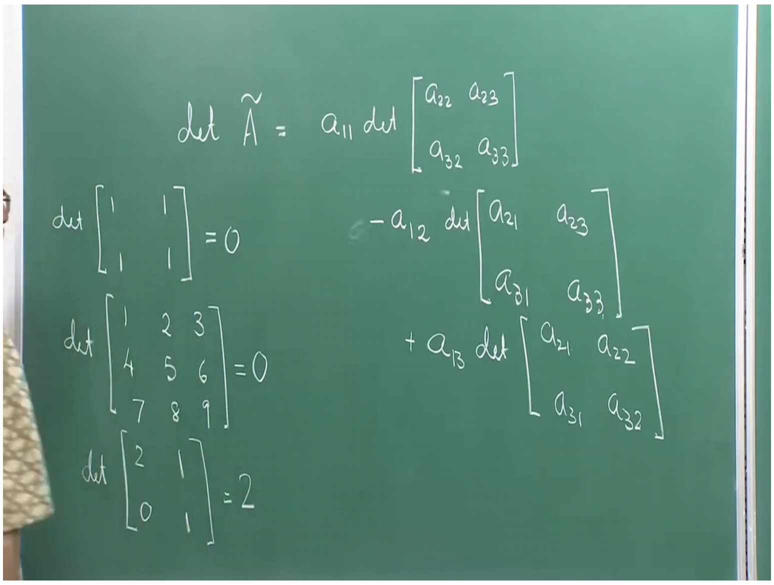


For example, let me take an interesting matrix determinant of all the entries are 1 you can easily calculate that this is 0. This gives me this gives us a suspicion that if there are two rows. Which are same 2 by 2 matrix, then they must give me 0 has the determinant value. But that does not mean that if you have all those different you cannot have a 0 value in the determinant.

So for example, determinant of this matrix which I learned as an undergraduate student, and I was very surprised when my teacher wrote this on the board, because it seems to have some kind of symmetry, which I enjoyed 123 456 789 is a 3 by 3 matrix on the determinant of this amazingly 0. Of course, there are matrices whose determinant is non zero and I need not give you an example for this for them will determinant 21 01 is determinant is 2 so and so forth.

Right I am not getting into the larger details of it. But there are some properties of what determines which we should know is the following. Suppose, you exchange suppose you have a 2 by 2 determinant and you exchange the first row in the second row. Usually in the first row, you put the second round in the second row put the first row, then what happens to the value of the determinant when same model changes.

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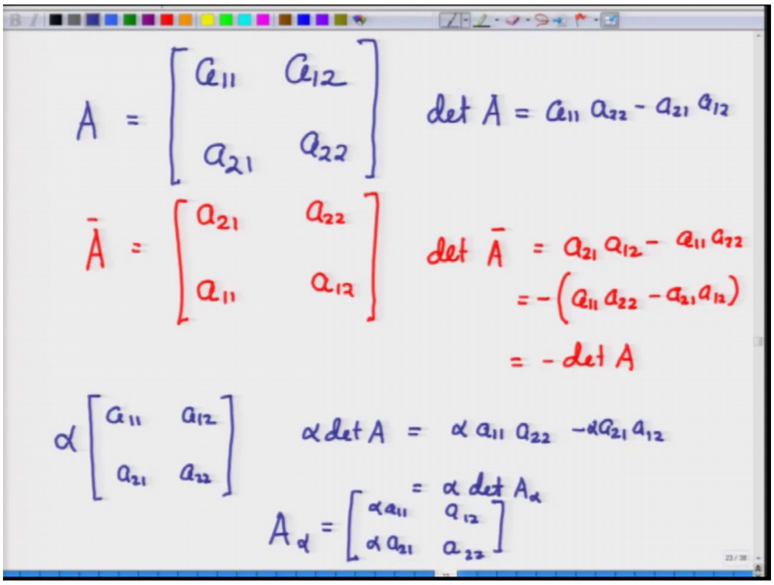


So you take A again, which is a 11, a 12, a 21, a 22, and then you choose an A bar for example. So here what I do, I take the first row and place it in second row and second row is taken to the first row. So, I will have a 21, a 22, a 11, a 12, let us compute the determinants of this. Now, determinant of A, which you already know, is a 11 a 22 minus a 21 a 12 now similarly let we calculate the determinant of A bar. So determinant of A bar is equal to a 21 a 12 minus a 11 a 22.

Which is same as minus a 11 a 22 minus a 21 a 12 and that is nothing but the negative of the determinant of A. So if you change one row, and exchange two rows of the determinant, the sign changes. Suppose I change in this 3 by 3 matrix I bring it here and take it here my sign will change. But then if I do again, I will bring this here so I took the first row in a thought during the first row, but now again, take the whatever was there in the third row in the chain matrix.

And take it to a second and bring the second row there. Again there will be a negative sign will get back. So two changes in a 3 by 3 matrix would give me back the first one determinant value. So this is a property which you really have to keep in mind. It also works for three dimensions, but let us now get into a single property. If I take a number alpha and multiply it by the matrix a 11 a 12 a 21 a 22. Then my question is alpha det A.

What does it mean? Actually, it simply means that if I am doing so alpha of det A. So this is nothing but alpha time's a 11, a 22 minus alpha time's a 21, a 12. So I can write this as alpha time, determinant of A alpha, where A alpha is the matrix, say alpha a 11, alpha a 21, alpha a 21 a 22. So, you are multiplying, basically, you may take a matrix multiply any of the rows or columns by the alpha scalar multiplication, division is a vector ultimately every column or row is a vector.

And then basically it is nothing but alpha times the determinant of that value. So, it does not matter wherever you please it because of this, the same thing happens for 3 dimensions, you really, I would leave you as an exercise to really check for 3 dimensions, which are not going to get in because now we are going to tell you something more important. Suppose I do the following operation, I have a 3 by 3 matrix.

Which I would not do on the board right? Let me now I would these 3 by 3 matrix A Tilda. And what I do, I multiply every element of this third row with a number k and add them to the first row and I construct a new A double Tilda new matrix, which is a11 plus ka 13 a 21 plus ka 23 that my first row as first column completely gets changed a 31 plus ka 33. Now the second or third column remain same a 12 a 22 a 32 a 13 a 23 a 33.

Now my question is I am asking what is determinant of A Tilda Tilda answer is that it is same as the determinant of A you can play this game. So now let us know constructor I will do some operations on this matrix. So what I will do, I will multiply by a number k, this vector. So it will become k 13, k 23, k 33, and add that new vector to this first column leaving the other two columns on change.

Then I will get a new matrix and are asked.  What is the determinant of the new matrix? and what is this relation to this old matrix.

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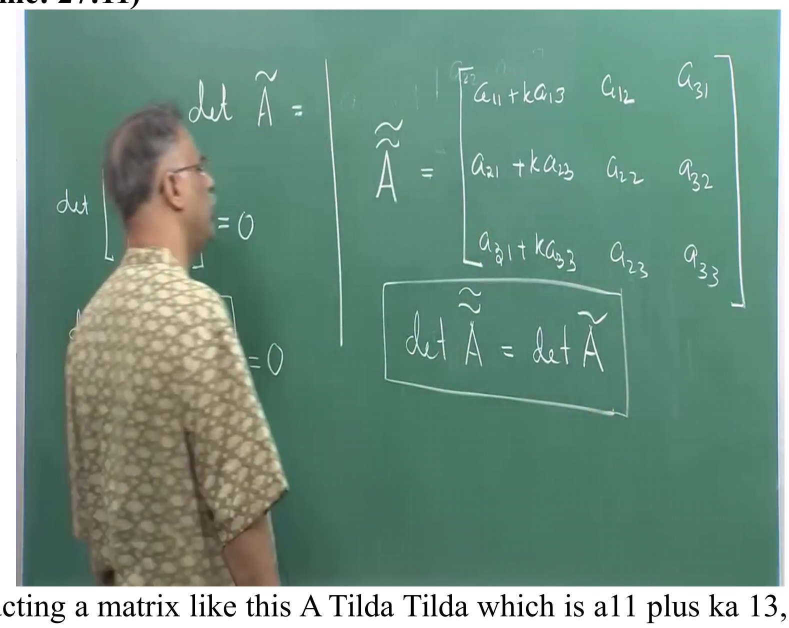


So I am constructing a matrix like this A Tilda Tilda which is a11 plus ka 13, a 21 plus ka 23, a 31 plus ka33. And the remaining two remain same, a 12 a 22 a 23 a 31 a 32 a 33. Now I asked the questions, what is determinant of A Tilda? A Tilda Tilda Sorry? The determinant of A Tilda Tilda same as determinant of A Tilda.

And that is the key. So it does not matter whatever operation you do, you can possibly multiply some k1 into this, k1 into this and added to the second row and calculate the determinant will get the same thing. This has got a lot to do with linear algebra, which you can later on, think about. So now we are going to talk about adding matrices subtraction is just quite similar condition and multiplying matrices.

So here I am going to explain to you first on the board I am talk about multiplying now here, let me explain to you what is adding a matrix.

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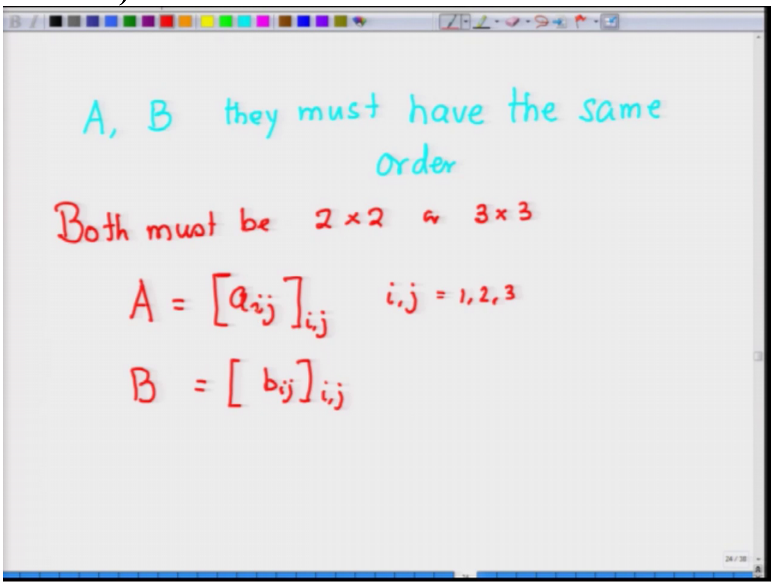


So when you want to add a matrix to matrix A and B, they must be of the same time you are adding to square matrix says they must have the same order means both must be 2 by 2, or 3 by 3 any matrix A can be shortened. For example, have a 3 dimensional matrix, I can write it as a ij and ij where i and j take the value 1, 2 and 3. These are entry of the matrix and B can be shortened b ij so, both are varying from 1, 2, and 3.

So i is the indicating the row number and j indicating the column number just as we had done it. So I am just trying to make you work a little more compactly making it a little more mathematically slightly sophisticated looking to you or just being very simple things. So A plus B is a new matrix where you are adding you're a ij plus b ij elements ij which means if you have a 11 so basically what it what does mean.

So it is the same ij. So, if you have a 12 then you add it i is 1 and j is 1 then a 11 plus b 11. So you take the corresponding positions. So if you want to add some another matrix where a b

11 and b 12 b 13 on this add with a 11 plus b 11 a 12 plus b 12 and so forth. So, these are short way of demonstrating. Now I am going to talk about multiplication of matrices. Right? You must be asking why you are only talking about 2 by 2 matrices 3 by 3 matrices why not 3 by 2 matrices and 2 by 3 matrices.

Of course, we are going to talk about 3 by 2 matrices and 2 by 3 matrices, but we will not concentrate on them on this particular lecture. But after 2 lectures, we will be really getting more into details about them. Because those kind of matrices would actually play a fundamental role in starting vector value functions. So anyway, let us take A 2 by 2 matrix and B and try to multiply them. I will show you a very simple way once you have learned about the dot product things are as simple as they can be.

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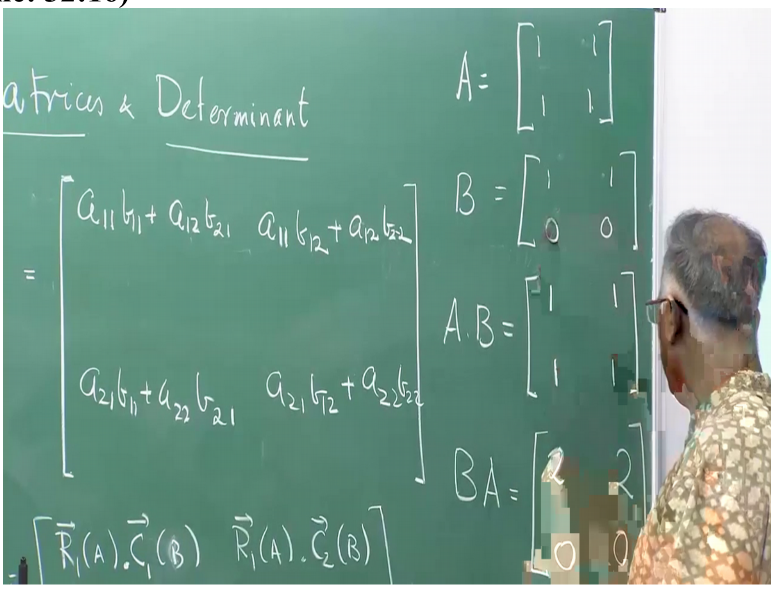


So, simple our matrix A 2 by 2. So, if you only 2 by 2 matrix can be multiplied by with a 2 by 2 matrix, a 2 by 2 matrix cannot be multiplied with a 3 by 3 matrix. Please keep this in mind and why we have to restrict to this will come after two classes, right. So, here I have, say a 11 a 12 and a 21 and a 22 and B for example, that we take as b 11 b 12 b 21 b 22 the how do I compute A multiplied with B it is all a game of taking in a products.

So, let me show you when we start with AB that is you multiply A with B not B with A in matrix world multiplying A with B is not same as multiplying B with A will show you an example, but let me just show this first.So, what we do is the following will fill up this place and what let us see how I will take the first column. When you start with A you take the first row away and take a dot product in the first column of B.

So, what is the product a 11 into b 11 a 12 into b 21 a 11 into b 11 plus a 12 into b 21. That is the definition. So, now consider the first row and then take the second take the dot product with a second column of the first. So this is row1 and this is row2. So you take the row1 and take the dot product with the second column. So that will give me a 11 b 12 plus a 12 b 22 now you come to the second row away, and then do the same thing with the two columns of B. So, you start with the first column of B.

So, you will take the dot products a 21 into b 11 and a 22 into b 21 a 21 into b 11 plus a 22 into b 21, and then take the second row and take the second column of B and take the dot product it is a 21 into b12 first into first, plus second into second, a 22 into b 22 look at the kind of symmetry, a 21 b 12 a 12 b 21 a 11 b 11 a 22 b 22 to their symmetry. And I love this kind of symmetry. And here is a 22 here is b 22 b 21 a 12. So this is pretty nice. So it is the same thing you will do for 3 by 3 matrices.

Let homework just let just it out of assignment homework for your trial home try I would say multiply 3 by 3 matrices. What are you doing if you take these rows as vectors, if you write them as R 1 vector R 2 vector and these are C 1 vector and C 2 vector? so row1 of A row2 row2 of A C 1 of A and C 2 sorry, C 2 of first column A and second column of B. So these are vectors then what do I get it? I Get I can write AB simply as write them simply as R 1 A dot product.

C 1 A R 1 A sorry C 1 B dot product C 2 B then you write R 2 A row2 of A dot product. The column C 1 of the some writing there was vectors R 2 of A dot product C 2 of B that is exactly what matrix multiplication. Now, give you an example A into B is not B into A. So let us take a matrix A which is 1 1 1 1 you know what is this determinant is 0 and then take a matrix B. Which is 1 1 0 0. So what is A into B.

A into B is take the first column and take the let us take the first row of A first column of B. So 1 into 1 then 1 into 0 you get 1 and then take the first row and the first column you will again get 1 here. Here also if you do the same thing 1 1 into 1 0 in a product and 1 1 into 1 0 at 1 1 or what is B into A. B into A is the game changes. Now, I will talk I have actually I will start with the rows of the matrix.

Which is on my left side, and I will use the column of the matrix which is on my right side. So here now I will start 1 1 in a product with 1 1. So 1 into 1 plus 1 into 1 which is 2 similarly keep 1 1 and also take with this 1 into 1 plus 1 into 1, it is 2. Then I will take this second row of AB and multiply with take the inner product with the first column away, which is 0. Similarly for that is also 0. That is first, second row of B, and second columns of A. So you see AB and BA are two different matrices so product of two matrix is a matrix.

But where A into B is not B into A we are in a world, a community. We committed DVD which you are so. So happy with when you are talking about real number A B is equal to BA is invalidated here. It is not valid here. So with this little interesting feature that mathematics can for you some interesting tables, some interesting facts, we end here and in the next class will talk about a very important.

Another very important vector operational, very different type of multiplication. We have talked about dot product of two vectors. Which give us a real number. Now we will talk about a cross product of two vectors, which will give us a vector. So with this, I end my talk today. Wish you a very good evening and happy I would say a happy this video thank you.