Calculus of Several Real Variables
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Lecture – 36
Gauss Divergence Theorem

So that is it the end of the course that we are coming into this is worldwide I guess would be the end of the first introduction to multivariable calculus after that the various aspects go into various things means somebody can be interested in the LaGrange multiplier rule or may be interested in differential geometry but everywhere you will see the role of calculus and multivariable calculus and that is essential to understand and that is the message of this course that calculus is stupendously useful for our modern civilization.

Today, we are going to speak first about Gauss divergence theorem and then I will speak about miscellany. There are interesting facts will put in and things and we will talk about several issues pertaining to something about fields something about curves something about divergence.

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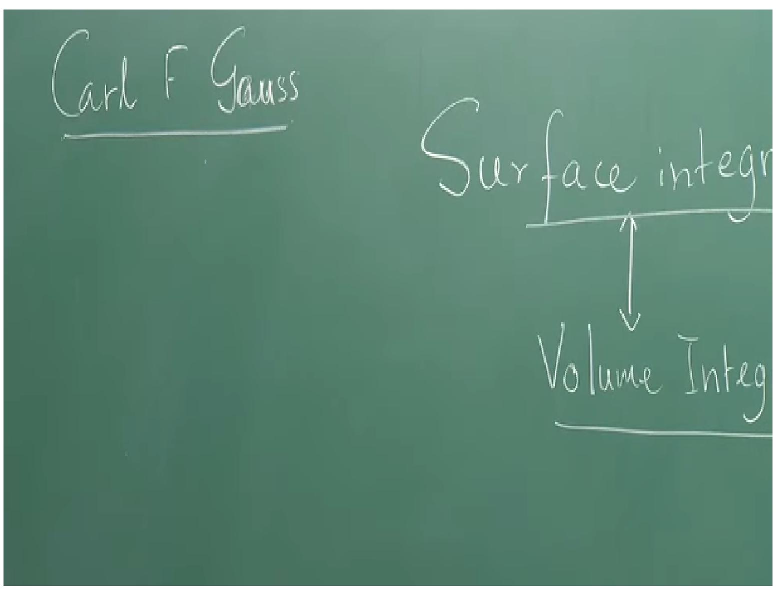

So we are going to talk about Gauss divergence theorem which relates surface integral with volume integral volume integrals are triple integrals. So they are they are done over dv or related to surface integral is related to volume integral. Anybody who has done some mathematics and says that he has never heard the name of called Carl Friedrich Gauss would be a sin against mathematics.

Carl Friedrich Gauss was probably one of the greatest mathematicians of our times and he I hope you would not mind if I take a sip a cup of tea. So I am this would be a much more relaxation so he taught in a place called Gottingen and I had been lucky to visit at least stand in front of the observatory of Carl Friedrich Gauss I have visited his grave and the city of Gottingen which is the university city which has a very famous university.

If you walk through the streets you will see all the famous names appearing out written in various houses Hilbert stayed here Gauss stayed here so and so. So Gauss had an immense contribution to mathematics whether it is algebra whether it is geometry whether it is about numbers and of course Gauss had a inkling of the idea of Non-Euclidean geometry. A revolution which was led later on by Rahman Lobachevsky many other mathematicians.

So unfortunately many overseas mathematics try to do research try to do something try to we think that we are trying to make a progress making a little progress of order subject to order ideas trying to answer the questions we have. But I do not know maybe fortunately or unfortunately I have no idea what to say but I want to say that mathematics only celebrates his greatest best and Carl Frederick Gauss is definitely one of the greatest ever.

Carl Frederick Gauss is one of the immense contribution of this divergence theorem is because they all come from a geometrical angle this contribution of divergence theorem has immense applications to fluid mechanics to electricity to several other things. So let me just write down so our aim would not be to prove the Gauss theorem because we have shown one proof of stokes theorem.

What we will be doing is state the Gauss theorem give one example of the Gauss theorem maybe 2 if you want 2, 3, 4 more in fact and that would be our aim and once that is done we will go into miscellaneous topics. So what does the Gauss divergence theorem says it is sometimes called the divergence theorem. So what does it say is that?

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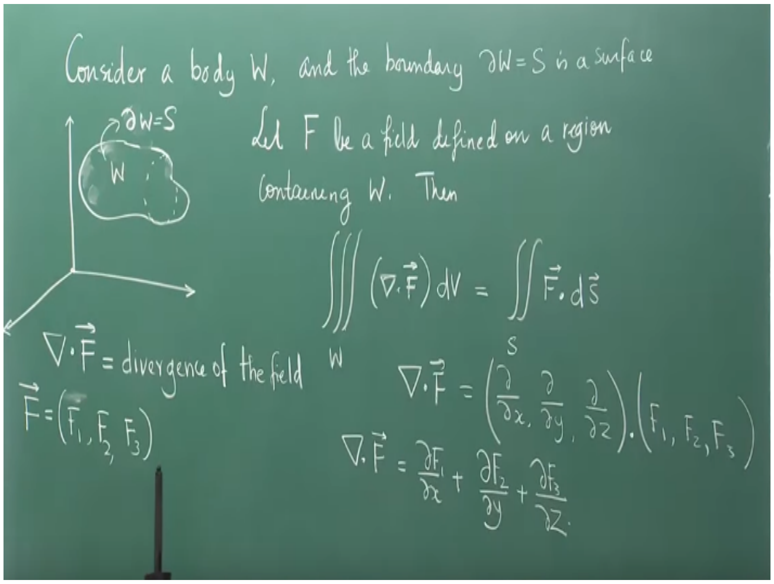

So consider a volume W consider a body W and the boundary del W = s is a surface. So it could be a body like this it is much more better to teach math on the board than down on this strange screen certain things for certain things old is gold is very true for certain things not. So here is a body like it is a kind of 3-dimensional body. A 3-dimensional body and the 3-dimensional body W and there is a surface the surface is del W beside del W which is s and you have a field continuously differentiable vector field defined on W.

So whenever we are talking about field I already mentioned a field is a vector valued function whose all components are continuously differentiable that is when I am talking about a field I am actually talking about continuously all the components should be continuously differentiable that is what I have mentioned when I have spoken about fields. So fields are vector valued maps but we will call those vector value maps fields which have all their component functions continuously differentiable.

So this is a criterion that we have so let F be a field defined on a domain on defined on a domain containing W. So maybe you defined on W whichever you want to say F be a field defined on a region containing W then the Gauss divergence theorem says why the name divergence comes is this. So if you take the divergence of the vector map you take the divergence of this and you integrate.

So this becomes a scalar function because divergence of V is a scalar function I will tell you what is it and that is equal to the S. Now what is this, the; we have spoken about dot products but we have not spoken about divergence. So delta dot F is called the divergence of the field. So how do I why do I write it like this. So let F the field F has 3 components by now you are habituated to the fact that these are all functions from R3 to R F of F1 of X, Y, Z F2 of X, Y, Z and F3 of X, Y, Z.

So I write F2 and F3 that is what I have written and now what I will do when I am talking about dF i will write down the del operator which you already know from yesterday and that dot product with I am not writing in the form I, J, K it does not matter. The del operator and dot product with F1, F2, F3 and you know you think as if I can do a dot product operation of course these are not these are not numbers but I assume that you can do it so you multiply this with this so del del X of F1.

So this means that this by definition del dot F is del del X of F1 + del del Y of F2 plus del del Z of of F3 this is actually the definition of divergence of F. So this is equal to the surface integral of F so the volume integral diverges volume integral of the divergence of s is equal to the surface integral of s. So it could be that sometimes the surface integral is difficult to compute but it is much more easier to compute this one.

So let me first start by showing you some simple examples related to the divergence theorem I want to also say that F dot ds so ds can be at any point s. So if you take a normal at this point it is pointing outward like this but this normal can be viewed to have 3 different angles alpha, beta gamma. So with the x axis it has alpha angle with y axis it has beta angle with z axis it has gamma angle alpha, beta and gamma. So it is alpha, beta and gamma.

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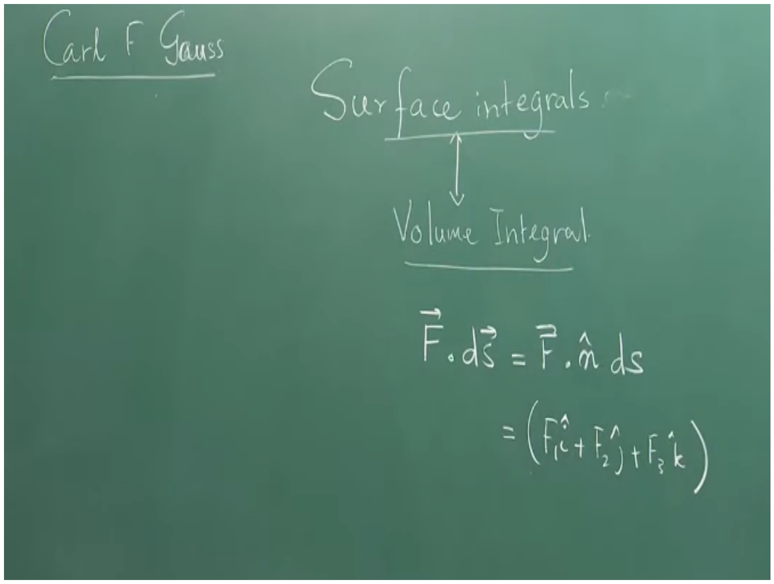

So when you write here F dot ds vector you already know that this is nothing but F dot n hat ds but in hat now can we actually written as in as a component wise. So if F is of the time F1i vector + F2j vector + F3k vector. So, the coordinates of the tip of n.

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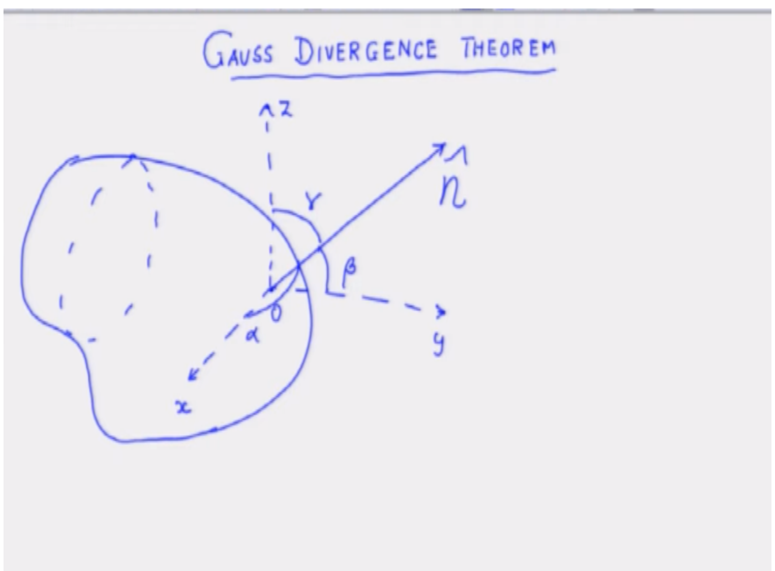

So here is your surface here is my n and here I have I think by my imagination I can bring the x axis y axis and z axis. Now so there is angle alpha with this angle beta with the y-axis angle gamma with the z-axis. Now if you keep on dropping to the tips you will get back x axis y axis z axis. So you get the distance of y if I drop from here a perpendicular on the y axis you a draw a perpendicular on the x axis.

I will get the x coordinate or so if I view it like this as if the 0 is I have drawn the normal at 0 then what happens is then I can write this normal in the following way dot product with the normal can be written as. So what would happen if I drop a perpendicular here on the x axis so what is the length it will cut off it will cut off a length which is n which is same as the Norman of cos alpha.

So Norman is 1 because it is the unit vector so other cases it will have Norman of cos beta a Norman of cos gamma and Norman is 1.

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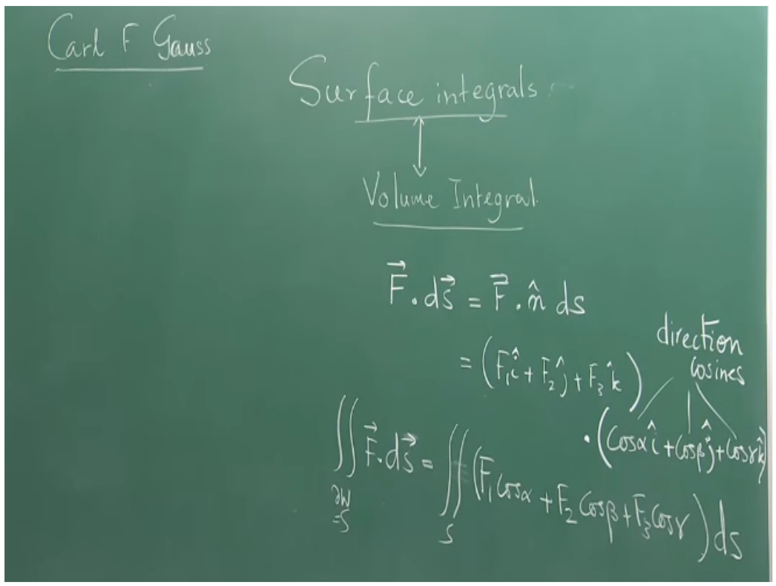

So the 3 components so the 3 coordinates are cos alpha cos beta, cos gamma. So I can now write it in terms of 3 coordinates basically this is nothing but the coordinate representation cos alpha, cos beta you need not write in terms of the coordinate also this cos alpha, cos beta, cos gamma these are called direction cosines. This actually tells you which direction the normal vector is pointing to at the given point and that will give you F1 cos alpha + F2 cos beta + F3 cos gamma into ds.

So that is also another way of deriving this vector so you can write it as so if you write the integral. So integral of so double integral over del W which is s which is equal to s F dot ds is the same story over you can have this alternative form of writing also. Now let us look at some examples we will not give the proof of the Gauss divergence theorem here if there somebody wants a proof if you can make a request of that on the thing we will put up the proof in the portal.

I think you already received your proof of the Taylors theorem for second order derivatives in the portal okay so I take an example from the book.

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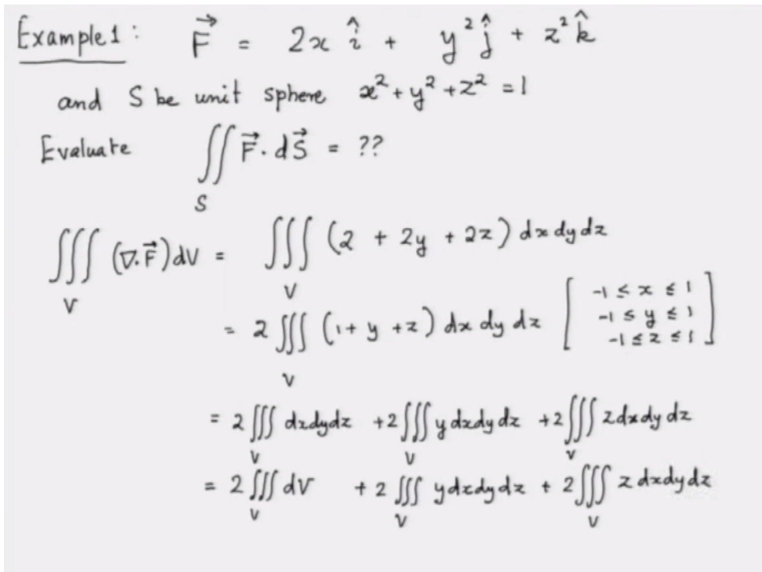

So example 1 says okay F vector is given as 2xi vector +y square j vector + z square k vector and s be the unit sphere and s be the unit sphere. So it volume is the unit sphere and everything inside it that is the v the W. So del W is this unit sphere that is it so I ask you to evaluate the line in integral. So that is what the book asks you of course you can evaluate the line integral yourself.

So suppose I want to evaluate the line integral myself. So I have to first find the n vector so I represented by spherical coordinates x, y, z and then I write the n vector you know or the n vector is phi cross u or or ds vector is the phi cross phi u cross phi v vector into du, dv. So I have to do that parameterization and do all those tremendous calculations. So that would be a mess of course you would not like to go into that sin theta, cos theta cos phi that kind of misses some people can enjoy it the trigonometric playing with geometry.

But many people might not. In this scenarios where you have to write things can go out of your hand the calculation the Gauss divergence theorem somebody knows we will try it using the

Gauss divergence theorem. That is all okay let me compute the volume integral of the divergence of F dV so what is my F so divergence of F is very simple. I will take del F del X here the divergence of F is very simple it is 2 + 2i + 2z and this is dx, dy and dz.

So what does it tell me I can take the 2 outside 1+ y + z dx dy and dz.? Now what are the range because it is a unit circle what are the range in which they are varying x is varying from 1 to -1 y is varying from 1 -1 to 1 so z is also varying in this zone maybe there are much simpler ways of doing it. Now we will see what happens so you have 2 times integral integral v dx, dy, dz which is dv actually + integral integral v y of dx, dy, dz + integral integral v z of dx, dy, dz.

So we let us; now I know what is this integral so its 2 times integral v dv. So 2 times the volume of v it will become so there is a 2 everywhere sorry 2 times now integral integral v y dx, dy, dz. See once I can evaluate this integral y dx, dy, dz I can evaluate the other one also. You can write W here instead of v if you want does not matter.

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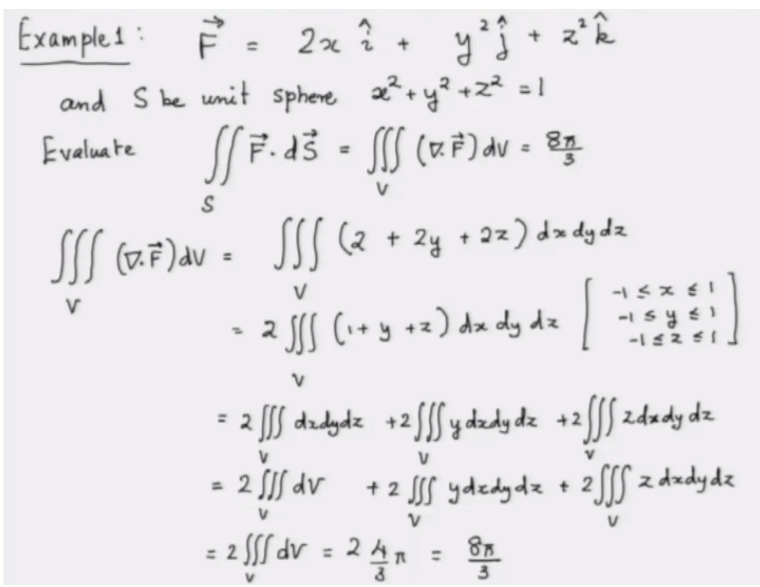

Now let me try to evaluate this so let me just evaluate first with respect to x and maybe with y does not matter it is the same. So y dx, dy, dz let me whether I put the things correctly here so even a mistake by putting -1 -1 at every point sorry it that is that we are in a hemisphere as a sphere. So we have to first look how z is varying that is varying over these - root over 1 - x square - y square to root over 1 - x -y square then how is y is varying in terms of x and then x is varying from -1 to +1.

So we have done this is the kind of thing we can easily do now then if you integrate in first dz then that is what you will have as y is constant. Now let us look at this part of the integral - i root over 1 - x square - y square dy. Now let us set a = 1- x square sorry root over 1 - x square. So a square = 1 - x square okay. So I can write this whole thing now as this can be written as minus because x is fixed in this case -x because it is just integrating over y 2y root over a square - y square dy.

Now I will make another substitution set y square = t it means 2ydy = dt. Now when y square = t so t will have how many limits when y square = -a it has the limit a t is y square and when y square is all a it has a limit a. So this integral becomes a to a, a square - y square is t and 2ydy is dt. So it is a to a sydy is 0 so ultimately this integral would be 0 so this is the important thing that integral integral integral Wydv is 0 and by symmetry we know Zdv is also 0.

So once you know then this integral so these 2 parts are 0 so this is nothing but 2 times integral v dv this is nothing but 2 times integral v dv and what is dv here it is a sphere it is so 2 times 4/3 pi r q r is 1 so 4/3 pi r so the answer is 8 pi/3. So that is the answer of the surface intern now this is and by Gauss theorem I know that this is equal to that this is nothing but the volume integral of and that is exactly the answer.

So here we have the answer okay so we have done one example let me show you another important result which is called Gauss's theorem and that is used for gravitation potential and electrostatic potential. So it is called the Gauss's results this is a result is due to Gauss.

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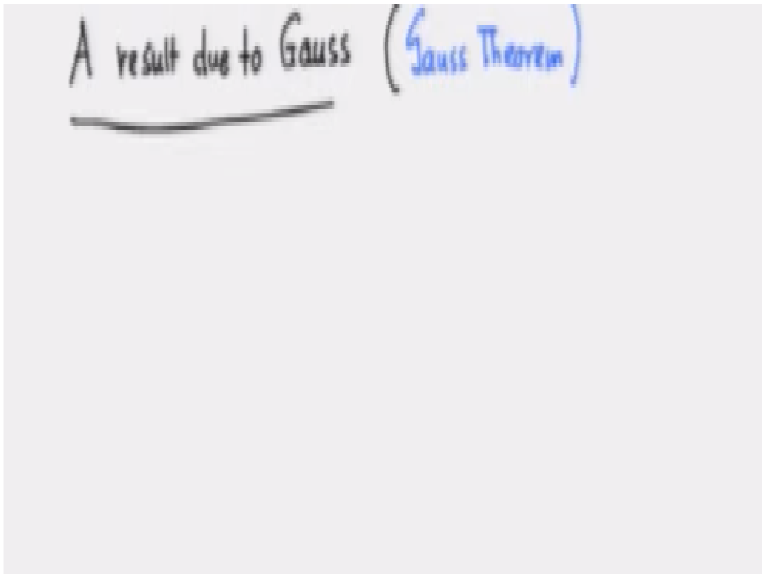

A result due to Gauss sometimes called the Gauss theorem. So this is given in a book the famous book of vector analysis by Spiegel.

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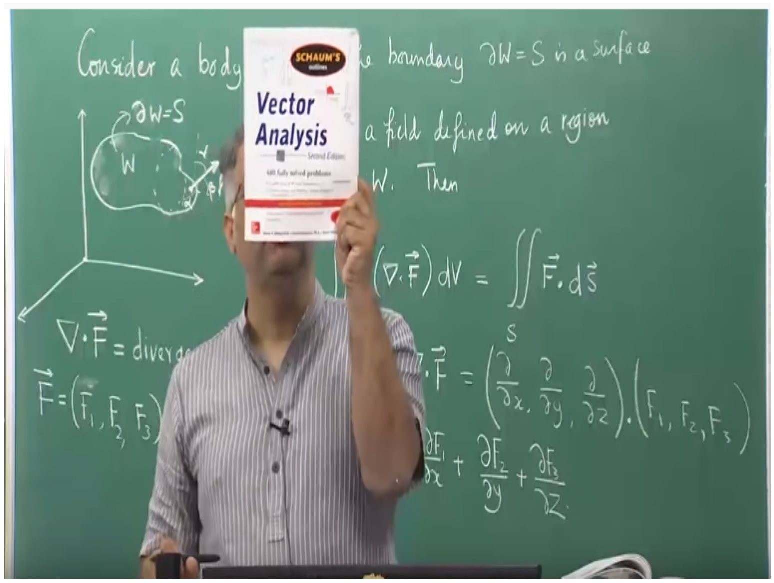

I think this is the book which you can always take for practice vector analysis by Spiegel from SCHAUMS series published by McGraw-Hill this is an extra exceptionally good problem-solving book. So there are a lot of problems and you can try and practice these problems. Please take a look of this book I will just write the name on the top.

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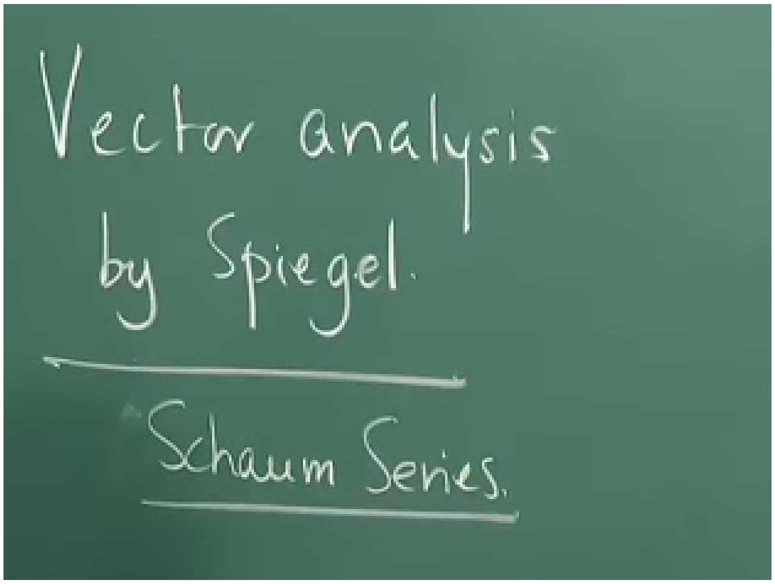

Vector analysis okay.

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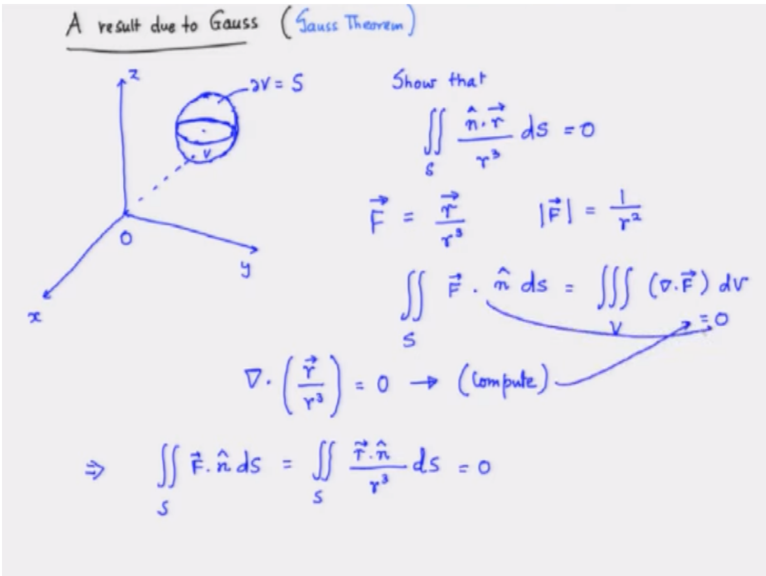

Now let me look at this problem the problem says that I suppose there is a spherical body. So here I have the coordinates and there is a spherical body outside the coordinates so their spherical body. So essentially so this is the origin and the origin is outside the body. So this is your volume v and this is the surface s is del v the spherical surface the sphere.

Now the question is that so that and this integral of s it is n dot r, r is a radius vector at of any point on the surface of course since O is outside none of them is xyz but none of them has value 0 xyz none of them has none of them are 0, 0, 0 r is not a 0 vector in vector or vector or cube this looks like gamma. So let me just write it again r vector ds so they are asking to show that this is 0.

So here F is a field r vector/r cube so the value of F the numerical value of F is 1/ r square so suppose there is a force field which whose magnitude is just inversely proportional to the square and let inverse the portion to the square of the distance right. Then so field which is whose magnitude is just 1/r square then I have to show that this is 0. So basically the whole thing is written as F dot n ds/s and this is nothing but by the Gauss divergence theorem.

Now divergence of since r is non 0 these divergence turns out to be 0 it is not very difficult to compute this I leave it to you as an exercise to compute and hence by Gauss divergence theorem what I get so this implies integral s F dot n ds when what is F dot n if I write down F dot n it is simply r/r3 so it is nothing but r dot n which is same as n dot r with you all commutativity of the inner product a dot b same as v dot e ds and this is 0 because this is equal to this and now this is 0 because of this fact.

So this is also 0 and hence I have this now there is another question were also the Gauss divergence theorem will play a role but the very subtle one and we learn a lot of things while doing it. It says that no now let us face hemisphere spherical region be such.

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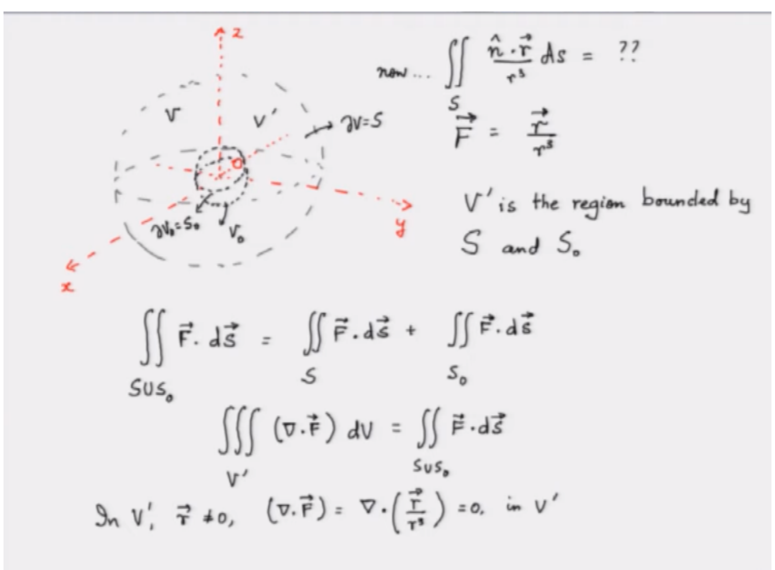

Let the spherical region be such that 0 is inside that is spherical region so the origin lies at the centre of the spherical region so here is a volume I draw a volume v. So at the centre lies the origin O now I want to now now I want to go and ask the questions what is the next question goes of is what is this now what is this. Now evaluate now find the same question the same surface integral what is this is it 0 that answer unfortunately is not, it is not 0.

Because when you are talking about finding the divergence you are finding the divergence. If you look at this theorem beforehand look at this application here in page or have skipped to 221 okay. So look at this application here when I am computing this grad dot F I am computing it at every point of the volume. I can do this here and every point of the volume because at no point of the volume are become 0.

But here I have a problem because 0 is the centre; the centre of the sphere is itself the origin. So 0, 0, 0 is there so when I compute grad d dot F is not defined at that point. So grad dot F here, F is the same F is but it is not defined. So what is the technique by what technique I can overcome it and here is a technique which is typical to physicists and that what we are going to do we draw a small ball around O small sphere whose centre is the origin.

And basically we target out and we now look at the volume v dash. So we call this spherical we call this the bigger big volume v and this small volume we call it v not and we add v dash. So v dash is every element in v but nothing in v not so the kind of thing like that iso it is a region bounded by that v so I should not write it like this but v dash is the region bounded by S and I will call the boundary of this sphere as S0 and S0 is del v0.

And this is del s is a del v is s is this idea is linked to gravitational potential we are doing physics actually mathematical physics. So what I am having this bounded by the region S and S0. Now I want to know integrate so now if I want to talk about the integral over both S union S0 of F dot ds and this is actually because these are all 2 disjoint surfaces. So over s F dot ds just like you have for line integral if you have disjoint curve 1 curve then another curve then another curve so the whole thing is path is broken into some pieces of curves then you do it over each of the curve or each of the holes.

So you do the line integral over each of the holes separately and add them okay but by this definition but we also know by Gauss divergence theorem v dash which is a region bounded by this volume. So del dot F and in this region there is no x, y, z with all the coordinates 0 r is not 0 here dot dv is same as integral S union S0 F dot ds. So what we know is that in because in v dash the vector r is not equal to 0 del dot F that is del of r vector/ r cube = 0 in v dash. So what do I get?

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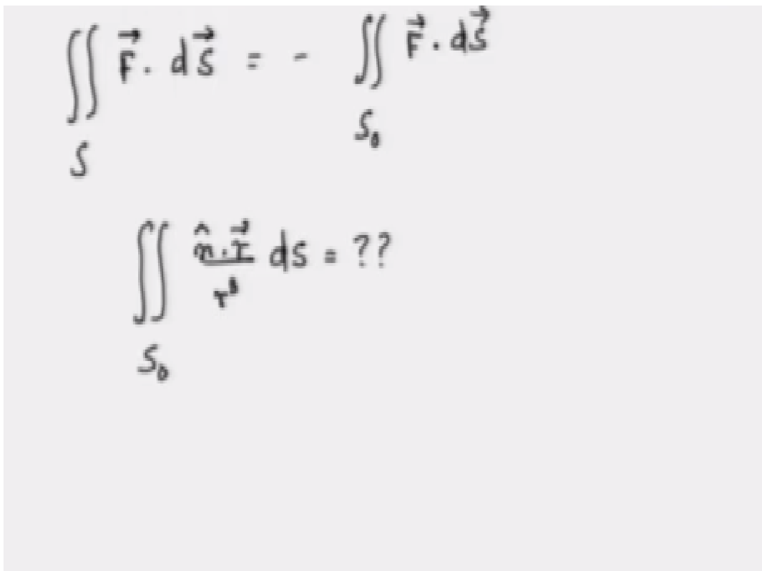

Now this is written as a sum of 2 integrals so basically I get integral integral s F dot ds F is obviously r vector/r cube =- integral integral S0 F dot ds. So I have to find this so if I evaluate this so basically now I want to evaluate S0 n vector dot r vector /r cube ds and that is what I am going to now evaluate. If I can evaluate that I have actually made a move forward okay now let me look at it in a slightly different way.

So now what I am trying to do this is a problem described in Spiegel but I am doing it stepwise and describing you in very simple way. Now let the radius let us look at page now let the radius of this small sphere S0 this be a this be observed so here I had 0 I had taken a small sphere out and then now a big sphere. So I am looking at the region in between.

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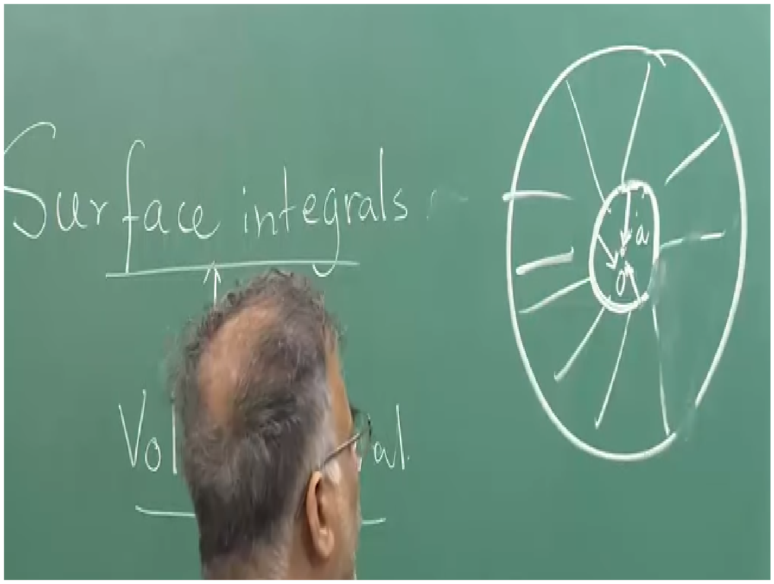

So basically if you cannot take a cross-section so you it would look like this so 0 is sitting here and you have taken this out. So you are essentially bothered about this zone. Now this is my radius a here value of the radius but any Norman on this surface corresponding to the given volume is in the direction negative to the radius the direction of the radius is always away from the origin this way.

And here it is towards origin you have to understand this part this hole here is not in part of the volume in the volume is here. So this is a surface this is the surface so if you have a cylindrical thing like a you must have seen coils which travel by train you know the aluminium sheets coiled up like this and there is a hole so or you can take some papers you can take a lot of paper together some 7,8 papers and make a hole like this.

So there is thick inside there is a bigger biggest boundary and a smallest boundary so there are 2 surfaces one is outside surface one is the inside surface but if you go it inside the inside surface and look at the inside surface the normal will point towards you so normally point inside and that is that is the key thing the normal will point inside that is the key thing to remember.

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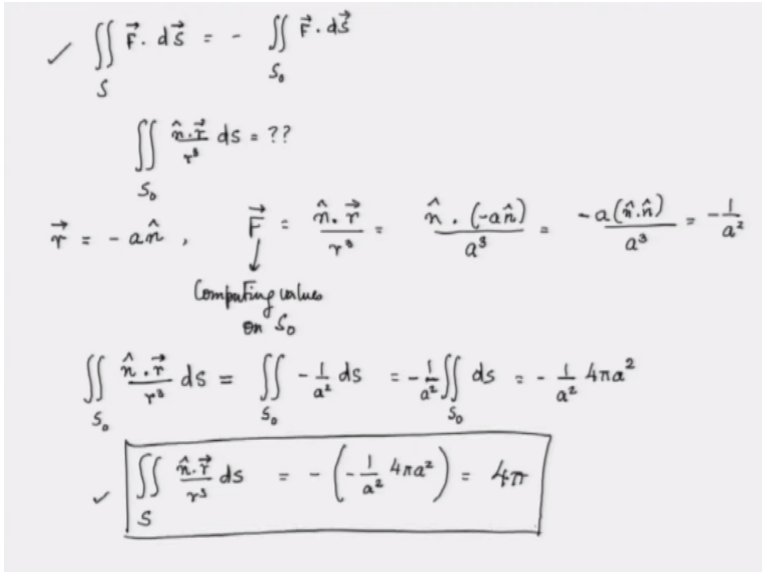

And once you know that then you know let me write the r vector is nothing but -a of n vector n vector is unit vector. So normal is a n vector is a unit vector along the radius so in this case because of the symmetry of that spherical symmetry you have all the n vectors along the radius. So this is what is there so F vector n dot r/r cube so r cube so the radius r in this particular case I am looking at the field values on the boundaries found because I want to find on S0.

So this is my S0 and this is my boundary S. So I am trying to find the field values on the boundary so finding F computing on computing values on S0. So I can write this as n dot -an dot and r cube r is a so it is a cube, r is a on the radius on the other on that S0 sphere r is a. So it is -a n dot n/ a cube but this is one n dot a is a unit vector. So it is -a/ a cube that is -1/a square. So integral of S0 n dot r/r cube ds = integral r – 1/a square S0 ds.

So what is that area of that surface 4 pi a square because it is spherical thing -1 by sorry -1/a square ds of S0. So that is -1/a square 4 pi a square so now integral of S n vector dot r vector/r cubed ds. So I am writing this part here is equal to minus of this integral which is minus of 1/a square 4 pi a square. So that is the volume of the area of the surface 4 pi r square so this is nothing but 4 pi.

So this is the answer, so when 0 is inside there is a beautiful simple answer which is 4 pi it is amazing with how pi comes out at many places. So we have a pretty good description here of

Gauss divergence theorem of the application of the Gauss divergence theorem so we have 2 good examples showing you the application of the Gauss divergence theorem which I have computed in total and we have given you an explanation of the Gauss divergence theorem.

There are many things associated with the Gauss divergence theorem which may be one of them I will speak in the next class which is a kind of you know there is the Gauss divergence theorem can talk about flux which I have not spoken about a lot of things about physics but I because I do not want to get into physical problems of physics then I will be teaching a physics class.

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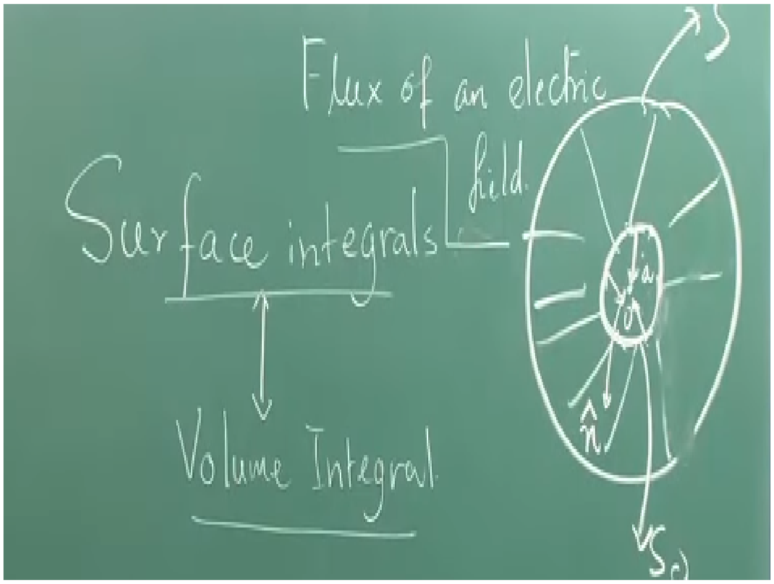

Flux of a fluid of flux of an electric field all these things are important and Gauss divergence theorem plays a very important role in such cases. So my job as a mathematician is to really to tell you the theorem and tell you some of its some of the problems where it can apply but I am not going to discuss too much of physics or continuity equations or know all these kind of things so that is left to the physicists to discuss.

So because there can be a lot of questions that will come up which we might need the help of physicists. So we are not going to get into that that kind of domain at all we are stick to our mathematical domain. In our next class where we are going to talk about how we are going to sum up everything that we have spoken about we are going to do on our way and we prove something at the end or do something just to talk about fields and this whole idea that we have come our fields have played a very important role.

The notion of a field have been very key to our understanding here and that is what we are going to talk about a bit how we can get a little further idea what fields and how we can mathematically handle them maybe we will speak a bit about it in the next class and which is a sum up class. So that ends which this ends the course actually this ends the course.

So you have done a quite an amount of stuff its a huge amount of material told to you in 8 weeks' time which I think in its a maybe 23, 24 hours lectures. So I think this is not so a lot a short time and a lot of material being pumped in into this course so what I want to say is that it needs a lot of practice in many cases you might get stuck you might get certain okay you write the wrong limits which can happen with me I also wrote the wrong limit today

So because you can just get carried away but suddenly you understand and you put the things. So think over the problems never hurry with a math problem if you are stuck with a math problem for 2 days does not tell you that you are mathematically incapable maybe you are very actually very capable maybe you are making a deeper thinking. Maybe you have formed you will find a much better way to do it rather than others.

So with this I would rather end my talk and I thank you for all of you for actually very patiently going through this course hope that you have enjoyed it as I have enjoyed delivering this course to you it refreshed sometimes as we researchers keep on focusing on our own research problems looking at research papers current research papers trying to ask questions trying to answer them. Research is a very different ballgame I do not know whether I am myself a researcher I maybe I am more on interested in understanding when you and I read new research.

I try to I have some questions and I try to answer them sometimes I fail sometimes I partially and sometimes possibly I can get some nicer ideas. So who knows maybe the whole of my life I would just keep on doing incremental work in mathematics but whatever little work that I do in research gives me a lot of joy and but as we keep focused on our research sometimes we get lose touch with basic mathematics and if we are stuck in something we have to rush and look at our books this is very common.

Please get out of the feeling that okay if you have learned a particular suppose if somebody has done taken some expertise in electrical engineering and expertise in communications he should remember each and every problem that he has learned in his first calculus one course or a multivariable calculus course. Please do not think of that if you need it go back to the books but once you have done a course these names like Gauss divergence theorem or stokes theorem or greens theorem will never leave your memory.

These names should remain in imprinted in your memory the curl, divergence, gradient dot product cross product this all should remain ingrained in your memory and that is what the message that I want to give; if these names are there if you need them just open up you can see them. So thank you very much we will come back for a brief sum up or what we have done maybe a 15 minutes' lecture trying to tell you some things that we have done here something that we have learnt here.

So thank you that is essentially the end of the course and there will be 15 minutes of some summarizing what we have done and hoping that you have enjoyed it.