Calculus of Several Real Variables
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Lecture - 33
Green's Theorem

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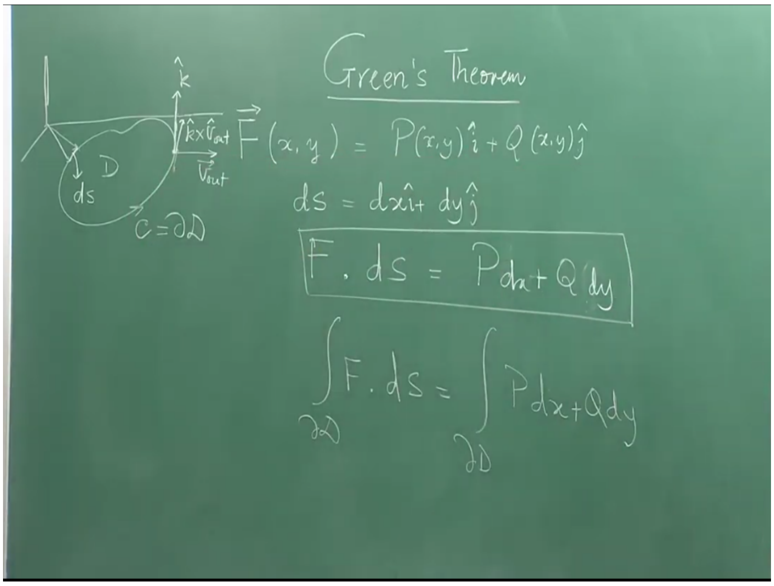

Okay Green’s theorem, we start what is called the integral theorems of multivariable calculus. I am not going to do a proof here, we are not going to prove Green’s theorem. The proof would take half an hour, so maybe Green’s theorem proof I will put it in, put up in the portal but maybe I will try to do a proof of Stoke’s theorem or a Gauss divergence theorem if time permits.

So, what is Green’s theorem? Green’s theorem actually relates a line integral with the surface integral that is what Green’s theorem does and we have so suppose I consider in the a vector function from R2 to R2 given as, then given a, so here the line integral is a special form of a line integral. So, suppose we are looking at a region bounded by a curve, so these are domain D which is bounded by a curve C.

And I will always evaluate the line integral along the curve C if I move in a direction which I will call the positive orientation in the sense that if I when I move along this curve, the area of this domain should always be on my left hand that is the criteria that I will follow. So, if I am moving along this curve, so if I am moving counter clockwise along this curve, then the area if I am this guy here, then my area is always on, the domain's area is on the left hand.

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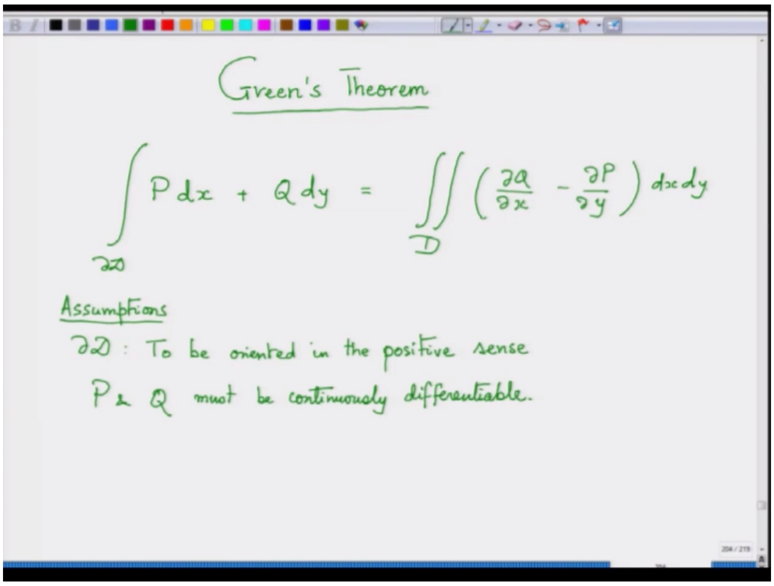

So, Green's theorem states the following fact that for such a kind of domain, we will get into more details. So, this C, this curve C is the boundary of the domain D. So, they write this as del D, so I will follow the book's approach. So, it is del D P dx + Q dy, this is same as integral over surface integral over the domain D del Q del x - del P del y dx dy. Q is associated with y, so you differentiate it here with respect to x and P is associated with x when you differentiate it here with respect to y.

So, this is the thing that you have to remember. So, how do I decide which direction is my positive orientation. In these kind of curls, you see I will take the, suppose this is my domain and the domains, this is outside the domain. So, this is any vector V from a given point pointing perpendicularly normal. If I draw a kind of tangent line here, there is a normal which I call the V out vector, this is pointing towards outside.

Now, if I am in two-dimensional space, so if this is my K vector right which is pointing, so this is my two-dimensional space in which I am lying and K vector is pointing in the perpendicular direction. So, the vector, this vector is K cross V out, that vector that direction is the direction of positive orientation. So, if you look at that direction, you immediately understand that if I walk down, then my domain has to always remain on the left-hand side.

So, the Green's theorem always expects del D to be covered or oriented, to be oriented in the positive direction in the positive sense means if I move around that would be the way I will move around. So, I am writing in green because this is Green's theorem. Green's theorem actually arose because of some computations in electrical potentials and gravitational potentials. Stoke's theorem was actually told to Stokes in a private letter as the book says.

I have to check out the history a bit more by Lord Kelvin famous for its absolute temperature, absolute zero, so the Kelvin scale. So and Stoke's gave this problem in a math examination in 1854 and gradually it became famous as Stoke's theorem. Now, so you see here there is a link, now how do you know that this is a line integral? So, if I have written F like this, so you know here I can take the r curve and then you put r + dr so this is your ds.

So, ds is measuring, the ds here is almost same as dr, very small because dr is very small. So, you can write this as dx i vector + dy j vector and that is exactly the way you compute ds, ds square + ds square root over ds square + dy square, you know this already. So, here if I do my F dot ds and this is nothing but P of i vector, I am not writing xy and Q of j vector sorry making a mistake, P of dx + Q of dy.

So, this will give me a scalar, so this is what it is. So, if I write integral, so if I write integral del D F dot ds that is same as writing integral del D P dx + Q dy. Say if it is difficult to compute this, you just compute the surface integral that is the message of Green's theorem. Of course, there is another requirement, P and Q. So, these are the assumptions under which Green's theorem is true.

P and Q must be continuously differentiable; of course we need continuity here for the integration to go through smoothly. Here, all integrals are meant when you take 1, 1 the single integrals, they may meant in the (()) (09:14) sense or most cases in the standard Newtonian sense that is really the effective thing. That is now let us start giving 1 or 2 examples and see how we can understand. So, we will take examples from the book and one by one we will try to work them out.

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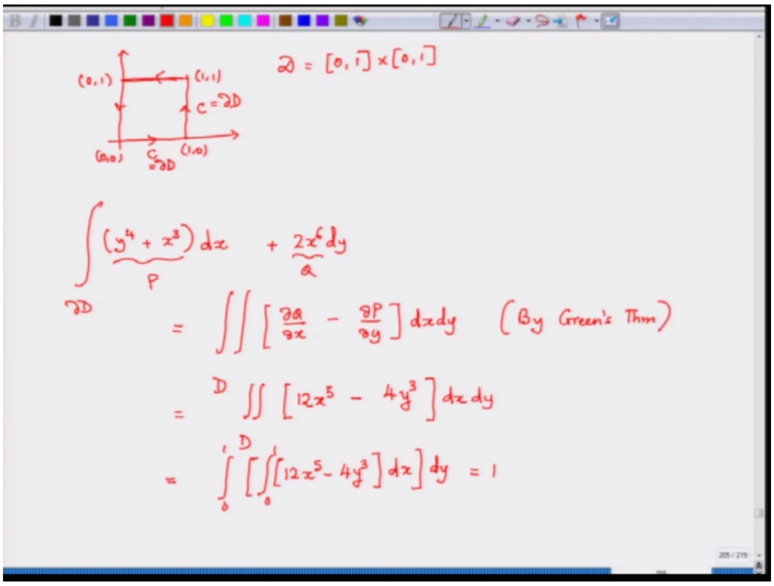

So, here is a problem that says okay you take a curve which is lying along the boundary of this square formed by 0, 1 0, 1 that is the domain D is formed by 0, 1 cross 0, 1. So, it is 1, 0 0, 0 0, 1 1, 1. Now, I have to traverse in this direction. You see I have to traverse in, so this is my C which is del D. So, this is the direction I should be traversing along the 4 sides. So, when I am traversing along this, I am starting from one point and coming back to the same point that is the meaning of del D that you take a complete round of the boundary.

So, start from one point and come back to that point that is the meaning of del D. So, they are asking you can you integrate on this del D boundary of this particular domain integral y4 + x cube dx + 2x 6 dy. So, here is my P, here is my Q but I go from one point and come back to another point, so you might feel that I have gone from the same point to the same point. So, it is a kind of very strange.

So, how will I actually, I am in a full circle, it is not going from A to B and so I have separately I can work them out. I will work out what are the limits of those 2 points, limits of the t the parameter and then I will work it out. So, here if I want to talk about such kind of line integrals, then the best way is to use the Green's theorem. So, in this case, I will now write integral D.

So, it is del Q del x here, so what I am going to have here is del Q del x - del P del y dx dy. So, what is my del Q del x, is 12x 5 - del P del y that is 4y cube into del x del y. So, here x is running from 0 to 1, y is running from 0 to 1, so that makes life much more simpler okay. I will not write down answer here, I will just write down answer here, it is 1. So, Green's theorem allows you to calculate this line integral and gives you 1 which is not so easy.

If I give you a complete circular path, if you go and look at the line integrals chapter where you have studied line integrals for the first time, see all the paths are usually open paths especially if you have, how do you compute a total path? So, you go by one path, so you go from A to B by one path, so you compute this. If you have to compute the line integral here, first you compute on this path, then you compute on this path, then you compute on the third path, then fourth path.

So, you start from 0, 0 compute up to 1, 0; then you start from 1, 0 compute up to 1, 1; then you from 1, 1 and compute up to 0, 1 and 0, 1 to 0, 0. That is the 4 different line integrals you need to calculate if you want to evaluate the line integral that we are choosing but just by applying Green's theorem, just by applying Green's theorem here, you have just solved the thing in 1 shot and that is the applicability.

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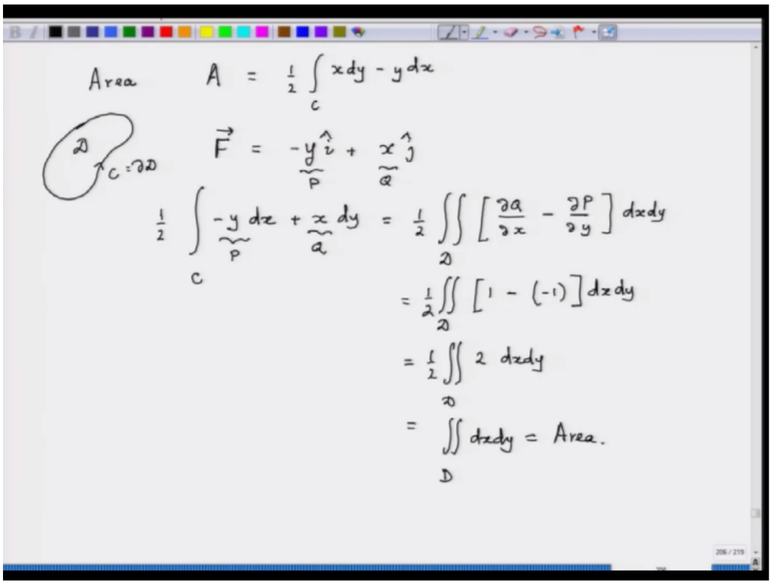

A very interesting application of Green's theorem is the area. So, if let me tell you that before I get into this area business, let me tell you that there can be regions which are much more complicated than the simple region that we have drawn there.

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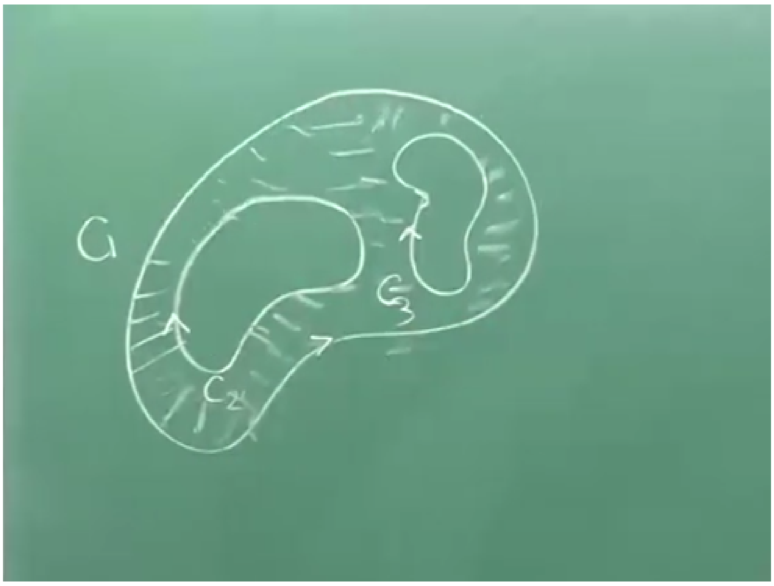

There can be region like this with holes inside. Now, here what you mean by positive orientation? See positive orientation would always mean here when you are encircling the region, you have to go, so the region is actually here, this is the domain. So, these are holes, so this is C1, this is C2 and C3. That C1, C2, C3 all 3 together make up the boundary. So, in such cases, how will you do?

These are the cases where this Green's theorem become useful, you do not have to bother too much about it. See here along C1, I move in the counter clockwise direction but along C2 I have to move in the clockwise direction and C3 have to move in the clockwise direction because my positive orientation has to follow 1 very simple thumb rule that my area the domain has to always even on the left side as I work on that curve.

So, here on the internal curves, you have to go on the clockwise direction; on the external curve, you have to go on the counter clockwise direction. That is something you have to keep in mind and now area, so suppose you have a kind of closed curve C covering a domain D and the area A can be computed by this simple line integral and how do we know that area can be computed by this simple line integral because suppose now we construct a vector field F where it is minus y i vector and xj vector.

So, let me look at this integral, 1/2 of integral C – y dx + x dy. So, this is your P and x is your Q. So, this is my P and this is my Q. So, this is your F dot ds and then I integrate by Green's theorem, I will integrate over for the domain D and Green's theorem should give me del Q del x - del P del y. So, what is del Q del x? It is 1/2 of, so here a line integral is being computed just in one shot so double integral.

So, what is del Q del x here? It is 1 minus, del P del y is minus 1, so minus or minus 1. So, this is, so I will get 1/2 of this is not infinity, this is the domain D. So, 1/2 of integral domain D 2 of dx dy and that is equal to integral D or the domain dx dy and that is exactly the definition of the area, that is the magnitude of the area. So, you see how beautifully this has worked, how beautifully this has.

Now, there is another interesting way of expressing the, I am not giving the proof, proof I will send across. Expressing the Green's theorem, it is called the vector form of the Green's theorem that is what we are going to now learn.

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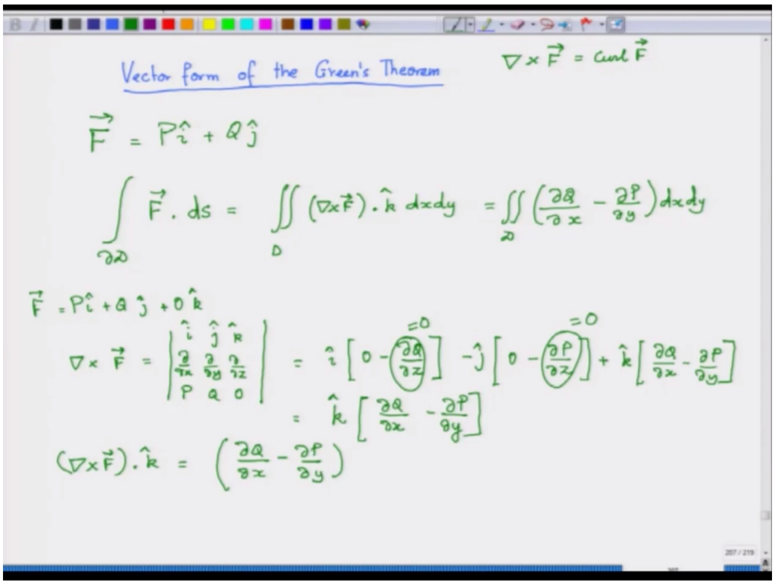

Vector form, so again the same story, so you have F vector given as P i vector and Q j vector, I am not writing the x, y's. So, what does it here, they say that you can write this line integral as integral curl F means curl as a sign this is the curl sign, curl F has this sign, curl of F. So, let me just remind you, this is the sign we use for curl F. Curl of F dot product with k vector dx dy. This is what another form is called the vector form of Green's theorem.

The other form was written in the form of a scalar. So, we have to see whether this vector form has some meaning or not. So, let us see how do I prove it. So, F is P i + Q j. Now, curl is in terms of 3, so basically here I have to write it in 3 components 0 k. So, curl of F is by i j k del del x del del y del del z P Q 0. So, it is i vector into this part which is 0 - del Q del z – j vector which is 0 - del P del z.

And this del Q del z is 0 and del P del z is 0, k vector into del Q del x – del P del y. You see this is completely 0 because there is no z in Q, it is in terms of x, y. So, basically this is 0. This thing is 0 and this thing is 0 because P is a function of x, y only, not x, y, z. So, I am writing everything stepwise, so I wrote this. So, it is what we have here is nothing k dot del Q del x – del P del y and what do I have here?

So, I have here F vector, if I take a dot with k vector, I will simply have this as del Q del x - del P del y and this simply gives me back the form in which I know about Green's theorem. So, this is another way to integrate the curl of F dot k. This curl of F will become important when we will be talking about Stoke's theorem right. So, there is a kind of result where we will not talk much about it, it is called the Gauss divergence theorem in the plane.

Because Gauss's integral, Gauss divergence theorem relates the integral of the divergence of a function, the line integral basically sorry surface integral. The surface integral of divergence of a function over a given surface is related to the volume integral over the volume or the triple integral, volume integral is a triple integral with respect to the volume enclosed by the surface, so that is how what Gauss divergence theorem is that.

That is very important in electrostatics, is also important in mechanics, so of course in fluid mechanics naturally, so fluid and classical both.

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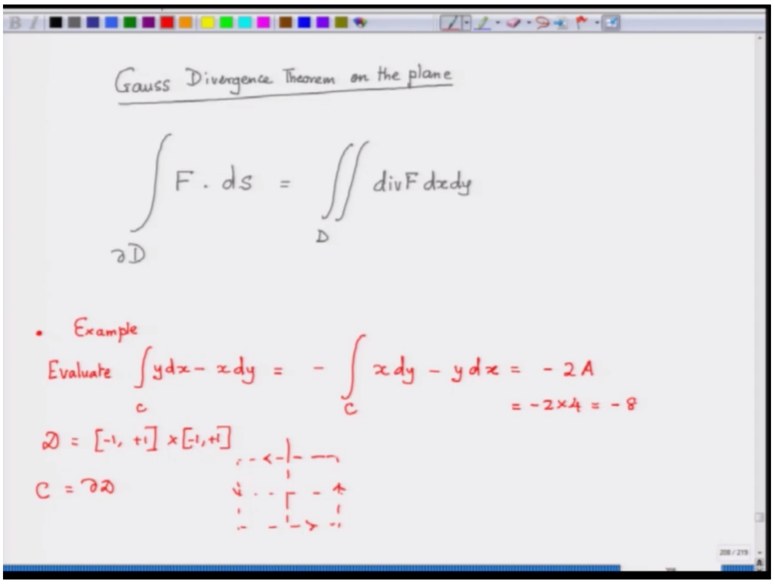

So, there is something called Gauss divergence theorem on the plane. So, I will just make a statement of this and end the talk here, Gauss divergence theorem on the plane. So, it says that this line integral, here it is not a surface but a small arc, arc length, is a surface integral in terms of the divergence. The proof here is slightly not simple; I would not like to give the proof here at all.

So, it needs some more ideas from vector calculus which will confuse you, so will not go to all these things. So, there are several, for example, just as an example I will end before I end the talk. For example, there is a problem here, evaluate integral C, y dx - x dy. Do not immediately jump to the conclusion that this (()) (27:30) area because look at the area, look at the area stuff here.

Area is x dy – y dx, here it is y dx – x dy, so kind of negative of area right. So, here you have the domain D as - 1 cross - 1 and your C is the boundary of the domain oriented in the positive sense. How will you evaluate it? There is one simple thing to observe, first observe that I can write this let me write down that this integral can be written as minus integral of x dy - y dx over this C.

So, what is area? X dy - y dx, so x dy – y dx is area, in fact twice the area. So, this is minus twice the area. So, what is the area here of this domain? One side is 2, another side is 2, it is 4. So, - 2 into 4 is - 8 but if I go in the clockwise direction, the opposite direction because this would be if I go in the clockwise direction, the value of the integral would have a negative sign here, so it will become 8.

So, the problem says you have to go in the clockwise direction but let me just keep it in the anti-clockwise direction, make we go and so the area of here this is nothing but minus or twice the area and this is - 8 that is the answer. So, that is how you apply what you have learnt here. So, with this I end my talk here and let me tell you 7th week is finished. There are 4 lectures in 7th week because the 2 lectures on surface integral has been combined into a single lecture which you know is pretty involved.

Green's theorem once you know the formula, you have the statement of the theorem you know how to apply it and that is what we concentrate on that, that is what we end, proof of Green's theorem will come later as a part in the portal, it is not a part of the whole thing. Let me see what I can do if I can give a problem. If the proof of Stoke's theorem can be done in the class but I do not think it can be done in the class.

Maybe I will just for you know kind of a mathematical work to be done so that you do some math and think about how proofs are done here, maybe I will try to do the Stoke's theorem here in the class but Gauss divergence theorem I do not know how much I will have time to do but will do something about it. So, let me tell you here the exact divisions may not be maintained.

So, we will cover Stoke's theorem, its applications, Gauss divergence applications and we will speak about fields in the 8th week because fields has lot of link and how differentiation and integration is linked here in this vector situation and how these ideas are called gradient divergence can be applied to several physical situation, those things will be discussed. So, overall I have to provide your content of 2 hours and 30 minutes.

So, I will try my best to cover the thing in 3 hours which could be in a talk of 4 which could be in a talk of 1, 2 maybe 4 and 5. So, it could be in talk of 4 or 5 but the 7th week also end with 4 talks because 2 talks have been combined in 1 the surface integral. Idea is that why I combined them was because if I just separate them out, then it loses the link, so I cannot talk about line integrals and suddenly I do not speak about orientation.

Because that is how the idea of sorry I do not speak about surface integrals and I do not speak about oriented surfaces because otherwise the parallel that you have learnt in case of line integrals would become lost. So, that is the way this or if I divide this thing into 2 parts, a little bit thing for 1 lecture, then another lecture little bit thing, then the link gets disturbed. So, it is better to go in 1 flow, so 2 talks have been jammed into 1.

So, there will be 4 talks, 1 having almost an hour, so thank you very much. Thank you for being patient and we are ending the course and hope the journey was fine and hope many of you find something useful from this course to take, anyway YouTube is with you and so you can always go back and (()) (32:53) and you can always write on the portal anything that you want to discuss with us. Thank you.