Calculus of Several Real Variables

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Lecture - 32
Surface Integrals - I

Welcome as we go along and we are in the last lap of the course that is something we have to understand that we are actually gradually coming up to the climax, we are almost near the peak, we were climbing a mountain and we were near the peak. So, with these 2 lectures, we will now introduce surface integrals and then we will start what is called integral theorems in vector analysis.

So, these integral theorems of vector analysis, so what is vector analysis you can say multivariable calculus, whatever you want to say. These theorems actually unite all these things that we have learned, they link all of them. Is there any link between divergence called gradient that we have learned in differentiation? Is there any link between the integrals that we have learned line integrals, surface integrals, other multiple integrals? What is the link?

All of these ideas are linked in that particular form and they have a lot of applications in the physical sciences and so this part of mathematics has got a huge application in physical sciences and many questions have come because you are trying to understand problems in physics. So, today our first thing with that we are going to learn about surface integrals is not surface integrals as per se what we call surface integral.

But we will talk about integrating a scalar function over a surface. So, you see here written down, here on the screen that there is a function from R3 to R which is a continuous function and there is a surface parameterized by phi.

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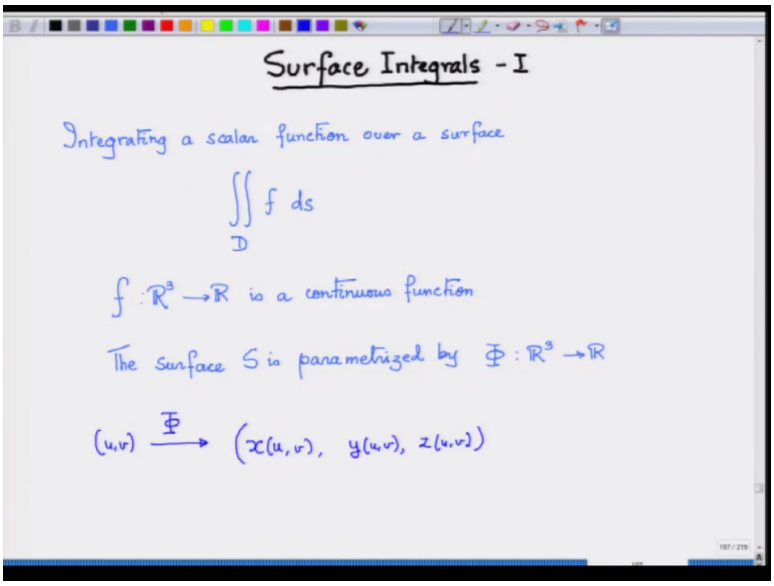

So, the surface is given by coordinates, so basically u, v is mapping to some and z u, v. So, this is the parameterization which we have been speaking about. So, by this time, you have got some handle on this idea parameterization, some hold rather. Now, what do I mean by this integral f ds. So, let us try to understand integral f ds.

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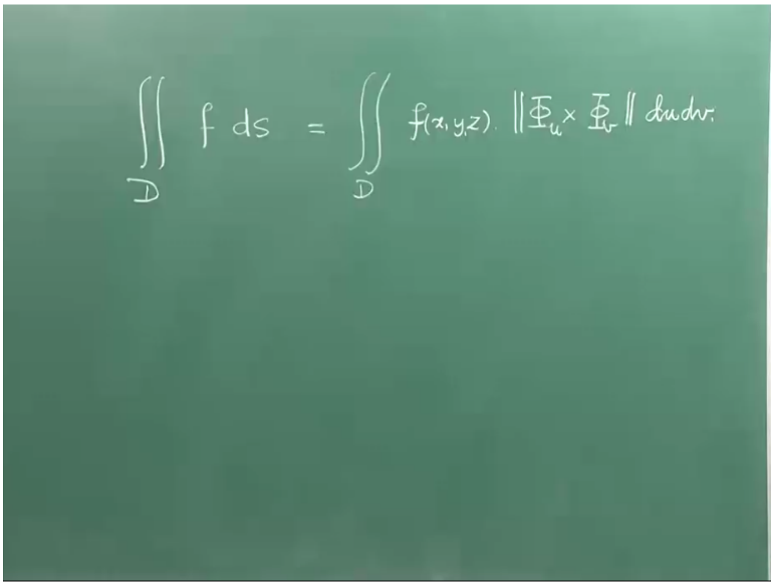

So, there is a domain D which is always there and integral f ds. So, because I have parameterized, you really know that integral f ds is nothing but and I am not going to do any calculations with you here because calculations are all done. So, we are not going to get into this issue of calculations. So, let me see how we proceed and then I will keep on adjusting the number of lectures, possibly we have to give some more lectures on integral theorem.

So, maybe I will cut this surface integral lecture in one talk. So, let us see what happens. So, this simply means that what is ds. We already know about ds. This means f x, y, z into norm of phi u and you know what is phi u, the vectors, phi v this norm into du dv. That is what is the meaning of this integral, so I am not going to write down the expressions of phi u cross v in terms of determinant of those Jacobian which you know already.

So, example where which talks of something called helicoid. So, I will just tell you what a helicoid is and I will just draw the helicoid and give you a parameterization and then I will just leave it to you to find that integral. So, that we just first start giving you that picture of a helicoid or rather let me show you the parameterization.

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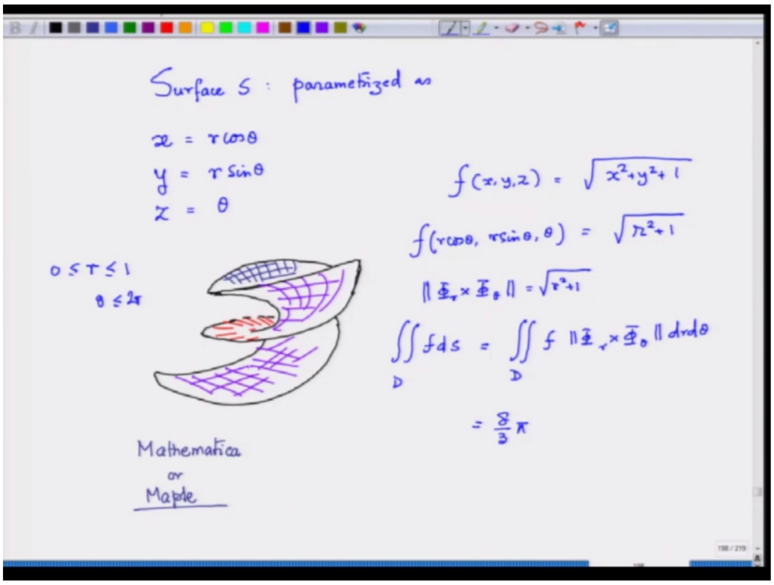

So, I have a curve or surface S parameterized as x = r cos theta. So, parameters here are r and theta the u and v; y = r sin theta, so they are lying on the circle and z = theta. So, what kind of a stuff it would look like that is the question? So, it is very difficult to draw a helicoidal surface actually but you can go and check, you just in type Google helicoidal surface, there is a by name Wolfram who has this software Mathematica, they have drawn it.

It is a kind of very strange kind of 3D surface and I do not think this is a good idea to even make an attempt to draw it. So, helicoidal surface looks like kind of like this, comes up I am just kind of making an attempt, I am just making an attempt to draw it, so you see how complex this structuring looks. So, this is one sheet and this is another sheet, which is mixing up and this is the different sheet and this is completely a different stuff.

You can see the helicoidal structure here, it is very difficult to make a drawing actually. I had a drawing done before I came here and so I am just I guess looking at that drawing and trying to draw here because it is a very difficult structure, you use computer algebra-based software like Mathematica or Maple and draw this. You will have lot of fun doing it. I assure you that you would have lot of fun doing this thing here.

So, here I am asking you to find the integral of say if you want to do something, so the problem given as an example in the book ask you to find an integral over this. Obviously, instead of x you will put u and v, so what is f here? So, f will be replaced by r cos theta, y is replaced by r sin theta and z by theta. So, root over x square + y square + 1, so then this is equal to root over r square + 1.

So, similarly if you compute this norm phi r cross phi theta, take the norm that would also be root over r square + 1. That is the interesting part here. So, then the integral I leave it to you to calculate with dr is actually integral f, you know the domain now, r is between 0 and 1, let us of course r let me put between 0 and 1. This is the drawing from r between 0 and 1 and theta is going between 0 and 2 pi because we have taken a circle of radius r.

We are going within that zone, so we are keeping, we also changing r. So, change r but as you change the theta what is, so here is r, so here so there are various thetas and based on that z is coming, so it is like this. So, it is a kind of helicoidal structure that goes up. So, here you have f of phi u cross sorry phi r cross phi theta dr d theta and the answer that you will get is 8/3 of pi, it is quite amazing that such a complex looking object right.

So, it is a complex looking object. So, as you move theta from 0 to 2 pi, this is the whole thing that it will give. So, such a complex looking object would have such a simple looking that is irrational the number, that is also one point, 8/3 pi is irrational, so but such a complex looking object would have such a simple looking expression for the area and that is what the beauty of mathematics is and that is a power that you can really understand such complex shapes using mathematics.

Then, now we come to what we will call actually the surface integral. It is the same story but a little different than what you have learned in line integral.

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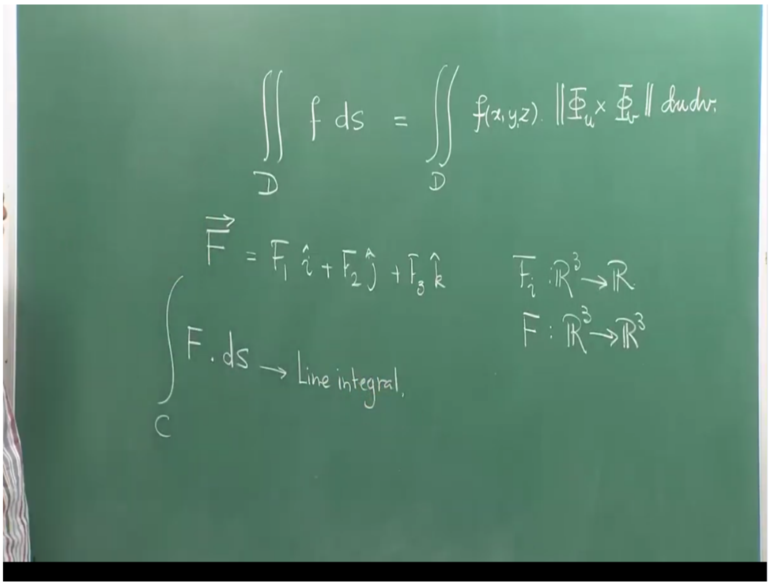

So, in line integral you have a vector field, F1 i vector, F2 j vector, F3 k vector where each Fi was a function from R3 to R. So, F can be also viewed without writing the arrow form because we will not write in the vector form as a function from R3 to R. So, it is a vector valued function. So, what did you learn? That if you take a small arc length, kind of dr basically, so F dot ds with a small arc length over a curve, this is what is called the line integral. So, the same idea can it be done for the surface?

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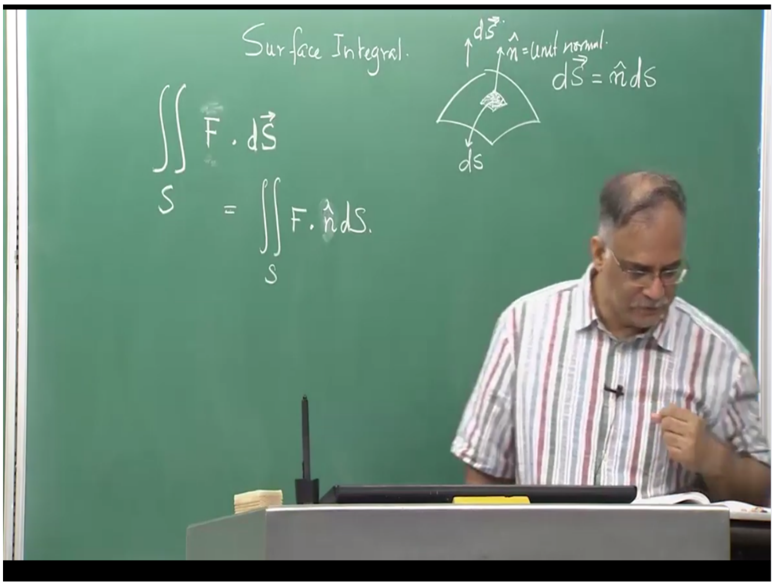

So, that would give me a surface integral. So, instead of integrating over a curve C what if I take this vector field F and integrate it over a surface. So, I am now trying to define an integral of this form F sorry F dot ds, so maybe I should put the vectorial form at ds and if you want to be F also just to make it look more. So, this is what I want to do, you may not put

F here, you can put, here because I have already started putting the surface area vector in that form a dS vector.

So, I am just trying to put F also to keep a kind of symmetry. If you do not want to put, do not put. So, it is understood that is a vector valued function. So, that is what is I am trying to understand what is the meaning of this but if you look at it what is the surface integral or what is this dS vector? So, here is a small elemental surface area, here is your normal vector and the dS vector is the unit normal vector, there is a unit normal.

So, it is n vector dS, so dS is the value of the surface area and this is the dS and area vector is always perpendicular because area parallelograms is given as cross products, so this is your n, so dS is actually along this direction, there is a dS vector. So, I can write this F dot n vector, n vector is unit normal, so because it is whenever I talk about a unit normal, I will give n hat, so i hat, j hat, k hat these are all unit vectors.

So, whenever I am talking about unit vector instead of putting arrow, I will put a hat just to signify that it is a unit vector and I need not keep on repeating that it is a unit vector. So, this is what is the meaning of our surface integral but how do you now ask me then why did you teach this at all because if you ultimately going to talk about this but I say that I have taught you that.

Because I will use that idea, that type of operations to actually compute surface integrals, so this is what I would try to show you. So, now we will see that how can we use this idea of parameterization? So, take a problem from the book and try to see how we can handle it. Now, there is a problem in the book which says that so here is a problem, I write in black.

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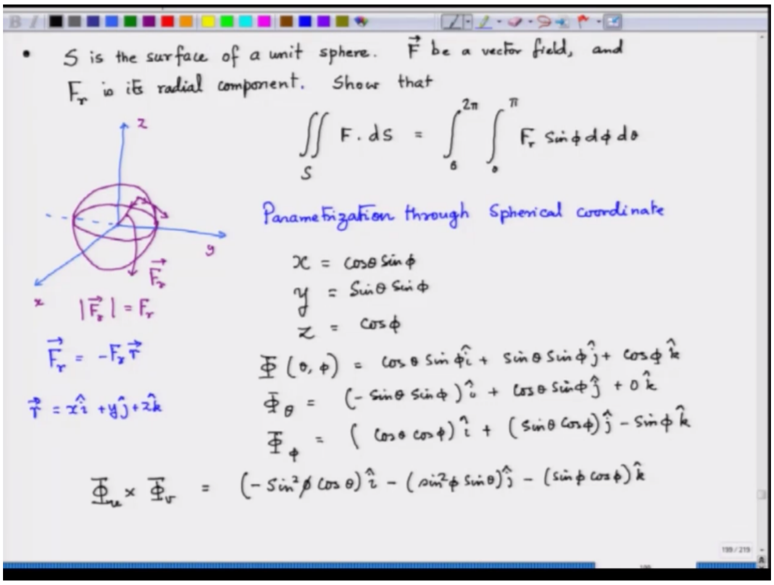

S is the surface of a unit sphere and it says that F be a vector field and Fr be its radial component, so Fr is radial component. Radial component means I will tell you what; Fr is its radial component. So, here I have my unit sphere, so I draw the axis in blue and then I will draw the sphere in some other color maybe maroonish. So, at any point here on the curve which is x, y and z.

So, here any point on this sphere, so the vector field F that is there that you calculated at that point F of x uv, y uv, z uv. There are 2 components; one component can be taken along a tangential direction and one along a radial direction. So, that in the direction of the radius towards the center, so this is your Fr component and of course magnitude of Fr is Fr and what the problem says is that prove that, problem says the following.

Show that F dot ds is 0 to 2 pi 0 to pi Fr sin phi d phi d theta. So, this thing, this expression can be possible if you express the coordinates on the surface of the sphere in terms of what is called the spherical coordinates right, that is very important. So, let us see what happens if we take the spherical coordinates. So, if you take a spherical coordinate, spiracle coordinates, so rather I should write parameterization through spherical coordinates.

Actually, this Fr vector that you see here is basically minus of Fr value into r vector, that is r vector is xi vector + yj vector + zk vector. So, now we will parameterize through spherical coordinates and in this parameterization through spherical coordinates you know what is how to parameterize. So, if I parameterize to spherical coordinates, I have x = cos theta sin phi. You have to understand here my radius is fixed.

Here, I have taken a unit sphere, so radius is 1. Actually, my radius is fixed. So, in this case, my radius norm of r is 1, so mod of Fr is Fr, mod of vector Fr is just Fr. So, this is actually the radial component. So, x and y is given as sin theta sin phi and z is equal to cos of phi r cos phi which you know. So, what we have to do? Here, I have my x uv, y uv, z uv, so my phi theta, phi is cos theta sin phi + sin theta sin phi + cosine of phi.

So, here I am going to write dS in terms of that. So, dS vector would be in terms of that phi u cross phi v. So, my phi u is equal to minus sin theta and this is crossed by k vector sorry i vector, j vector, k vector. So, minus sin theta so phi u I would not say, phi theta I would say, phi theta, so theta means I have theta here, theta here. I do not have theta here. So, minus sin theta sin phi i vector + cos theta sin phi j vector and + 0 k vector and phi of small phi, capital phi of small phi is now you take this.

Cos theta cos phi i vector + sin theta cos phi j vector - sin phi k vector. So, I have this phi u cross phi v calculated which I am not going to have separately checked. It can be, so phi u cross phi v turns out to be the following. So, let me just write down the expression of this so that I do not which is already checked up and calculated is that this expression turns out to be - sin square phi cosine of theta i vector - sin square phi sin theta j vector - sin phi cos phi k vector. Now, what is direction of F?

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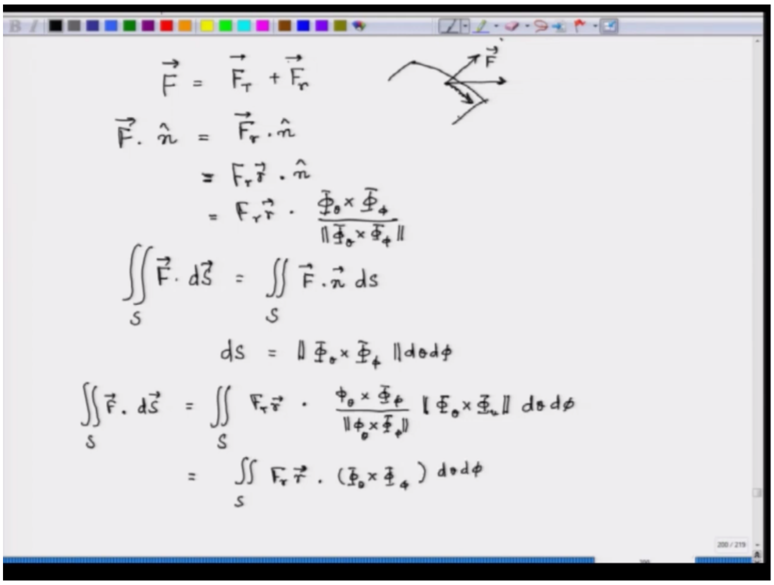

Now, F vector can be decomposed into 2 parts, you can always do for a vector, one is a tangential part at a given point x, y, z into tangential part and another is a radial part. Now, when I am doing F dot n is n vector is only along the radial part, the n vector does not have any tangential part. So, this would just become Fr dot n vector. Now, what is n vector by the way, n vector is nothing but so it is decomposed into parts Fr + Fr, Fr dot n vector.

So, it just becomes Fr dot n vector. So, n vector does not have any radial part. So, when you are taking F dot n, we are writing F dot n, that also must have a radial part but n vector does not have a tangential part, it is along the radius, so it has a radial part. So, F dot n here, if you decompose it into tangential part plus radial part.

So, whenever you have some at a point F vector, if you have a F vector given and at any point on the surface, it should have a tangential part and it should have a radial part or the radial part could be, this is my F vector, so this is the radial part and this is the tangential part okay. So, I am writing it like this I mean it depends on what is the orientation of F here okay, F orientation could be different.

So, there are 2 parts, Fr dot n vector. Now, what is n vector? So, what is Fr? So, Fr is Fr into r vector dot n vector. So, n vector if I go by my rules, then it can be written as phi of theta cross phi capital phi of small phi by norm of the same vector phi theta cross phi of small phi. This into d phi d theta, this is n vector, this n vector into dS which is. So, my surface integral is F vector into dS vector which is integrally written as integral S F dot n vector dS.

Now, what is dS? Ds we have already learned. In this particular case should be phi theta cross phi of small phi into d theta d phi. So, this is your n vector, so what is your, so your integral S F dot dS vector is integral S. What is your F? F is nothing but r into when you are writing F dot n, so what is your F dot n, it has become Fr into r vector dot phi theta cross phi u by norm of phi sorry not phi u phi small phi, small phi phi theta cross product phi capital phi of small phi norm this thing into dS which is norm of phi theta cross which is the ds.

Now, it is not the vector dS, it is the scalar ds, n dot n vector ds is the dS vector which we have already discussed here, you just see it here in the board. So, it is phi of u right, so this thing into d theta d phi. Now, these two cancels assuming that is a non-zero of course, that is the standard assumption and it is non-zero in the case of our particular problem in the surface of the sphere.

So, what I get here is integral S Fr r vector, it should be a scalar multiplying a vector. So, Fr r vector dot of sorry not Fr into r vector, Fr into – r vector, so I am coming in the direction opposite to, if you want to come in the direction opposite to minus r, then minus r or if you keep as Fr r vector, then let us see what it comes. I am just seeing what, so Fr into r vector, I am taking Fr into r vector as the, you can take it, it depend on the type of force you have basically.

Fr into r vector would let us take it along the outward direction. So, F is the force component, so Fr into r vector into phi theta cross phi small phi into d theta d phi.

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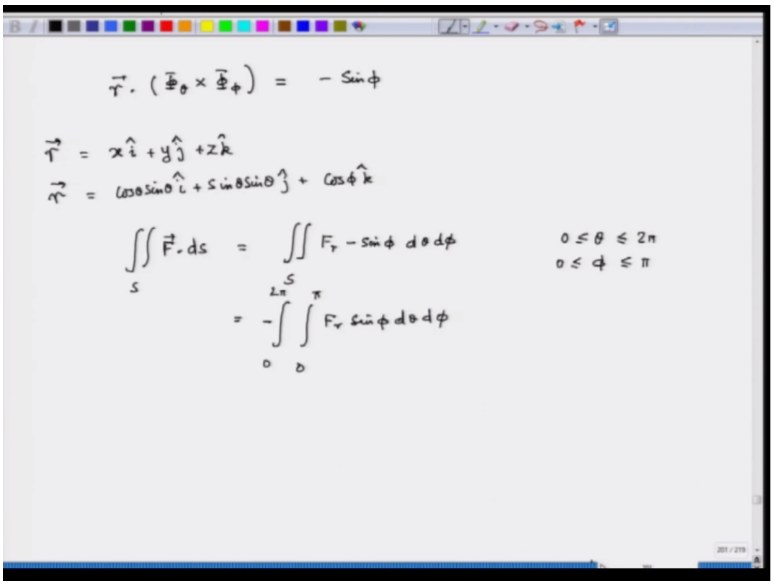

I just want to tell you that now once you have this r into, now you have to do r into. Now, what is r here? R here is xi vector + yj vector + zk vector and so in this case r vector is cos theta sin phi i vector + sin theta sin phi j vector + cos phi k vector. So, if you do, now if once you have this, now if you do the inner product, it gives you - sin phi okay. So, what do I have? So, I have integral S F dot ds = integral S Fr - sin phi d theta d phi d phi.

So, you know what is the angles things, how you know where theta varies. Theta is varying from 2 pi to 0 and phi is varying from 0 to pi. So, you have sin phi, I can take the phi, integrate over phi first. So, 0 to pi 0 to 2 pi – Fr sin phi d theta d phi. So, I believe that there would be a minus sign here in the problem; I do not know why they give it.

I do not understand why they had not put it minus Fr, there should be a minus Fr. So, this is the calculation of a surface integral. Now, certain things has to be understood when we are talking about surfaces, it is called orientation of surfaces.

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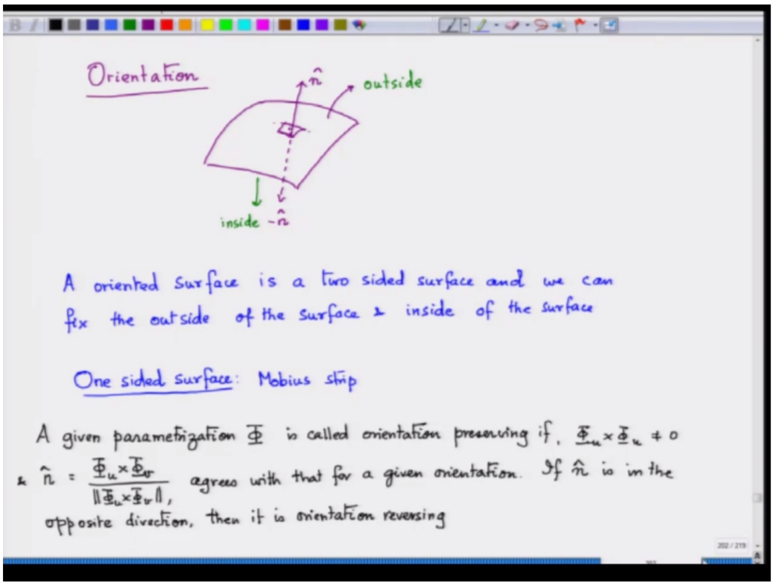

So, here is the surface and let us consider a point, let us consider a small elemental area, elemental area on the surface and from one of the central points here center point I draw normal. So, you see this surface you can (()) (34:43) I can draw normal on the other side also is minus of n. So, it depends, so the surface has an orientation. So, when I am talking about an oriented surface means I can say which is outside of the surface, which is inside the surface.

For example, here I can say this part is outside while I can say this part is inside. This idea is somehow slightly important if you are calculating surface integrals, just like in line integral if you move along the curve from, you have a curve from A to B, see a curve joining A to B. If you move from A to B, you have a value of the line integral, some value but if you move from B to A, the value will change, it will become negative.

So, orientation matters which direction you are doing, so here orientation whether you are looking at the surface from inside or surface from outside that would matter. So, a oriented surface is a two-sided surface and oriented surface is a two-sided surface and we can decide which is outside and which is inside, we can fix the outside and inside of the surface. So, you can now you say that for example the n vector, when n vector is positive I say it is outside.

In this case, n vector is negative, I say it is inside but all sides, so you if I move n vector along this surface in the positive direction, this will continue to remain positive. So, for example if I have this surface here, so this is my n vector and I am moving it, so it continues to point in the outside direction. So, here I put it here, this continues to point in the inside direction but there are certain surfaces called Mobius surfaces like I can show you by some demonstration, an interesting surface.

So, now I am going to introduce what is called a one-sided surface or a Mobius strip just for your curiosity that every surface need not be two-dimensional, I mean two-sided.

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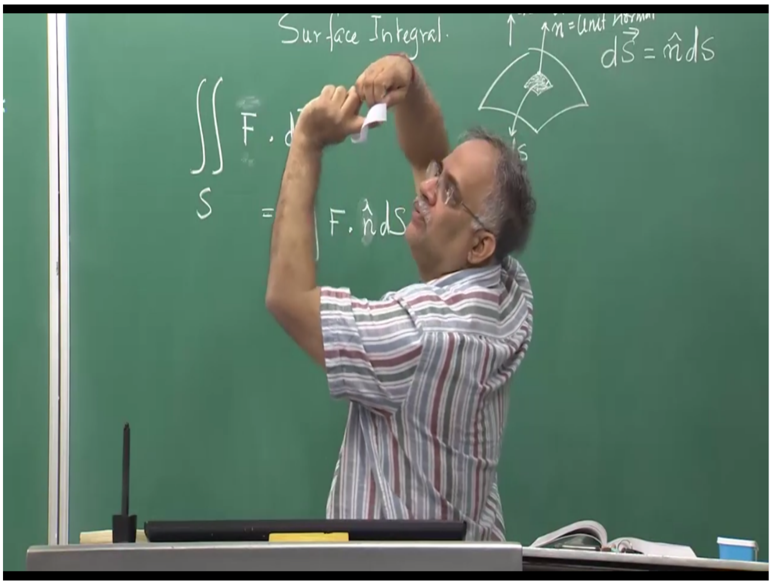

So, take a paper, it should be a slightly longer paper and give it a 180 degree twist and then attach it. So, it is something like this. So, if you start say from a point here, so from a point here if I point outward, so I go along this pointing outward, pointing outward, pointing outward, pointing outward in this direction and then suddenly here I come and point inward. So, it is my vector which was pointing outward, it is pointing outward from here.

And if you go like this, you put your hand like this and move it and then you have come to the inside basically what you feel come to the other side, your direction which was when you come back to this at this point from where you have started, so you have started your, you are pointing in this direction. So, now let us move, so here you see your finger direction has changed.

So, means for a two-sided surface if I move, if on a surface of a sphere, if you take a normal vector and keep on moving it, then when you come back to that same point around the circle, you will be pointing in the same direction. Here, I pointed along the opposite direction. So, it is an example of a one-sided surface Mobius strip and so there are these kinds of things in mathematics and this is pretty interesting.

That every sided is not two surface but we here we will largely work with two surfaces, two-sided surface which has an orientation. So, a given parameterization phi is called orientation preserving if the unit normal which is given as phi u cross phi v cross phi u cross phi v. This so given parameterization phi is called orientation preserving, of course you are assuming that if sorry I should write maybe I should write it in a much more better way than what the book has also written.

If phi is not a zero vector and the n vector agrees with that for a given orientation, n vector agrees with that for a given orientation. So, if I have given a n vector in a given direction and that is what I think is my outside direction and if the n vector turns out to be positive, then n vector is called the orientation preserving vector. If n vector turns out to be negative, it is called orientation reversing.

If n vector is in the opposite direction is in the opposite direction then it is orientation reversing. Try out this for a unit sphere.

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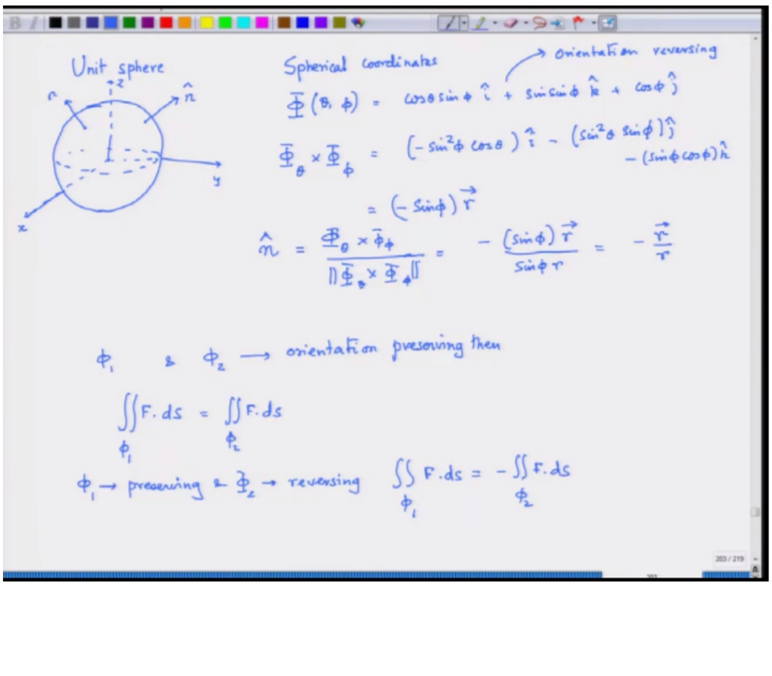

So, when I am putting n vector, so this is my outside, you know we have intuitively understand what is the outside of a sphere. So, you know what is the outside of a sphere, so this is another n vector. Now, is this, so suppose now I parameterize it. So, how do I parameterize a sphere? So, I parameterize a sphere using spherical coordinates. So, I use parameterize it using spherical coordinates.

So, I have my phi u, v here or theta, phi here. So, if I do this and you know what is phi, so it is cos theta sin phi i vector + sin theta sin phi k vector and cosine of phi j vector and you know that what is phi of theta cross capital phi of small phi turns out to be something like this cos theta i vector - sin square theta sin phi j vector - sin phi cos phi k vector. If I take the sin phi out, remaining what I will have is – r vector.

So, basically it is minus r vector is sin phi into r vector minus sin phi the scalar into r vector that is, so what I have, here my normal which is phi of small phi, capital phi of small phi so it does not matter what is this. So, if you take the norm of this, this is nothing but the radius sin phi radius into sin phi. So, it is theta, so this is my normal vector. The normal vector turns out to be minus sin phi r vector by sin phi r.

So, it turns out to be minus r vector by r and that has a negative sign. It is pointing inwards, so my parameterization, the parameterization in spherical coordinates, this parameterization is orientation reversing. So, if I have 2 parameterizations, phi 1 and phi 2 giving the same surface and both of them are orientation preserving, then phi 1 F dot dS = phi 2 F dot dS.

Suppose phi 1 is orientation preserving, but phi 2 is orientation reversing, phi 1 is preserving and phi 2 is reversing, then the whole game will change. So, what you have learned about line integrals appears to be true in the case of surface integrals. So, basically I have clubbed both my lectures on surface integrals into one hoping that I will have some more time to speak about Green's theorem and that is why I did it. So, Green's theorem will be done very carefully and with this I end my talk.