**Calculus of Several Real Variables**
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**Lecture – 31**
**Area of a Parameterized Surface**

As we have learnt earlier and I have mentioning in next, earlier class before this one that we are going to today talk about how to find surface area of a parameterized surface. Again, I want to recall, what is the meaning of parameterization that every surface that you see may not be the graph of some function it may not be immediate but it might be easier to write in the follow.

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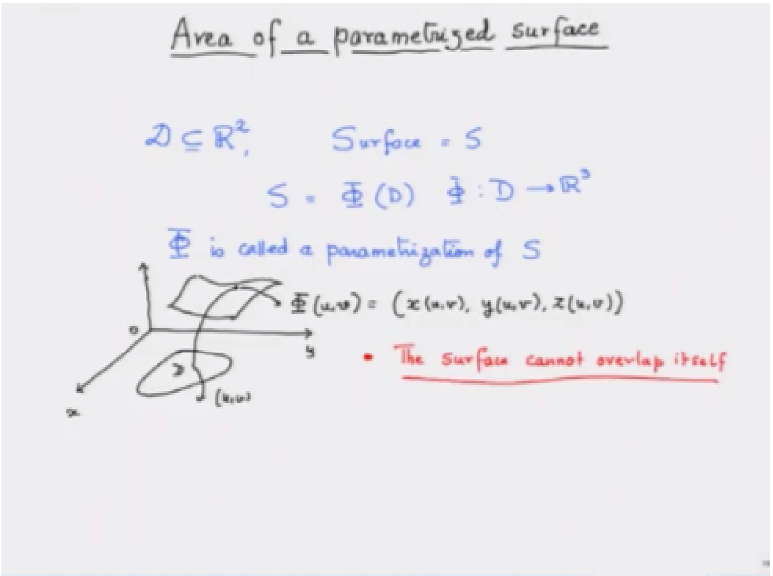


So, if you have a domain D in R2, right then and if S is the surface where denote the surface as S, then the S should be; if S can be written as phi of D, where phi is a map from D to R3, then this phi is called a parameterization of S, so the idea is this that here is the domain D in 2 dimensions which we have already discussed earlier and also I have drawn pictures and if this is the kind of surface, then if you take a point here say, it is u, v it get maps on to some point phi of u, v in R3.

So, it takes; u, v is taken to a point phi of u, v where phi of u, v is actually these 3 coordinates; x u, v, y of u, v and z of u, v, so this is how we represent. Now, in attempt to compute the surface area, you try to imitate what we have done it; done for the graph for the surface which is given by the graph of a function but we have to understand that it is not easy to find surface area everything.

Suppose, if you take a surface area; parameterised surface area but it is like this which when you folds upon itself like the graph that we had drawn when we were talking about parameterization of surfaces, finding surface area of such things are slightly difficult, so that means we need to talk about orientation and all sort of stuff, so we are not going to talk about those right away.

So, here we will consider those kind of parameterization where there the surface itself does not fold onto itself and it can maybe at the boundaries of D but we; over D it is a surface but we do not know to what function is graph, it may be a graph of a function may not be a graph of a function, so we are taking a very general view that we will talk about such surfaces which does not fold it into itself means it for that; for every for a given x, there can be 2 y's, we will not have that sort of situation, given xy there can be 2 z's sorry.

So, given xy, there can be 2 z's, we will not have that situation, so but still we will write it in terms of a parameter in the parametric form rather than writing it as a graph of function and we may not know whose which; whose, are we may not know whether at all it is a graph of any function. So, how do I find something like that? So, here I have to keep in mind the surface right cannot overlap itself, this is something very important thing that we have to mind, keep in mind; surface cannot overlap itself.

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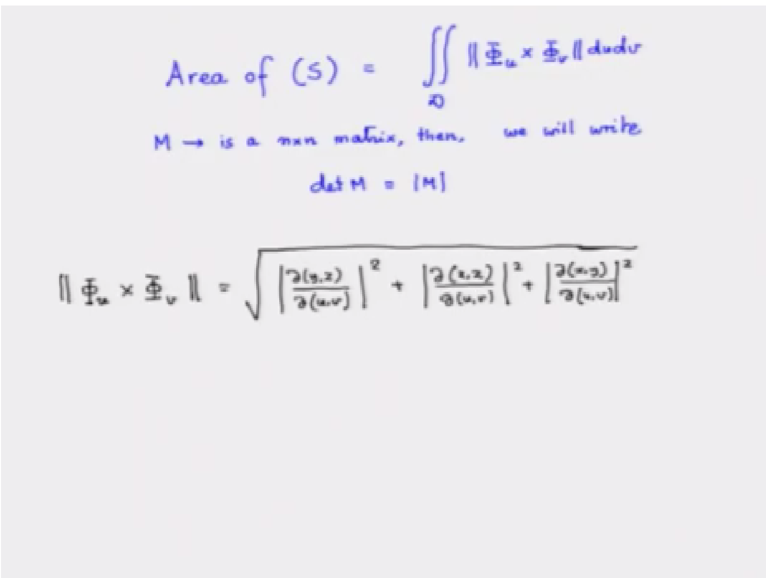

So, once these things are kept in mind, the surface area which we write area of S is given as integral over the domain D, the norm of phi of u and you know what is phi of u; cross phi of v into du dv. Now, we will make a convention that if I take any matrix M, so if I take a M is n cross n matrix of course, you might ask me how do you get this, we will get to that. Then, we will for short, we will write, then we will write determinant of M, we will write like this, you might be thinking why this is each time to change symbols at this moment.

But okay it is just for shorthand that we will use here and you know when you give a talk you can change your notations a bit while you go along because people are going along with you, while in a book a uniformity has to be maintained. So, if I write a book on this, then we maintain a uniformity but there are anyway good books, I do not have to troll for that.

So, let me see now, it is interesting that if you look at the structure, if you look at the norm of phi u cross; cross means a cross product of phi v, these vectors, we have already, what is this? This turns out to be a very interesting kind of expression; the expression that it turns out to be is the determinant of the Jacobian matrix, del yz you know what is this, del u v square of that plus determinant of del xz del uv square of that plus the determinant of del x y del uv, take the determinant and square the value.

It looks very, very surprising but how such a thing happens here okay, so first let us try to figure out what is this, why it is like this and then we will try to argue why we are writing, why the area of S is given by this integral and then one example we can do within our allotted time.

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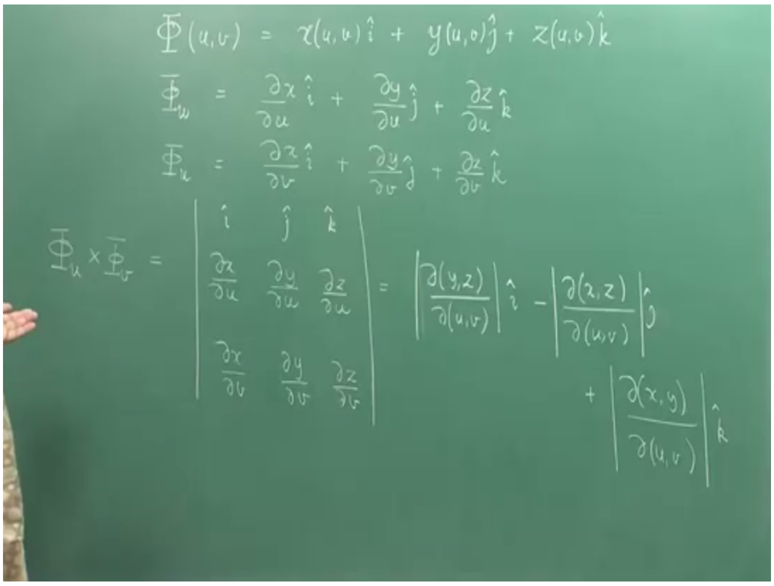

So, let me just go and write down what is phi uv, so your phi u v is x uv, i vector plus y uv j vector plus z uv k vector now, when you write del phi u, you are essentially looking at the derivatives like this; del del u of x uv, I am not writing uv, I am assuming that you understand, just for short I am doing this, del del y of uj vector plus del del z of uk vector. Similarly, you know what is phi v, if the vector of partial derivative is a gradient basically kind of; not a gradient.

But some sense, since in a sense that basically, this actually symbolizes that you take the partial derivative of this partial derivative of this and partial derivative of this and form a vector in terms of u and that is phi u and del x of del v i vector del y of del v j vector del z of del v k vector, okay. Now, once you have these notations with you, you know that what is the cross product, let us write down a cross product, so the cross product of these vectors.

So, we write down the cross product, it is i vector, j vector, k vector del x del u, del y del u, del z del u by del x del v, del y del v, please follow the calculations; del z del v, so if you know how to compute the determinant, what does this mean? This gives me determinant of this, so what does this; if you look at the determinant of this, this is del y by del u, del z by del u, del y by del v del z by del v, it is a kind of Jacobian of the transformation.

So, it is actually the determinant of which I am not writing, I am writing it like this, determinant of this, this, this, this which is same as writing determinant of del yz del uv i vector minus; now if you take j vector, then del x del z del x del z, so del x del u del z del u del x del v del z del u, it is again the Jacobian of the transformation which you have learnt, you change of variables, this xz del uv is the determinant.

And now, plus now, this is into j vector plus now, the determinant you have to; k vector, so this row goes, this column goes, so you are left with this part, so del x del y del x del u del y del u del x del v del y del v, so this is the determinant of del xy by del uv k vector.

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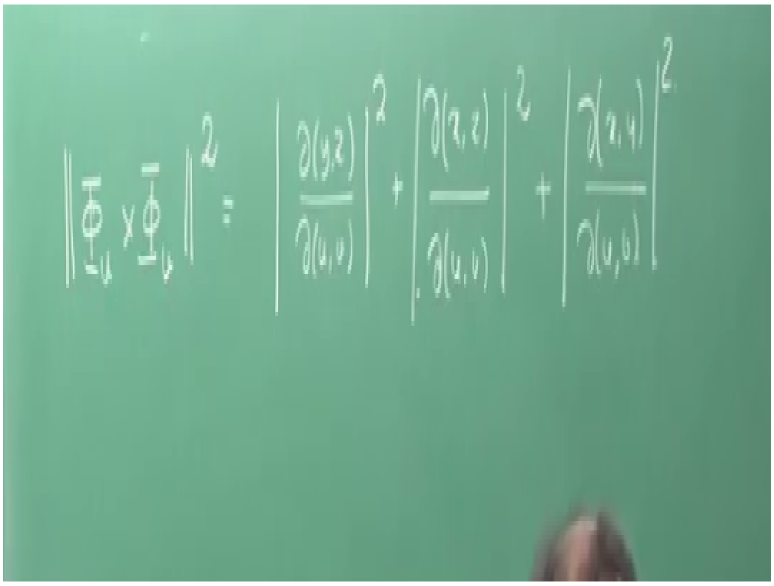


Now, if you look at the norm of this, you know what is the norm of this, the norm square of course is del yz del uv whole square + del xz + del uv whole square + del xy del uv whole square. So, you know this is a norm square, the standard Euclidean norm, so you know how this expression has now come which is the root over the positive root of this. So, we have actually computed the cross product but how does this cross product at all arise?

That is the question that we are going to tackle in the; in this discussion, so how does this cross product at all would arise, so again we will talk about an infinitesimal area or an elemental area, I would; I somehow call it elemental area, so that is a very basic building block element; element of an object, the basic element of an object.

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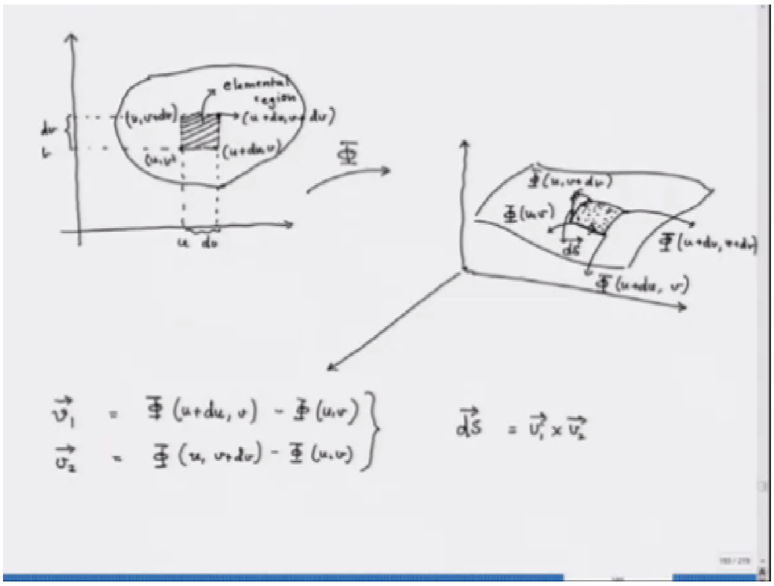


So, here is my domain D and here is my point uv, 2 dimension u and v, so here is a point uv and I make a small change along this direction that is u + del u and v, then I make a small change along both x and along this direction but keeping the u fixed, so it is giving me u v + del v and then I have the last point where I make both a change along x as along, change along y.

So, these distance this is my du and this distance is dv; infinitesimal distant, these are infinitesimal is a very strange object in calculus, Euler said that it has to be 0, you want to talk really about the infinitesimal, infinitesimal means it is a small quantity smaller than any quantity that you can think, any positive quantity that you can think so, it is a strange kind of notion actually, trying to talk about very small lengths or very small surfaces so or very small volume.

So, this is a very, very this; using very small quantities, very small things is this name infinitesimal got you know a kind of; it is a kind of lingo which calculus uses and though (()) (17:20) you have a math for infinitesimal but it is not so easy to comprehend them but it is very important to understand that when some length is very, very small, so when del x; when del x is very small, the del x distance, the change in x, we shall mark it by dx.

So, of course this whole idea of differential is not so very clear, maybe I should put a note here, given note to be given for Taylor expansion so similarly, I can put a note for what is the differential this is something I think students should try to realize first. So, many people including Bertrand and Russell, the great mathematician and philosopher said that he found calculus logically sloppy, does not matter it works it seems.

But you know the whole idea is that in the limiting version, see calculus a is a study of change and not only a study of just a change, it is a study of continuous change and calculus looks everything in the limit, so what happens in the limit is the question, how you are getting on and what sort of mental you know conceptual ideas you are taking in to reach the limit is secondary thought but what is happens in the limit is the key idea which calculus dwells on.

And so now, here we have this is my domain D, so this is my infinitesimal domain or elemental domain, this is the big one is a domain D; elemental region. Now, this by map; by the map file is getting into a 3D surface, there is a; I have a tendency to write or draw everything in the positive part non negative part, does not mean you can draw it at any part but it is easier to draw when you keep our drawing like that.

Now, here is a corresponding small elemental region, kind of parallelogram or (()) (19:50) parallelogram something a little curved lines that is there, that is formed here and this is your ds actually, the elemental area ds. So, how this form u and v is mapped here, so this is this point gives me your phi of uv, this point gives me this this one; phi of u + du and phi of u + du and v, this point is mapped to the one here, this one which is phi of du.

So, this point; this endpoint sorry, which I have not marked as a total elemental region, this point here is actually u + du and v + dv, carefully look at this, you should stop this thing and carefully look at the picture, stop the video here this point is phi of u + du, v + dv and this point corresponds to phi of uv + dv. So, I have been speaking about the area vector, so the elemental area vector which should be perpendicular say at uv, so this is my elemental area vector ds vector.

So, this elemental area vector, I can think of it as a cross product of these 2 vectors which is; so the cross product of phi u + dv, cross product of these 2 vectors that is; so I will; the vector v1 which will be; so the ds vector will be the cross product of these 2 vectors, v1 which will be phi of u + du v - phi of uv, so here basically we are putting x of du, you are putting du + du u + du.

So, when you do a first order Taylor's expansion, forget about the errors, so what it remains; del x del u u, del x del u this, du del y del u du del z del u du, so I will just do that and the v2 vector whose cross product are also take and that is phi of uv + dv - phi of uv, so let us see. So, our ds vector; the elemental vector, which is perpendicular normal to the surface to the area; surface area.

So, we are taking it as a; it is a minimal school zone, so the straight lines; this curvy lines are almost straight because a very small, so we can talk about this as a kind of a parallelogram, not really rectangle but a parallelogram and so essentially, the cross product of the 2 sides gives me the area, so the area vector now, let us look at the v1, what is phi of u + du?

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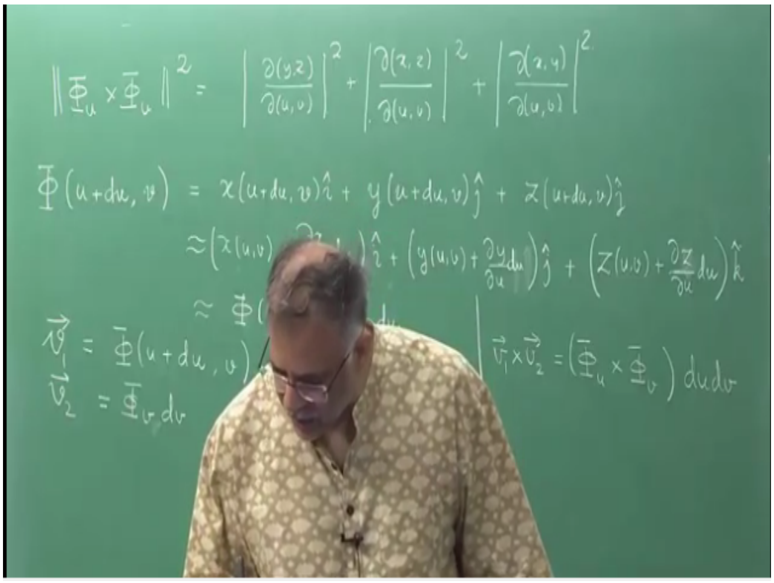


So, if you look at the phi of u + du that is x of u + vu j vector + z of u + du now, when v is fixed, all these x, y and z, these are function of the variable u, a single variable, when the functions of a single variable I can apply the Taylor's theorem but I will forget the errors because du and dv are so small that I can forget the errors for all practical purposes, so I can write approximately, I am not giving the equal to sign, please understand this.

In the limit everything goes, the error vanishes in the limit that is the key idea of calculus, the errors will vanish in the limit, this is the key idea, if you keep this in mind you will know calculus much better. So, you know math has to be done also in certain kind of intuition, it is not always that you are talking about rigour here of course we are trying to be very quiet rigorous here.

And now here, what happens is the following; now, here I can write this as a Taylor's expansion first order but forgetting the error, so it will be x of uv + del x del u del u, this i rector and similarly, y of uv + del y del u j vector, I am forgetting sorry; del y del u du j vector, I am forgetting, I am just completely throwing away the error that is why I am writing as the approximation.

And it is a good approximation when they are very small, so z uv sorry, y del y del u del z del u, I am sure if people were here they would have corrected. Now, what happens if you look at this if, you take this as i, this, this and this club them into one; xi, yi, zi and this then this one, then what I have got here is x uv + phi u du, this vector. So, when I am writing the v1 vector which is the difference of this vector with phi uv, so this is phi uv, sorry not x uv, plus xi + yj + zk gives me phi.

And this plus this plus this would give me phi u, this will give me phi u vector into du; du is the scalar thing; a scalar into a vector. So, what I would have; when I write phi u; phi of u + duv – phi of uv, I will get phi of uv cancelled and this will become nothing but phi of u du and similarly, v2 vector by symmetry can be written as phi of v dv. So, now when you are doing v1 cross v2 which is your ds vector, what you simply get is phi u cross phi v into du dv.
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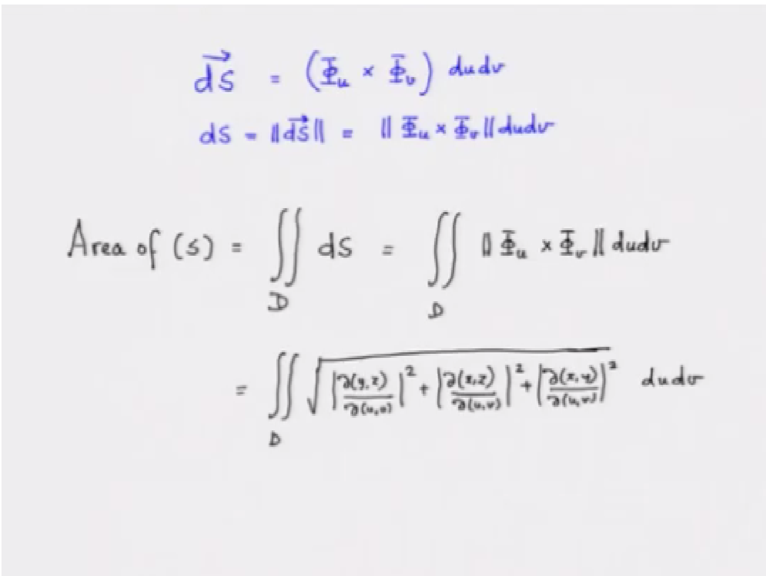


So, what I finally get here that ds vector is phi of u cross phi of v into du dv, so these are norms, we essentially take them to be very small positive length these are representing length, so we take them to be positive. So, ds; the surface area the vector ds, elemental surface area which is the norm of the area vector, so is nothing but norm of phi u cross phi v into du dv is positive quantities out of the norm.

And so, area of S is integral D ds which is nothing but integral D norm of phi u, so Taylor's expansion is the secret of many things in calculus, please understand this thing, du dv and that is exactly what you want and that is and of course you can make a more very strong expression because you already know that this can be written as because you already know this fact that is this is del yz del uv, you know this already, you know that norm of this is what I am just writing down what is the expression of norm in terms of given data, del of xy by del of uv.

Sometimes, I become quite while writing, it might pour your bit but just bear with me because I am also writing something pretty long, so that is the expression. So, now once we have done that a single example would show us that okay what we have done is pretty okay pretty decent, my Taylor's theorem game works.

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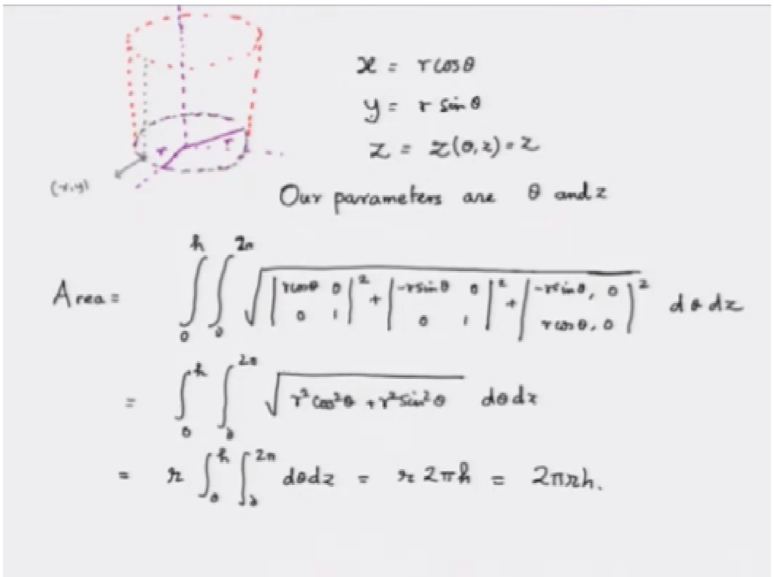


So, let us do a very simple thing, so now let us consider a cylinder which is centred at 0, as a radius r and as a height h, so this is my z axis which is going through, so this is; maybe my drawing might not be this is also r, so what is you know what is the answer of course, the surface area of a cylinder. So, basically if you cut through the cylinder and roll it off, you know that this one length side is nothing but 2pi r which is the circular circumference into h, the breadth; length into breadth.

Let us see a calculus can do us, they give us anything, so how do i parameterize this cylindrical surface? I do not know whether it is just for corresponding to every xy, there is one particular z, this is corresponding to every xy, here is a z, so here is a corresponding to xy here, there are more than one z's but they are not overlapping, so I said that okay, we will not consider surfaces where 1 xy gives to 2z's.

But I think that is not really the thing I wanted to mean; means I wanted to mean that when you overlap a cross like this, then when the surface overlap on itself, then corresponding to every such xy, you would have more than one here of course, you can say that I would have more than corresponding to any xz in the domain right. So, where is; what is my domain? My domain here is the circular domain, this circle is my domain.

And of course, here I am having corresponding to every x, there is a more than 1 z, so I cannot write it as a graph of a function but see I can parameterize it, so when I parameterize it, then that parameters will be r theta and z, so corresponding to r and theta, given r and theta there will be a particular z that is corresponding to x and corresponding to y, there is a z.

Actually, if you take any xy here for example, take any xy here on the surface of the cylinder all along this height h, there are fixed xy, the z's are part of the cylinder, so they cannot be expressed as a graph of a function but they do not overlap, so I would like to rather correct my sentence what I meant by that time that I will not consider those functions which overlap itself.

Because overlapping will always give me situations where corresponding to 1 xy, there are more than 1z and because it overlaps there are issues of orientation and which we cannot handle but when it does not have overlap but is not the graph of a function, we can still handle this situation using parameterization. So, what is my parameterization? Parameterization means x is expressed as a function of r theta, so given an r theta there is a fixed x.

So, given an r theta, there is a fixed y, given as it has a fixed z, so for r theta; for a fixed r theta, there is no 2 x's or 2 y's, that is not 2x, 2 different xy's there are fixed xy's for a given r theta. So, I can now express; what does parameterization do? One has to again remember at this stage that because I cannot express it as a graph of a function because this surface does not represent a function, I can represent it in such a way such that its x coordinates, y coordinate, z coordinates itself can be represented as a function.

And then, we can apply our things by using this transformation, this kind of change of variable, so we are making kind of a change of variable here but so that my x coordinate, y coordinate, z coordinate which are not linked through some function, can be linked through some parameters and with respect to those parameters, each of them are a function so that is the key idea.

Please remember this key idea, here x is equal to r cos theta, y is equal to r sin theta and z is equal to z, so here my parameters are r theta, right here my parameters is theta and z because r here is fixed, so I have been given a see, base of the cylinder which has a fixed radius, so x and y will depend on the theta, so if you give an input theta and z, sorry I am talking about r theta.

So, if you given input theta here given input theta here, so for a given theta, there is only one x, for a given theta there is only one y, not 2 y's and for a given z there is only one z, so theta z for example, here z is; z theta z which is z, so here also you can say okay, how do we express these are the function of theta; z and theta. So, z, theta z and that is equal to z, that is what we are doing, that is the meaning of this.

So, basically, so our parameters here are or parameters are not r and theta, so theta and z, i is fixed because the base of the cylinder is given or parameters r theta and z, right, the Z is changing and theta is changing, here nothing is changing. Now, if you go by what we have done earlier, so its area is given as; so theta is going from 0 to 2pi, this is going from; z is going from 0 to h root over, now we do those calculations.

So, the determinant would be r cos theta 0, these are thing given in the book but you can just write others yourself; minus r sin theta, you are taking the derivative of cos here, 0, 0, 1 whole square plus minus r sin theta 0 r cos theta 0, whole square into d theta dz, my parameters. So, if you calculate this; this will come up to 0 to h, 0 to 2pi root over r square cos square theta plus r square sin square theta d theta dz.

So, r is; this will just be r, so I will take r outside, so r times 0 to h, 0 to 2pi d theta dz and this integration is now simple to you, r 2pi h, so that is 2pi rh. So, calculus can give you; get you back what you already know from geometry of course, you can talk about much more difficult looking curves, so you can in fact also for example, if you look at a surface, so here is an example from the book but I would like you to try it out yourself.

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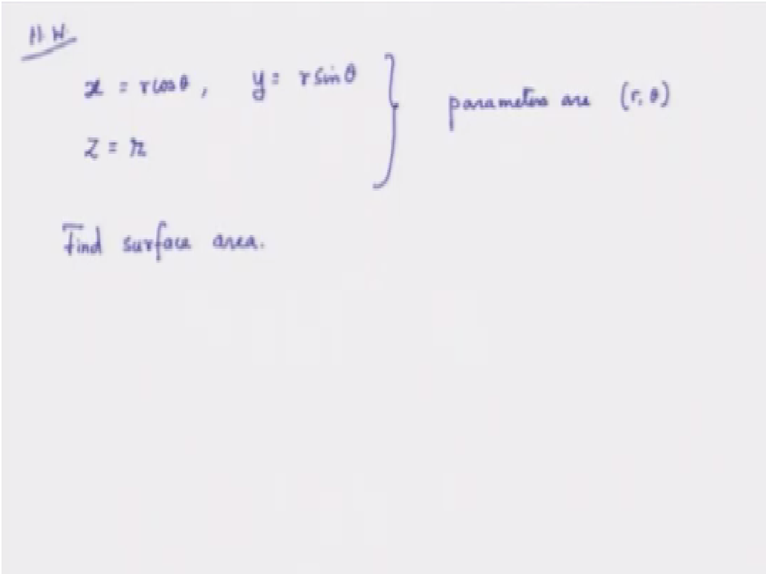


Take a look from at the surface which is character (()) (40:53) which is parameterized as homework, this is homework, so before I end I give a homework; x is equal to r cos theta, y is equal to r sin theta and z is equal to r, so here r and theta are your variables right, here r and theta here your parametric; you parameterize by r and theta, parameters here are r and theta.

So, now find surface area, the same story, same calculation, the same similar sort of calculation now you have to do with noting that r and theta is my; the same looking thing instead of z, I have met here r is not fixed here r is changing, so you may try to get into computers those you have an use MATLAB if you have and try to draw the surface and see what it looks.

So, with this I end this talk here and in the next class, we are going to talk about how to integrate a scalar function over a surface and then we will talk give a formal definition of a surface integral where we integrate a vector function over a surface that is instead of line integral f dot ds, we will have a surface integral right, so for a line integral you have done it over a line over a curve f dot ds where here the ds was arc length.

Here, the so f dot small d small s that is what you have learnt in line integral, here you learn in surface integral f dot d; capital S because it will talk about the elemental surface and they will talk about how to integrate that; that is so integrating a vector function over a curve is a line integral, integrating a vector function over a surface is a surface integral and that is what we are going to learn in the next class.

So, after talking about surface integral, we will go to a second idea about orientation and all those stuff about surfaces and then we will go into greens theorem, so which will be the last class of this 7th week and then we will get into the 8th week, the last week which will have interesting stuff.