Calculus of Several Real Variables

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Lecture – 30
Area of a Surface and Integrals - I

So, welcome to the 7th week, we are actually gradually going towards the end of the course and this 7th week contains 2 lectures on how double integrals can help us finding areas of surfaces secondly, we are going to talk about surface integrals, what is the meaning of surface integrals, like you have spoken about line integrals so, there will be surface integrals and also we will come to what is called greens theorem which would link the line integrals and surface intervals.

So, this will culminate into a good study of integrals and then their application where we will combine all these integrals with curl and divergence and all this stuff that we learn into in 8th week in the strokes and divergence theorem and at the end there will be more; the last talk would be more of a general talk giving you the idea of fields and how all these concepts can be applied for example, to physics and so this you might feel a little bit cowed down by this too much of computation but you need not worry.

Because ultimately, here we are getting into a lot of interdisciplinary domains we were; for example when we are talking about a parametric surface, we were essentially in the domain of a subject called differential geometry which is a kind of geometry of surfaces with whose surfaces where or parametric surfaces whose derivatives are present, right, so that kind of geometry we are getting into.

So, at the end when you; this is advanced calculus, when you are going into this advanced zone, you come and merge with different other aspects of mathematics, so or they emerge with calculus whatever we say, so calculus even central with that; that is what one thing we are going to show, here we are trying to emphasize here that calculus remains central to science and calculus even central to many aspects of mathematics.

It might look secondary now, people I have read an interview of a great mathematician called Paul Erdos and in which he was asked by the interviewer that Oh! sometimes we know that you go and teach a subject as lowly as calculus, it was very surprising to me that calculus been called lowly, while people like Steve strogatz have glorified calculus in their book and showing that it is so much central to our very existence, modern world would not exist without calculus.

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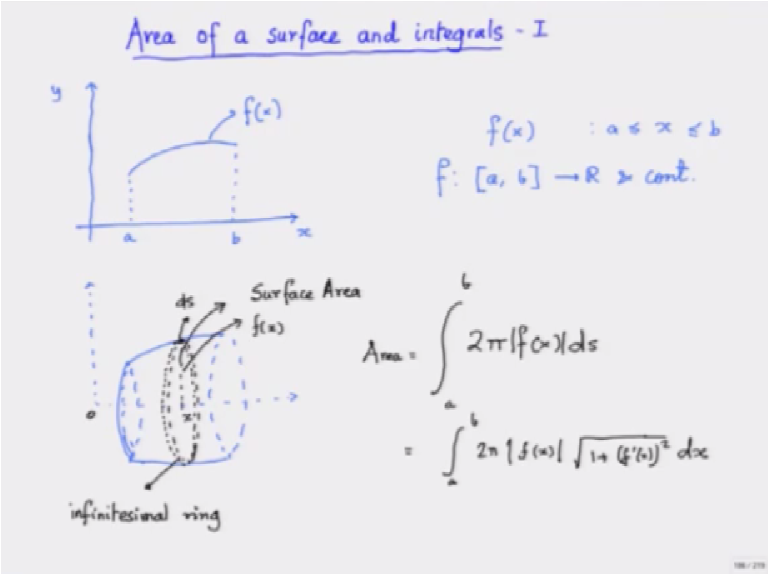

So, first we are going to talk about a very simple idea which you know from your one variable calculus that suppose, I take a non-negative function a and b and so, I have a function fx which is lying between a and b, function fx and this function fx is bounded function obviously, f is defined from a, b to R here of course, x is defined from between a to b and if suppose, it is a continuous function from a, b to R, I am written in short continuous.

Now, if I revolve this so, I revolve this curve around the x axis, then you would get what is called a solid of revolution of this form, a kind of solid of revolution of this form and suppose, we ask ourselves what is the surface area of this, and you compute the surface area. We go again by that idea of infinitesimal chopping, so you basically chop out and so from a distance 0 to x here, you chop out a small circular zone or infinitesimal cylinder and which has a length ds.

So, what I have; I have that for this small circular zone, so I find the circumference which is the length of the line into the area basically, into the; so that is the length of a kind of a rectangle and into the breadth. So, if I take; if I now I have made this infinitesimal cylinder; this infinitesimal cylinder is made on the body of that surface area now, if I cut out that infinitesimal cylinder somehow, take it out and then cut from one point.

What is it? It is a rectangle whose length; whose breadth is dx and length is 2pi R, R here is nothing but f of x, assuming that the function value, it does not change very much when dx is very small, this is f of x. So, the surface area of the infinitesimal ring or infinitesimal ring I would say or cylinder whatever you want to call but I would say it is a ring because I am just talking about surface area.

So, this is the infinitesimal ring and the surface area of that infinitesimal ring is 2pi fx; fx is the radius into dx, now this can happen I can now vary x from a to b and c and basically, sum up all the areas of; surface areas all those infinitesimal rings and that would give me the surface area of the solid of revolution that I have generated and that area I should say, mod of fx here sorry, that is a mistake please, please kindly correct it.

This is not dx but this is ds, ds would come on the x axis, so ds is the arc length that is on the boundary of the curve, it is a small arch line, so that is actually the breadth, when you take that infinitesimal ring and cut out from that cylinder and if you cut that at any point and straighten it out, it would be a rectangle with an infinitesimal breadth ds, not dx sorry, ds and you have an infinitesimal kind of; you have an infinitesimal and a length 2 pi R into fx.

So, but this can be now, this of course has to be written in terms of x, so 2pi and you know that this can be written in terms of x; dx can be written as root over 1 + d/dx whole square which is f dash x whole square into dx, so this is how the area is calculated. Can this idea we use to calculate the area of any surface not just a solid of revolution but a surface which is given to us in a 2 dimensional space.

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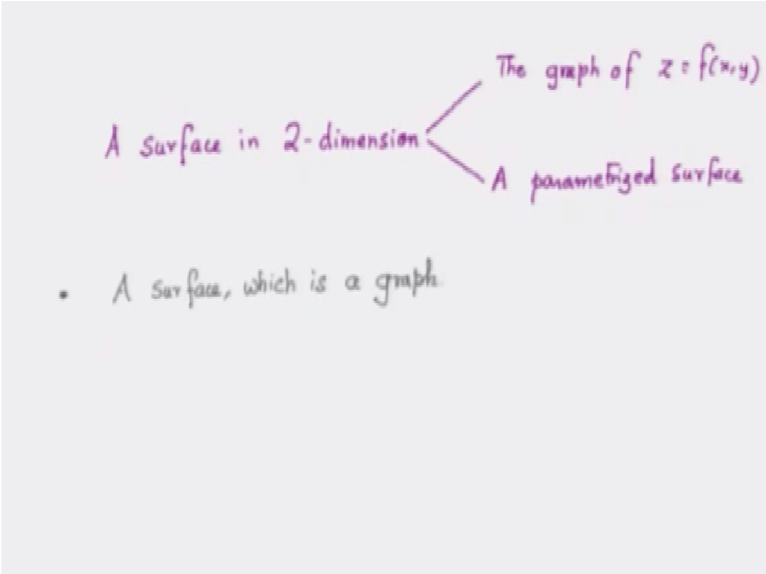

Now, a surface in 2 dimension can be 2 things, in the sense that it can be the graph of z equal to f x, y or a parameterized surface, need not be the graph of something. In this talk, we will talk about how to find the surface area in terms of double integral of the graph of f x,y, see all these calculations are required in the sciences, in natural sciences, so that is why these are very, very carefully calculated out.

And please understand these are not done by Newton, Newton was not doing any double integral or anything, these were developed much later by people like Cauchy Euler or Cauchy and other people were stressed, so all these things came much later. Now, so we will talk about the graph of a surface, which is the graph of z equal to f x, y and now let me see how do I try to compute its area.

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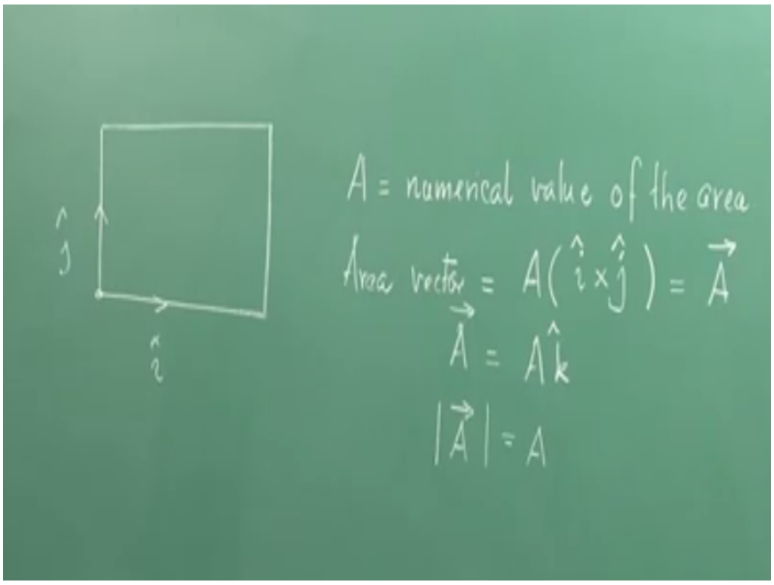

Please understand area can be viewed as a vector for example, if you take a rectangular in R2, suppose, you take a draw rectangle, so this is a x axis and this is along the x axis, this is along the y axis, so there is the i vector and there is a j vector, so this i vector lies along the length of the rectangle and j vector lies along the breadth of the rectangle. So, now suppose the length into breadth; A is the numerical value of the area, this is a non-negative number and in this particular drawing, it is positive.

Then, I can always compare now, I can always write down or rather represent the area in terms of a vector which I will called an area vector, you understand this i vector is lying along the length, j vector is lying along the breadth, so area vector; how can the area vector we develop? So, length into breadth in terms of vector is nothing but i cross j, okay and this i cross j vector when it is scalar multiplied by the area, numerical value of the area, this gives us what we call the area vector, okay.

So, we can use this idea, so the area vector always points; here it is pointing along the k direction, so area vector A is nothing but this, so here it is follow; it is along the k vector, along the z axis and its i cross j; so or if you want to say j cross i whichever, so if its i cross j is like this, so when you put the screw like this, it goes in that direction right, so the area vector that is pointing out towards the other side.

Or if you say j cross; so if you say see, if you have i cross j, so then it is pointing upwards, so if you have i cross j, then the area vector is pointing towards us, toward my nose right that is what is important, it is pointing towards my nose while, if you do the opposite point to the opposite direction and of course, the modulus of the length of the area vector is same as A because the length of this is 1.

So, this idea of the area vector becomes very, very crucial in understanding the surface area of; computing the surface area of the graph.

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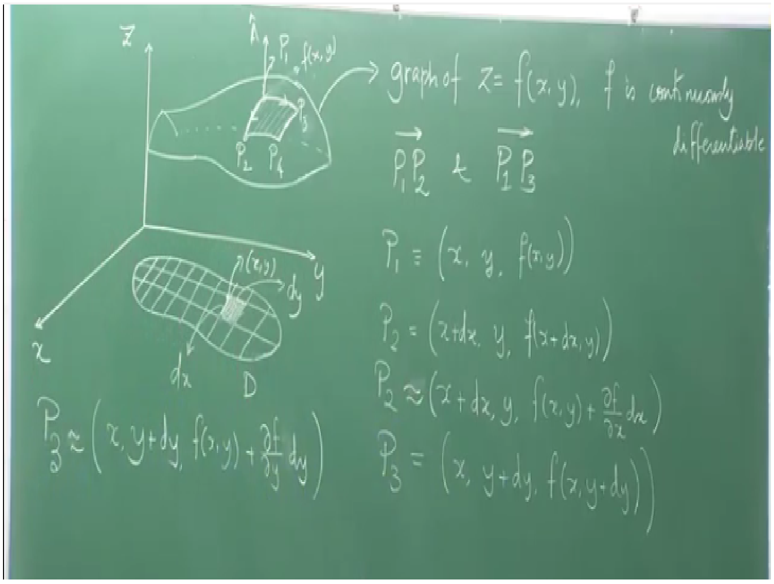

So, how do I look at this whole thing? So, consider some kind of a domain D here and then there is a surface area associated with the domain D, okay. What you do is you do, you take some grids here, make some rectangular grids in the domain D and make them very small so that I can consider one of them as an elemental area. So, let me consider this is your dx and that is your dy.

But so, this let me consider this elemental zone, so this is my x axis, this is my y, axis this is my z axis, so that is my elemental zone, so this side is dx and this side is dy and this gets is very small rectangular zone gets because of the surface is continuous, we are assuming nice continuous functions, so this is the graph of; I am just drawing me giving the thing continuous, f is continuously differentiable, so that is the kind of thing.

Now, what would happen that this part gets mapped to a small zone here, so if this was my xy point and I have around xy I have made some kind of small infinitesimal rectangle xy, it is a map to this point which I call P1, then I will call these corresponding points here, P2 which is corresponding to this one, this one; this point here gets mapped to P2, this point gets mapped to here which I call P3 and P4.

So, here it is a kind of an infinitesimal parallelogram not a rectangle but slightly a very small indeed but it is; though these sides are curve, we can for very small dx and dy, you can take them to state in this kind of parallelogram (()) (17:49). So, parallelogram also can have an area vector defined in a same way and I would ask let your imagination freely flow as to how can I do that for a parallelogram because this cross product; notion of cross product will come there also.

Because if you see, if you take a rectangle and a parallelogram lying between the same parallel lines and one of the length of the rectangle is shared with the parallelogram and then they have the same area, area is a very important notion, it is very intuitive but it is not so simple to really formalize. So, here I have something called an infinitesimal parallelogram maybe I will call this; I will circle it P1 P2, P2 P4, P1 P3, I am just going by the book if somebody need so, it does not matter, okay.

So, I am now looking at these 2 vectors, I am looking at the vector P1 to P2 and I am looking at the vector P1 to P3 and the area vector would become a kind of cross product of these 2 vectors, which will be normal to; so this point is actually f of x, y so it will be normal, so this will become the area vector normal to that point, okay. Now, let me look at the point P1; P1, the z coordinate is f x, y actually, the height.

So, if I write down the coordinates of the point P1 and the point P1 is described by x, y and f x, y, if I look at the point P2, it has x + dx and its y of course has not changed and f of x + dx, y. If dx is very small, very, very small, then this can be represented in terms of the derivative, we can forget the error term and we can write this as P2 as; P2 can be written as approximately you would say, I would go by your thing.

But we can for all practical purposes, we can write them as; this can be written as f x, y + del f dx into dx, this is what how P2 can be written as, then I have to write, talk of P3, P4, you may write, you may not write, it is up to you because I only need these 2 lengths. Now, how can I write P3? P3 is corresponding to this where I made a change in y but not in x, so P3 is actually x , y + dy and f of x, y + dy.

In the same way, knowing that dy is very small, my P3 can be now written as, if you want to be more precise, I will say approximated as but the approximation is a very good approximation I will use that itself as the difference x + dy f of x, y + del f del y into dy that is what is P3. Now, let me come to this side and look at the vectors, I will just rub this part, I am sure that you have copied it out or you can rerun the video.

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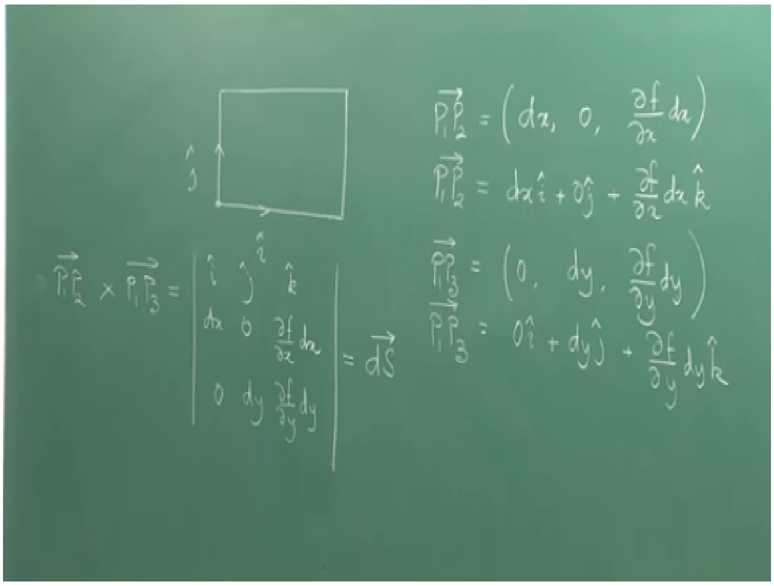

So, now what I will do; I look at these vectors P1, P2, cut; so now if I look at the P1, P2 vector and you know how to do it, is to just to subtract this differences, we want to P2, so you actually subtract from here to here, so basically P1, P2 vector would give me dx, 0 and this, so P1 to P2 vector is giving me dx, 0, del f, del x, dx. So, here I; you see here I am making a first order Taylor expansion throwing out the error term.

But in terms of the first order only in terms of the x, so I am just doing it for one variable, so I can write it very simply, so here because I have actually a function of 2 variables, my derivative in one variable is actually is a partial derivative and that is what I have written it in this term. So, I can write P1, P2 vectorially, more vectorially as dx of i vector + 0 into j vector + del f del x dx into k vector, this is what I can write, okay.

Similarly, I would looked at P1, P3 this vector, right P1, P3 this vector, so if I look at P1, P3, you basically again do the same subtract from here, I am going to take this vector not the; I am going to take this approximation from here and subtract this one, so I will have, x will obviously go, will have 0, dy, del f del y dy. So, again if I go by my vectorial writing like physicist, I will have 0 into i vector + dy into j vector + del f del y dy into k vector.

What is important that to get an feel of the area, first I have to compute the area vector, right, the area vector take its norm basically, so to compute the area vector, I have to take a cross product of P1 P2 cross P2 P3 okay and let me just try to do that. So, basically now I have to do P1 P2 , sorry P1 P2 cross P2 P3, so the vector will go on the top; the area vector.

So that is; that will give me what is called ds, elemental surface area, so that is the vector which will represent ds, the norm of that would be my infinitesimal surface okay, so that is the area of parallelograms are computed there, you take the were vector along the length, vector along the width, you take the cross product and take the norm that is the; okay.

So, P1 P2 cross P1 P3 cross product, so here I write down the definition of the cross product; i, j okay, that is the cross product you have dot product we had also spoken about cross product say if i, j, k so, you have dx here, so dx, 0, del f, del x dx, 0, dy, del f del y dy, so let me call this as my area vector; small area vector, ds vector, so this is what I will call as the ds vector.

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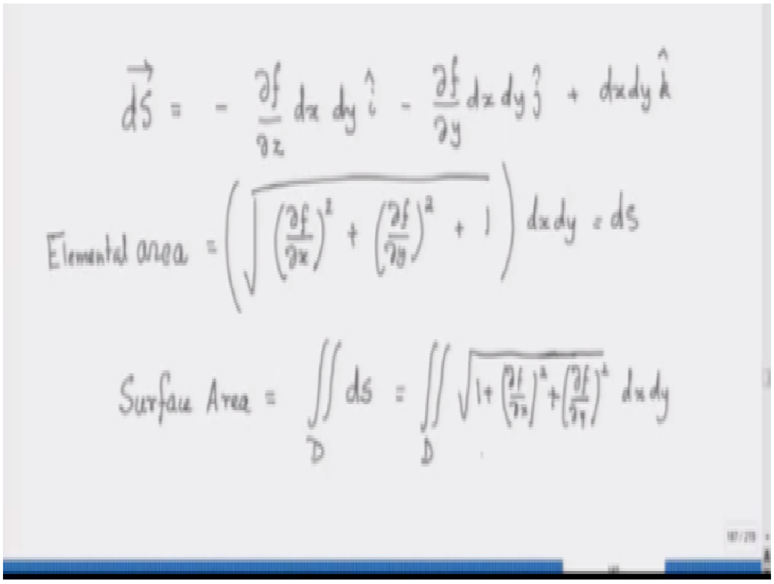

Because of the area you are talking about an infinitesimal area, so what we have got now is that the infinitesimal area vector which is now computed through this dot product which you is; cross product which you see on the board that will give you a final answer of this form ; - del f del x dx dy i vector - del f del y dx dy j vector + dx dy k vector, so I have to take the norm of this vector to get what is the value of the elemental area; I have to get the norm of this to get the value of the elemental area.

So, what shall I do? Basically, I take the norm of this vector and you know how to calculate the Euclidean norm, so elemental area is equal to root over actually, I can take the del f del x whole square + del f del y whole square + 1 this whole thing into dx dy, so this is your ds, so my total area; total surface area is integral over the domain D, where the x varies; x, y varies ds.

So, that second integral of the domain D root over 1 + del f del x whole square + del f del y whole square dx dy, so this is the formula for the surface area which is the generalization of what we learned for a solid of revolution, it is for any surface which is given by some graph function, this is the way one has to compute the surface area. So, once you have learned how to compute it, we give one more example.

With that example, we shall end our discussion and giving a very simple and I will give you 2 examples in fact and I will give you a very simple way to look a ds. Here is an example I would like you to solve yourself, it is in the book but try it out yourself, I will give you a hint not an solution but a hint how to do it. The hint is as follows that okay, I what I am trying to say is that okay.

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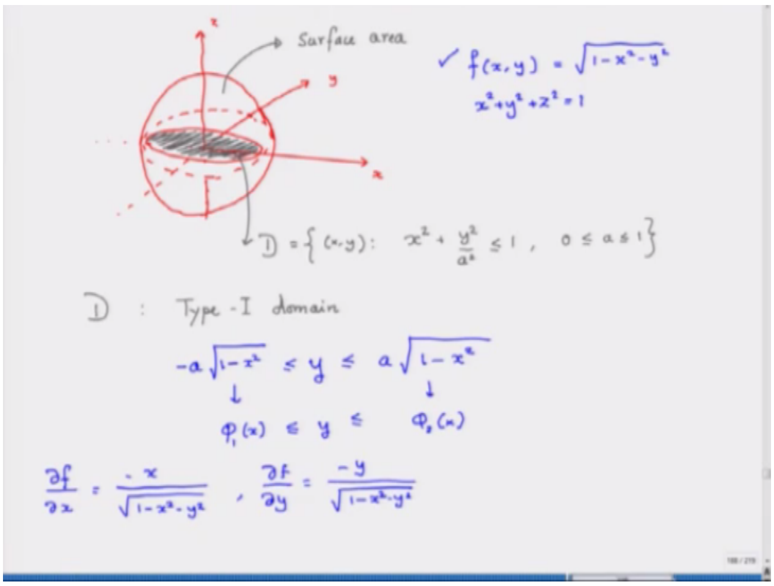

Here, you have a hemisphere; x axis, y axis, z axis and you have hemisphere of radius 1, now in this domain I will consider ellipse of this form; elliptical domain, I will consider an elliptical domain, this is my domain D, that elliptical domain will consist of all x, y such that x square + y square/ a square is <= 1; equal to 1 means the ellipse and less than equal to 1 means the region inside the ellipse with of course a lying between 0 and 1, some number, take it some number.

If it is 1, it will to become the whole circular domain now, if you see how to compute this, I would; this is a very general class of problem, this will actually allow you to even compute the surface area of the sphere you know that look at this region now, I have taken the domain

D now, I want to calculate to the surface of the hemisphere which is lying above the domain, so I want to you to compute this surface area.

So, this you take it up as a small homework because here first you have to observe that you can write down this D as D is a type one zone, this is a type one domain, why it is a type one domain? Because y can be; because I can write y between these 2 numbers, when these 2 functions, is a symbol to carry out this inequality from this expression and I will leave it to you, so this can be thought of as phi 1x and this can be thought of as phi 2x.

So, it is a type 1 domain and you know how to proceed in the type 1 domain, you know how to proceed, you know how to go in the type 1 domain, so in the type 1 domain you first integrate over the y, we did phi 2x and phi 1x and then integrate over the x, so here you have to first understand that here del f del x that is all your del f del x here, so what is here, f here, is a; so your f x, y, z sorry, f x, y, so I am talking about the hemisphere, so it is equal to root over 1 minus x square minus y square.

Actually, we were talking about the hemisphere of this form, x square + y square + z square = 1, so if you compute del f, del x in this case where our functional form is this one because here we are looking at the hemisphere x square + y square + z square equal to 1 but we are only bother about upper part where that is gone negative, so it is this part. So, del f del x in this case becomes minus x by root over 1 - x square - y square, while del f del y becomes has a symmetry here.

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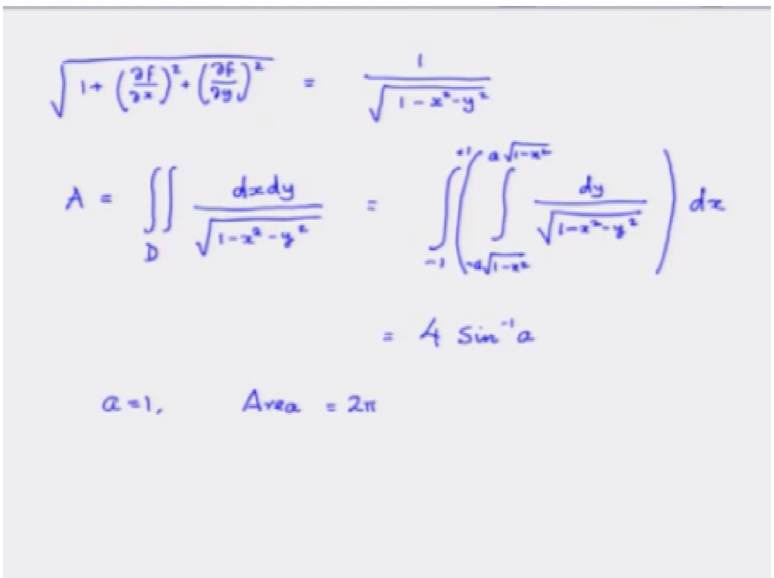

So, it is minus y by root over 1 minus x square minus y square, so now once you have that computing the area is not a big problem, so first let me just check it up, del f del x whole square + del f del y whole square what does it give me, so this will give us 1 by root over 1 - x square - y square, so this area A can now be written as integral D dx dy root over 1 - x square - y square.

Now, I can actually first do it over y, naturally because y is this is a type 1 domain, so minus 1 to plus 1 which is the outer part where x and here it is - a root over 1 - x square a root over 1 plus, say root over 1 - x square dy by root over 1 - x square - y square actually, you have to put 1 minus x square is something, so you can make the integral look, put 1 minus x square as u1 and make the integral look much more simpler, so this into dx.

I will not do the full integral, I will tell you the answer of this is 4 sin inverse a, now you understand that this is a very general kind of problem that we had given, if I put a equal to 1 here, so that would give me the whole circular domain that is given by the red dotted line and if that is; so then I am essentially asking about half of the surface area of the sphere of radius 1.

In that case, I will put a equal to 1 and sin inverse 1 is pi by 2 and then I will have 4pi by 2 that is 2pi, so 4pi R square, so 4pi here is the whole area of the surface area of the sphere and hemisphere means half of the sphere has 2 point. So, when a equal to 1, the area is equal to 2pi, so that is a nice way that is a very general problem where you can use what you have learned to compute the area.

And why this thing actually works because it returns me back the value of the sphere that I really know.

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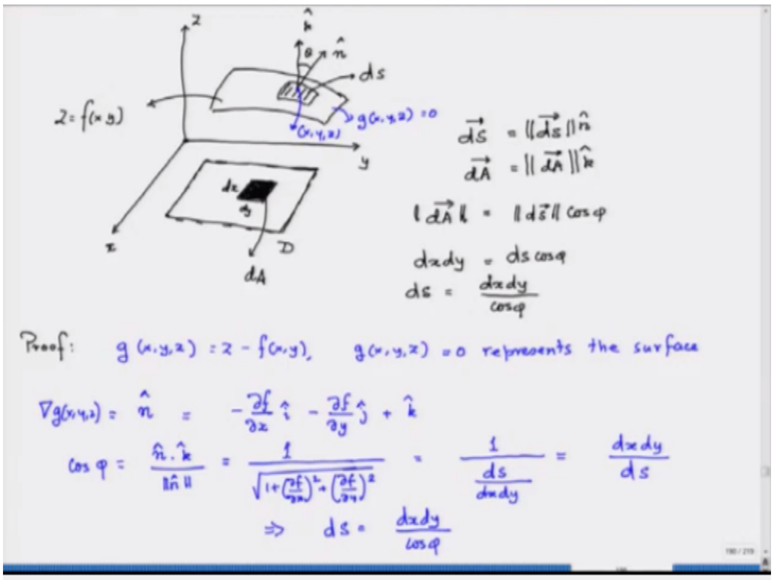

Now, I am going to give you using this idea of the area vector, I am going to give you a very simple expression of that of a relation between ds and dx dy; dx dy is the area of the domain that you have, so for example this is my just for heck of it, I am just drawing a rectangular domain D now, this if I take this rectangular zone and suppose, I have a surface area here which is my ds, so here it is my dx dy, so here it is my dx, so this is dy and this is dx.

So, here at this particular point, so this is my x, y and I made a rectangle around x, y, the normal to this surface is this, n vector, so whether it is a unit vector and the unit vector which is actually normal to this dx dy domain on the xy plane that is the k vector and let the area be the angle between them is some theta and this is my ds, so the ds vector; so what is ds vector?

It is ds, the actual the length of ds vector the magnitude into n vector, this is the area vector and what is k vector, so if I call this area is; small area is dA, I am calling it as small area is dA which is dx dy. So, dA vector is dA vector into k vector but you observe that what is cos phi? So, ds vector lies, so I can write it as ds vector dot dA vector into norm of ds norm into dA which is this actually, norm of ds or if you just think ds has the n vector maybe you can just think of ds itself has a n vector.

I am telling it has a unit vector, then just a moment I will just okay, so let me just think in a slightly different way, so what is this k vector, what is this direction? This k vector is; k vector by a nth vector is cosine of this theta, so k vector is n vector cos theta something like this, so where along k vector lies this dA area, magnitude of dA, this is dx dy that is along n vector lies what; along n vector lies the ds, I need draw ds.

And that of course phi that is what happens, so dA is dx dy, so dx dy is ds into cos phi, so ds is equal to dx dy into cos phi, so these are very simple idea which comes out of looking at these vectorial things by using the idea of an area vector, so because the dA lies along this direction, dk lies along this direction, we can write this kind of a relationship, right. Now, of course you can say okay these are just intuitive things that you are doing but how do you actually prove it.

So, let me give you a proof which is also in the book but the proof is very simple now, assume that here we are talking about only this is a graph of some f x, y, here z is equal to f x, y, so let me write down the proof; let g x, y, z is z minus f x, y therefore, g x, y, z equal to 0 represents the surface. Now, once you know this that this represent the surface, you know that the gradient of g is normal to that surface, right that the normal vector can be written as; because here if you take the gradient what is this; minus del f del x, minus del f del y and del z del x is 1.

So, this is the normal vector itself can be viewed as a gradient vector because this is nothing the gradient lies, if this is now my g x, y, z equal to 0 and my gradient is of course along the normal at the point x, y, z. If this is my point x, y, z and the gradient computed of g will be at will be exactly normal along the normal to the surface, so I can take this gradient vector itself; grad g x, y, z as my normal vector.

So, cos phi is nothing but n dot k, along if you look at the picture, by norm n norm k, norm k is 1, so norm n vector, so what is n dot k? See, n dot k is 1 because in k, this the components related to i and j are 0, so this is 1 divided by root over norm n is nothing but 1 + del f del x whole square + del f del y whole square and what is this; this is nothing but ds by dx dy. So, it implies that this is nothing but dx dy by ds.

And so, you simply have ds is equal to but I still like this intuitive reasoning, you can now first in simple situation write down the ds, if you know the angle phi, you can write down ds in this simple way and actually integrate it much more easily, so you do not; if you know the phi, if the phi is constant, it is very good, then you can very easily compute out the area.

Now, you just have to know not bother, you just have to bother out the limits of x and y and then you can compute the area. We will stop here, we have already taken a lot of time and we will go to the next one, after the next class we will talk about the parameterized surfaces and how are we going to work in this scenario, how can we talk about parameterized surfaces, how are; what would be our approach, what would be our area and that is the kind of thing that we are going to talk about, okay.

So, let us we will talk about the same thing, we will do the same operations, we will repeat the same operations but in our parameterized setting, okay and then we will find out what is the surface area and give a small example, thank you.