Calculus of Several Real Variables
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Lecture – 03
Application to Real World Problems

So, friends, Welcome to the third lecture on calculus have more than one variables, you see the heading today,

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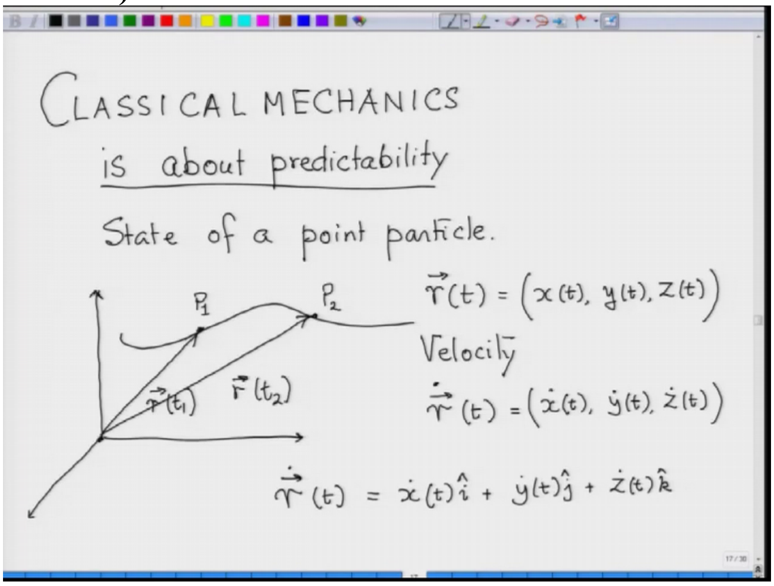

Application to real world problems. Also here I am written on the screen application to real world problems, you must be thinking that I must be talking about something very important very realistic, take some real problem and try to solve it with what they were little we have learned.

Now, the idea is known and hence, this name could as well as we thought of a misnomer. It is to tell you that real world problems are very tough. It means a lot of people come to come together. Then giving and taking various aspects from various fields, one has to really handle a real world problem. But there are certain tools which are required to solve real world problems.

Some tools of natural science, like physics and mathematics is the way where these rules of physics or natural sciences described. So here we are going to be concentrating ourselves on the use of Vectors in physics. And, you can see this can be done with whatever knowledge we have already got about vectors. About say we have learned over Tina products. See the idea is to show you with that this little knowledge.

That we have gathered over the last two classes can be sufficient to understand some bit of what I call classical mechanics or Newtonian mechanics. Newton was, of course, the most famous genius you know, other than Albert Einstein. Newton was born in 1727 in England and went on to become one of the greatest scientists who ever lived.

In 1687, he published a book called Principia. We just called Principia or Newton Principia. There he actually lays out laid out in mathematical fashion. The laws of motion or he studies the properties of moving bodies. So, mechanics is about the study mechanics is about the study of moving bodies, mechanics, studies, bodies in motion. So, this is what you have also learned a bit in your Higher Secondary classes or maybe even learning a first year classes in physics.

So, that is why bringing relating things to mechanics will make much more life much easier. In mechanics, we are often talking about point particles. I would again say that Newton mechanics talks

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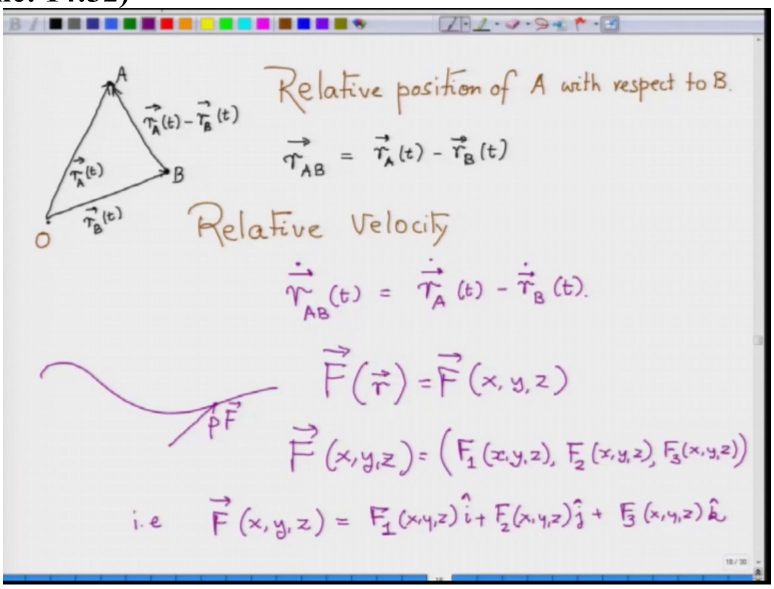

About motion of bodies around us which we can see it, it does not talk about things that do not make level things that do not make level does not conform to Newton’s mechanics. So and you will see the introduction of Mr. Bond particles in most books or mechanics and I want to really tell all of the students who are watching this, you should go and read the preface of the Principia.

And then you can understand the clarity of Newton's mind where he essentially tells you that you see in geometry he was interested in geometry. Euclid geometry was he said that we learn to draw circles we learn to draw aliens to learn to draw several ellipse parabola hyper bola for how does nature draw them. So, if, for example the sun is having a gravitational attraction on the art, then how is the art supposed to move around the sun, it is supposed to move around in an ellipse.

So, nature has a mechanism to draw those scores. So, in that sense, he puts geometry even as a part of mechanics. But he also says that the mechanics that he talks about is not really the mechanics of you know, doing nuts and bolts, but mechanics which is a mathematical modelling of nature. So, here is this important term that comes in is the mathematical modelling of nature.

Mathematical modelling of nature those would be physicists those who would be applied mathematicians, those who would be engineers, they should get habituated with this term called mathematical modelling of nature. Mathematical modelling, that is modelling real world problems. In the in a mathematical format is not an exact description of what is happening in nature. Because exact description is very complex, too many complex things happen for us to really fathom our brains to really fathom.

So what we do used to we take very similar situations in which we can use mathematics to describe them and describe them through some equations, solving equations and then try to interpret that phenomena. So, what we do is not exact. So, a large amount of exact science depends on non exact things are on approximation. So, this is the approximation is a key issue in science.

So, you have to carry on your scientific carrier or engineering carrier by understanding the role of approximations. Approximations play a key role in science will never ever forget this. So, what is the point particle So, you can see okay a ball rolling past me is it upon particle I can see the object but if you are for example, looking at the planetary system, then compared to the sun for example, compared to the sun, a planet moving around the Sun Over to the mass of the Sun, planet can be thought of as a particle.

So, here is a sun and here is the planet. In fact, if you are looking at an apple falling from a tree on that is what Newton had been famous for. So, here an apple is falling. So, compared to the mass of the art the apple can be viewed as a point particle that is all its mass Oh, you point by geometry means it has no dimension.

It is just a kind of very idealistic thing. It has essentially a thing which I cannot think materially exists. I agree. But the whole point is that we are thinking there is no harm to think and that is what science is all about to imagine that this apple that you thinking about can be viewed as something, which is so small compared to that is so tiny that it looks like a point. And the whole math is concentrated here. So you many of you might have heard about the term centre of mass.

When you study mechanics, the centre of mass actually gives justification to this idea that if you have a system of particles or even a rigid body, you can define in terms of other particles, a point is where all the mass of the body would seem to be concentrated. And that point particle can be taught as a representation of the body itself. And that is why it does not harm to talk about point particles.

So what should I know about a particle? Classical mechanics is about predictability. Classical mechanics is another name as Newtonian mechanics. So, I just get you habituated with the names is about predictability. It means that if I know the current state of a body, so what do you mean by the current state current state of a body or state of a point particle will tell you, what is it if I know the current state of a particle.

I can tell how the particle will evolve in the future and how or how it has evolved before. So it is a completely predictable system where no uncertainty is involved. So, Newtonian mechanics is a system where there is no uncertainty as far as so, this is something which is very important to understand. So, what is the state of a particle any particle that is moving, same space.

Changes it is position with respect to time. So, once you are talking about real world problems here, that we tell you Newtonian mechanics still continues to be applied in many cases including space exploration, so let me tell you that this point particle P. You see the position of this point particular moves along the curve in three space it changes position in time. So at time t 1 it is here which I call P 1 on a time t 2 to it is here so here it is position is given by a radius vector rt 1 one.

And here, it is position is given by the radius vector rt 2 obviously here rt 1 so in general the position of a particle is given by the 3 coordinates which also depend on time as time because the body is moving at every as time evolves, the body position keeps on changing because it is moving just like a moving train right and then you need to know what happens is that we are not just in what is the meaning of a state. State means, where it is at a given point of time.

And how fast the body is moving. So, both the information important to know about of future and also know something about the past of the body of the particle. So, we also have to know the velocity of the particle. So, what is velocity? Velocity is nothing but the time derivative of the vector rt. So, which we can now write as r 0 t, which is the vector x 0 t y 0 t z 0 t. Of course, you can write it is no harm to write this r 0 t as rt whatever as x 0 ti vector plus y 0 tj vector.

Plus z 0 tk vector know a state of a particle then is really governed by this 6 elements. So, state of a particle you see how we are using our symbol vector ideas to define things. So, we are talking about physics of mechanics and we are talking about the state of a particle is actually defined by a 6 dimensional vector. So, Vector is not just 3 dimension or 2 dimensional as we have been studying.

You can be more than that is we will also talk on innovation vectors it is very large. So here we need a 6 dimensional vector to define what is called a state. So, suppose you have 2 vehicles A and B they are moving right and with respect to some origin, the position vector of the vehicle A is described a r A t and the position vector of the vehicle B is described by r B t. And this Victor, we joins B with A which you know from very basic vector geometry is sorry r A t minus r B t.

Then the vector which we call r AB is a relative position of A with respect to B is given by the vector r A t minus r B t. So, this is the relative position of A with respect to B. I know for many of you must have worried about the concept of relative velocity. Relative Velocity of one body with respect to the other, see if this is a body moving in this direction or a ship moving in this direction and there is another ship moving in this direction. So, what is the relative velocity of A with respect to B.? So, what does it mean?

I am now on the ship B and if I think that my ship B that rest, why how do I what velocity and in which direction I see the ship A moving. I know my ship B is moving to. I want to make it as if that the ship B is rest thinking that the ship B at rest I want to see in the water in the sea or river whatever in which direction and what velocity my ship A that you see actually moves.

So do really think that the this one this ship B at a in a stationary position it has halted and you really have to so if this was moving with a velocity VB and this was moving with the velocity VA to stop the ship B you have to apply an opposite velocity so that the total velocity of the ship B is 0 and then what do you really have to do is now once you do this it is 0. Now with respect to the ship. How will this B how will this look like? So, basically you add this velocity with this velocity.

So basically I am doing okay if I apply minus VB velocity to the ship B my ship B has stopped. Now, my ship A would actually move look like look to move in this direction which is VA minus VB this all vectors of course and you see this is exactly the vector VA minus VB that is exactly. What is the meaning of relative velocity? So, it is nothing but the time derivative of the relative position.

So, basically what it says here, what is the meaning is a very important thing, what is the meaning of related position if the origin was O then the related positions of A and B the positions of related positions of A with respect to O is rA with respect to manipulate the position of B with respect to O is rB but then B becomes origin then what is the position of A with respect to B and that position is rAB.

So, the whole idea is making B the origin. So, when you are talking about this relative velocity, the whole idea is about making V the origin. So, if I make V the origin this is my V. V has now halted and now V is the origin. How do I look at my ship A in which direction the ship A is if V is the origin, then the direction in which my ship is this and hence, it must be moving along this direction which is exactly VA minus VB.

So, what is relative velocity so, again really velocity is a time derivative of the relative position that is it as simple as that. So, what happens what is the velocity of the ship A when my origin is not taught to B and not O that is the whole idea. So is a time derivative? Actually t I should write t here but you can fill that little error up with some little fun to say that the instructor is goofing up.

So now. So this you see, whatever little we know we have really made an application now comes the idea of force. Fore is fundamental to Newton's mechanics. And everybody of you know, without, I am sure everybody sitting in this audience would know, this famous law, which I am writing in the board in a very big way. F, the force is mass times the acceleration.

This is Newton's second law, or key to the universe I guess, because of the popularity of this popular science literature, this might be only second to equal to MC square if you really take up the popularity of equations that is known to the general public, but this is the key to the Newtonian Heaven, right? And what is this force? How do I write it? How do I mathematically express it? Let me tell you 4 suppose here is a particle moving.

Here is this P a force is acting on it right. Now, this force that is acting on it is also a vector and will change in general with the change of the position. Now, the force is itself a vector because it acts in all the 3 directions right it will have force on the body is not only acting in one direction, but even resolve it into 3 different directions. So, the force actually has 3 components. So, I can write either force of xyz obviously, these xyz are also functions of t.

At the end everything becomes a function of time, but we can we did not put time explicitly hoping that you understand whatever happens in mechanics or in Dynamics because things are moving has to depend on time. So, this F of xyz has 3 components. One component in the x direction to these are scalar quantities. Another in the y direction, another in the z direction. That is my force can also be written as F 1 i vector plus F 2 j vector coordinates of the force.

So, the force vector is acting on, on this particular F is acting, right, what are the coordinates of this force vector so that is about it okay. Now, how do you express this thing in Newton's law?

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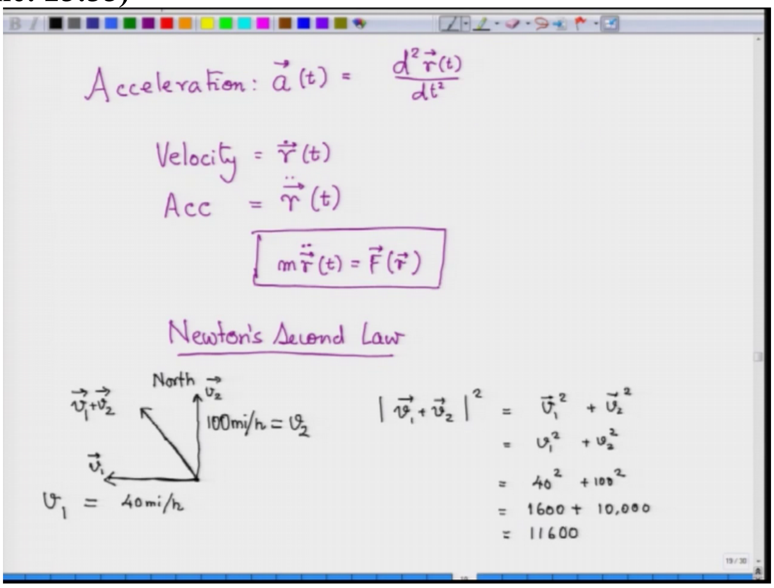

So, acceleration can also be expressed this a vector which is actually a function of time can we also express as a time derivative of the second derivative second time derivative of the rt vector, there is a position vector, because the velocity is nothing but r 0 t and so. Acceleration is nothing but again taking the time derivative of the velocity.

So, it is so, from your vectorial way, Newton's law can be put in a very simple format that is m and so, this is not of velocity see when we largest study particles moving in a straight line, but waning a particles moving in three space, this is the way to very compactly you write down your Newton's second law please understand that.

This is actually 3 questions combined into 1 that is the beauty of a vector expression that it compresses a lot of equations into one right. So, this is the Newton's second law the key to the universe actually Okay. Now, for example, let us take the book, which I have been working on or let me just take a problem from the book and try to see what happens let us try to do some exercises here.

So, suppose there is a, I am doing a problem right this time. Let me just concentrate on a problem. Suppose there is a wind velocity of 40 miles per hour and this is a British way of spinning miles per hour and all our you do not hear or mean over you want to say 400 kilometres for whatever So, there is a wind velocity. So wind is moving from east to west at 40 miles per hour.

So, this is what we call V1 and of flight a plane is moving in the not only direction so this is the north at hundred miles per hour. We raise slow actually for a plane with a speed of train nowadays, most in most countries. So, this is V2 so, in which direction the plane from the ground, I see the plane actually starts moving northward. But from the ground in which direction do I see the plane moving.

So, here what happens is here is the plane moving straight upwards here is the velocity which tends to push the plane towards the west. So, basically the plane would now move at this to meet will appear in the ground as is the plane is moving in a slightly westward direction, northwest toward the direction which is given by the vector V1 V2 okay V1 plus V2 so, this is an application of the ideas that we are saying so, you can easily figure out what is this vector V1 plus V2.

So V1 plus V2 the square of this by law of Pythagoras is V1 square plus. So let me do not V2 that V1 and V2 let me the magnitudes of the vector V1 and V2. So V1 is a velocity here to the right direction east to west and here is towards north so from the airport it is whenever I am observing that is going up in the sky in the northeast going up in the north, actually, but from the ground I am observing as if the wind is pushing the plane towards the west.

So what Pythagoras theorem I will know what is the magnitude of the velocity so magnitude of the velocity sometimes I can write just V1 square, this is just a real number. Pythagoras theorem just applied here. I do not want to go into teaching you what is Pythagoras theorem? So you can just applied so I immediately you know what is this 40 square plus 100 square. So, that is what you have 40 into 40 is 1600 plus 100 square 10000 10 to the power of 4.

So, I am so bad at doing this computations, so 11600 so bad. So now V1 plus V2 the modulus the actually the velocity with which the plane is going to move. It route is I am observing the plane as if it is moving now at a velocity of root over this miles per hour. So, here is a demonstration of what we have just learned, which can be applied. Another demonstration comes, work done. Work done by a force.

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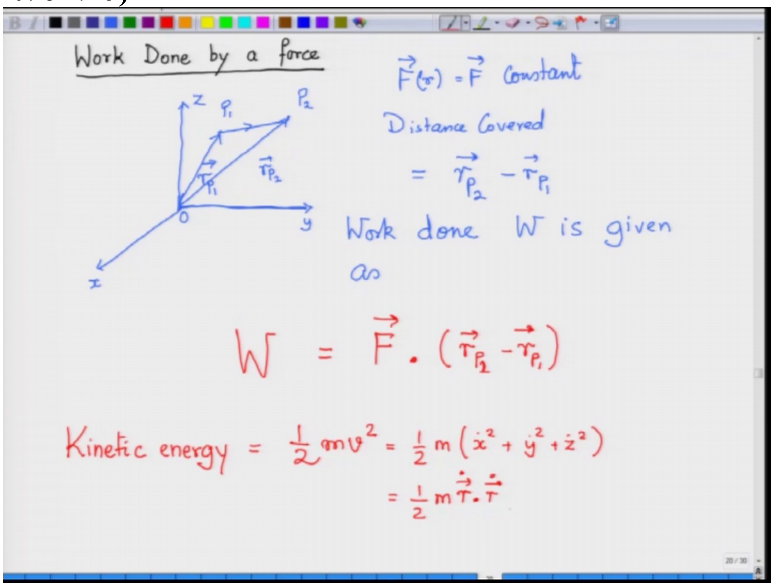

And you will see how you know product comes into play toward his work done by force, I will just remove it here and just try to explain it to you here, what is work done by a force and then we get into the vectorial form of it. So there is a block of wood here and there this is a very smooth table. I am ignoring the fictional force that there will be. But I am giving a force F and pushing it.

And I am starting safe from here where the centre of mass is say the point A, and I am moving this. Finally the body to point B that the central mass comes to the point B. I am not getting into the discussion of centre of mass. If you want to look in just google it Wikipedia would have it surely. So this distance that I have moved this body, basically the point particular centre of mass, if I call it S, and the work done is force into the distance.

So magnitude of the force is actually a scalar so magnitude of the force into the distance. So I am not writing F. F simply means the magnitude that is what is work done. But suppose so into the distance S Sorry, I have said S, so it is S. Now, suppose I be a big bolder rock, and I am trying to push it with all my strength then it does not move whatever force I am applying is does not move then my work done is 0.

For a work to be actually done by a force, the body needs to move a certain distance but what happens when I am in this setting of. So let me draw the axis first the best thing in this is to draw the axis first. I am not so habituated handling this electronic pen so you forgive me for this. You can definitely draw better than me. Suppose the body or body was moving and it has gone from P1 under the action of a force F to P2 let me assume that the force is constant.

So at for every positions are at every time t the same forces been applied on the body. So, Fr is a constant vector is constant. So now suppose this is rP1 is my position vector of P1 and rP2 is my position vector of P2. Now, you might ask me that what happens if I go through occur. Then, how do I really handle the situation? How do I really calculate the distance? Okay, here is a clocks which we cannot come in now, because this will lead us with something on the line integral?

We will discuss it later on and then work will be discussed in a much detailed way. Let us assume that this has moved in a straight line between P1 and P2 then the distance covered between P1 and P2 so, the distance covered under the force distance covered is r P2 minus r P1. So, now this force F acting on it constantly. So, how do I define the work done? So, work is nothing but multiplying we define work as multiplying the force with the distance.

But here you are in 3 dimensions you know, and you know when we are in more than one dimension to dimension. What is the general realization of multiplication? What is the way we generalize multiplication of two numbers into multiplication of two vectors? The way we know it till now is called the dot product or the inner product. So work done is W is given as W is equal to F dot product r P2 minus r P1 what so you see the things that we have learned.

The little things that we have learned in the two classes, have actual applications have actual meaning can be really used to talk about something meaningful, talk about something practical talk about something that we want to see and we want to discuss So, here is a gamut of things that we have done. I guess that with all these you have some idea another thing that I want to tell you for example, many of you know what is kinetic energy?

Kinetic energy is, you know, human evolution three dimensional bodies of mass m is mass time square of the velocity. Now, when the body is actually, in 3 dimensions square of velocity means square of the magnitude of the velocity, so, half of m into x 0 squared this all the functions of time by the way, y 0 square plus z 0 square. So this is equal to half of m times. If you look at the notations, this is our dot in our dot product with r 0.

You see the dot product is again used to describe kinetic energy. So when we learn more about derivatives and gradients, extra, we will see how this formulation would help you to very simply prove the fact that the total energy which is a joint which is the summit of the potential and kinetic energy is independent of the time that is the time derivative of energy is

0, that energy remains constant in time and hence, that is what we call the conservation law energy.

So all these simple this sort of vectorial writings will help us do things much, much more simply and give a much more clarity to whatever we have already known about in physics, in mechanics. So mechanics died, applications from mechanics would come enough, maybe you can save a personal life for it yeah I do I agree that I do, but this is something very important because mechanics is a very fertile field where vectors can be applied.

Of course, it can be applied in a more abstract sense in quantum mechanics the science of understanding the atomic the thing subatomic world, but that is what we are not going to discuss here we are talking about very concrete 3 dimensional vectors most of the cases. So, with this I close the talk for this time, this class. The next class we are going to talk about simple facts determinants and matrices and determinism matrices will become important as we move on.

Because determinant is a kind of tool, which talk tells a lot about what a matrix behold matrix behaves. And matrix is a kind of operator, which is all over the place. And for example, when you talk about derivative of a vector function we are talking about matrices. I want to bring your notice to this I want to bring a notice to this force vector that I have written. So, what is a force that in fact it is proven of velocity vectors?

So, force vector what does it means? So he did xyz and there were 3 components F1, F2 and F3. So, here is a function of several variables, but this function of several variables F is not carrying this element in r3 to an element in r it is carrying an element on r3 to an element of r3 such vectors such functions are called vector valued functions of vectors or other vector world functions of many variables.

When you are going to talk about derivatives of such functions, derivative of such functions will be matrices the things which will learn in the next class. So, thank you very much. Thank you for your attention. Hope you enjoyed this class. You can obviously put your queries and we will try to answer them. Thank you.