Calculus of Several Real Variables

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Lecture - 29
Parametrizing Surfaces

So as we have said in the last class that here we are going to talk about whether surfaces can be represented by parametrization.

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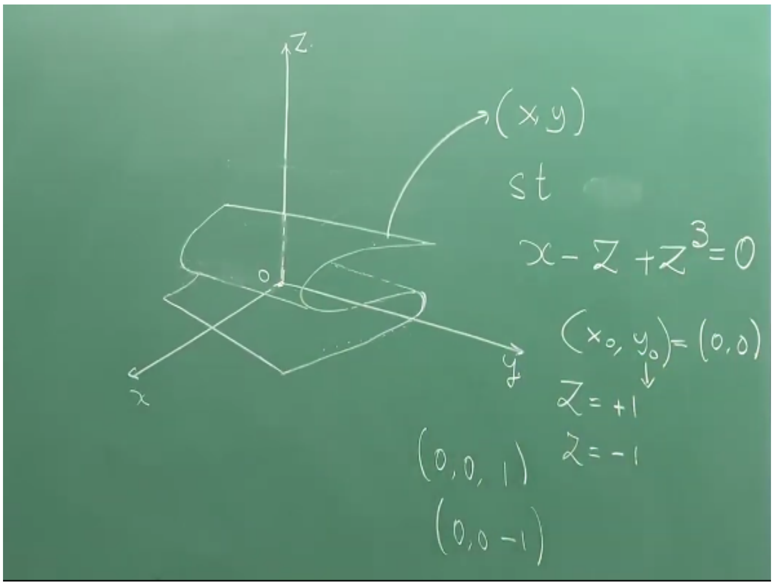

You are seeing something very strange drawn there. So we are going to talk about parametrizing surfaces and their associated concepts.

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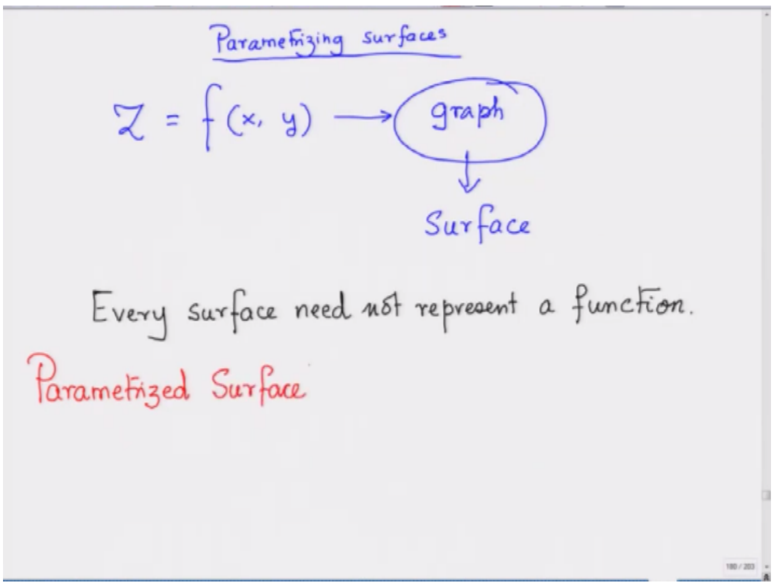

When we talk about a surface, we are essentially talking about a two dimensional object embedded in a three dimensional space; two dimensional object embedded in a three dimensional space. That is what I mean by a surface. Now please understand that if you have the graph of a function $Z = f(x, y)$. If you write z of $f(x, y)$ and draw the graph of it, then this graph is a surface. But every surface is not a graph, because every surface need not represent a function.

For example, this surface that you see, this kind of folded paper. This is like a folded sheet of paper, like a kind of a lasagne. You know some covering all some food, right? It is or a folded sheet of paper like this. You have often seen you know it comes out from a very big earlier you have seen old photos of you know computer printers giving out sheets of papers which fall down like this.

So this function for example, is the set of points x, y. This function here is a set of points x, y. So this graph represents, this surface contains points x, y. So this has points x, y such that x x minus so it satisfies $x - z + z$ cube equal to zero. Is this a function? The answer is no, because take put the point x 0 and equal to 0 and y 0 equal to 0. So take the point x 0 and y 0 and put them to be 0, 0.

Corresponding to this you see z can take the value plus corresponding to this points z can if I put z equal to +1 this equation is satisfied. If I put z equal to -1 this equation is satisfied. So 0, 0, 1 and 0, 0, -1 are both points of the surface. But corresponding to a if it has to represent a function corresponding to this 0, 0 point has to be only one z. There cannot be more than one z.

So every surface though it can have an algebraic representation need not represent a function. So this is a very important thing that one has to remember that every surface need not represent the function, even they have an algebraic representation. That is true about curves too. One of the key points that one needs to understand that there is a desire in us to actually work with functions, because that is what we understand best.

If we need to work with any analytical tool of representation, functions has to play a role. So in that way, how do I now represent any surface which need not be the graph of any function in terms of some real valued functions. That is exactly what means exactly what it talks about. So if there is a domain and there is a surface above it, take a bond on the domain and get a function, which will take it, take you to some point in the domain.

So get a function which will pick up every point and take it to a particular fixed point in the domain. Not fixed point, a single point in the domain. That is not in terms of the x, y, z's involved, but in terms of the some this x, y, z points. They can be represented in terms of other variables. And in terms of that other variable I can represent the surface of the function.

Means I can have a function which will give me back which will return me back for every values of the parameter one coordinate of the surface. So I will put a say u v in a function phi output would be some x (u, v), y (u, v), z (u, v). So this particular point. If I change the u, v the point will change. The same two different u, v's will not give me the same point. So that is what is called parametrization.

So now we will define what is called a parametrized surface. So our aim is to define the notion of a parametrized surface. So for this we will go to the next page.

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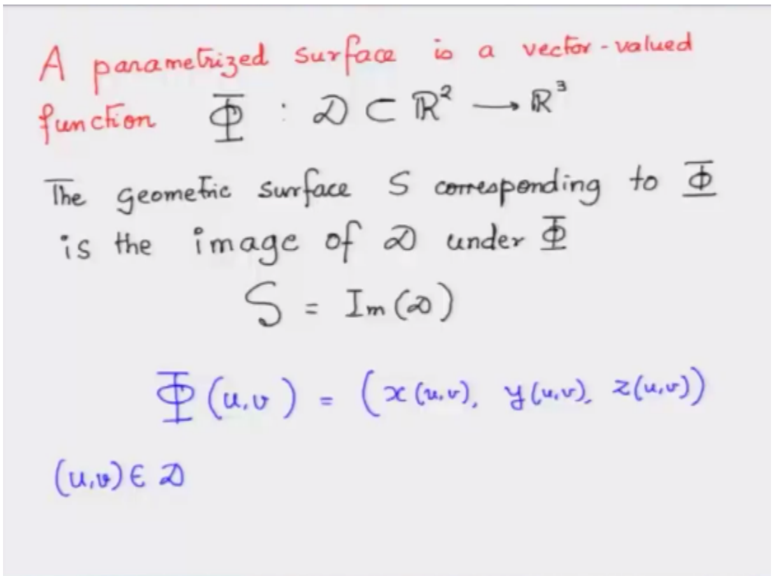

A parametrized surface is a vector valued function, right? But from R 2 to R 3 phase is a vector valued function on a domain D naught to phi which takes points in a domain D subset of R 2 to R 3. The geometric surface S corresponding to phi is the image of D under phi, corresponding to phi is the image. So basically S is image of D and phi (u, v) where u, v is some points on the domain D. So u, v belongs to D.

If you take a point u, v on the domain D, it will give me a point on the surface and if you change the u, v it will give me a different point; x coordinate, y coordinate corresponding to u, v, z coordinate corresponding to u, v. Let us first see if I have a function z = f (x, y) and this graph is a surface which we know, can we parametrize it in this way. So first step comes in this particular way.

So what we are trying to do is to try to use the idea of a single valued function to talk about surfaces.

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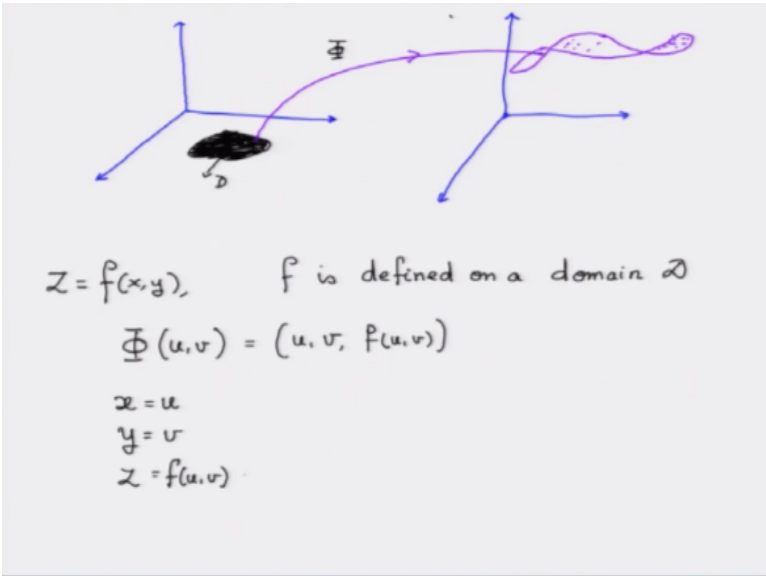

So basically what is the thing that we are trying to talk about here is that okay, here in R 2 you have a domain, right. You have a domain, I should color it up. You have this domain D and this a function phi is operating on it to give a surface that so it is a kind of twisting of D itself. So now that would give a kind of surface. This kind of maybe a very twisted kind of thing.

So this phi that you see, takes a point here and drops it here. That is what phi is doing. So this is what the idea of parametrization is. So let us first look at a function z = f (x, y) and we see whether we can parametrize the graph. So f is defined on the domain, defined on a domain D, okay? Then how do I write a parametrization? How do my phi is defined, So phi (u, v) is defined as u, v and f (u, v).

That is, this is nothing but x, y, z. So you define x = u, which is x (u, v) and y = v that is y (u, v) and z (u, v) is equal to f (u, v). So z = f (u, v). So x, y, z. So this is the parametrization. So a graph can be parametrized. So if you have function, if you define a surface as a graph of a function, then it can be very easily parametrized. Our next aim would be to talk about tangent planes on the surfaces.

These are very important ideas, because they would play a very important role when we are trying to compute surface integrals because we have spoken about line integrals a curve in 3d space, integrating along that as though something moves. But if something moves on the surface, how do I integrate? So once you have defined that phi (u, v).

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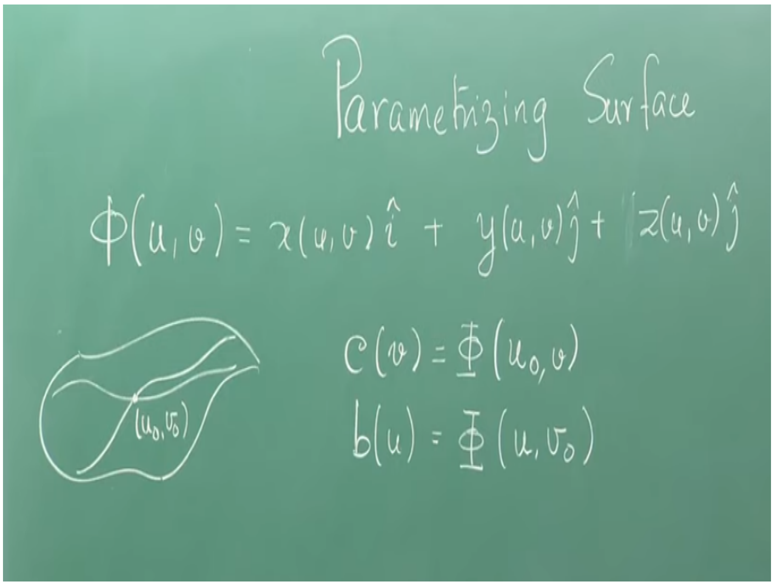

So this phi (u, v) function can also be written in a vector form as x (u, v). So these are very useful in differential geometry, what we are doing is these are the basics of differential geometry, z (u, v) j vector, okay? Now observe that if I have drawn a surface, so I have I have a surface here. So here is a surface. This is a surface.

Now if I fix my u say at u naught, if I fix my u naught, then c (v) phi u naught v represents a curve which moves through all the points on the surface whose points are of the form u naught v. So c u, c v represents a curve. Now suppose similarly I can do the opposite thing also. I can fix my u, fix my v at v naught and that would be some curve which is passing through all these points.

So both these curves will pass through a point, the point u naught v naught. So at any point v along this curve c (v), what is the tangent? How do I represent the tangent vector? So any point, at any point v, u naught v along this curve c (v), how do I represent the tangent vector. The tangent vector is nothing but the partial derivative with respect to v of this parametric function.

Because ultimately it is nothing but a restriction of the parametric function itself in one variable.

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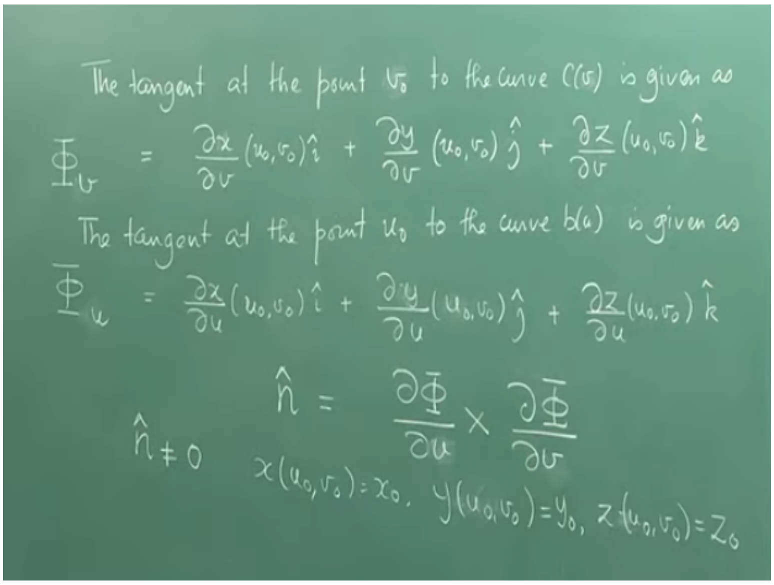

So the tangent at the point u to c the curve c (u) is given as phi of the derivative with respect to u, sorry the derivative with respect to v, sorry this is c is with respect to v. I make a mistake, v. So it is phi of v that has to be evaluated at u naught v naught, okay? That has to be evaluated at u naught v naught. Because suppose we are just looking at what happens at u naught v naught.

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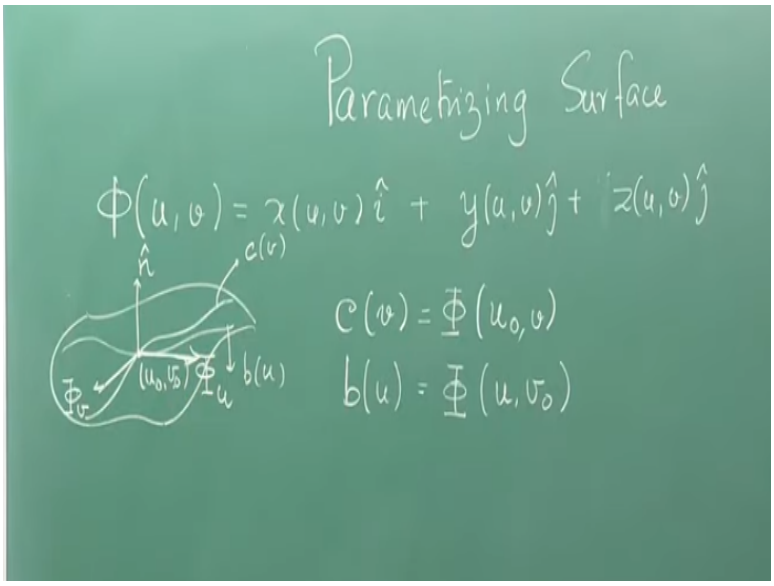

So then basically we are looking at this tangent to c. Then we can look at a tangent to the, this is the curve c, c (v) and this is the curve b (u). So I can look at the tangent to b (u). Then the cross product of these two will determine a vector whose which will determine the tangent plane itself. So we will see, this will determine phi of v and this will be phi of u. So phi of v is what?

Is del u sorry del x del v at u naught v naught. So I am trying to, you can write it, this could be at any v, right? So suppose I am just trying to it at u naught v naught, right? Because I am trying to find it at that point. So I parametrize it in both ways. I am trying to find it at that point, the tangent at that point. So that is our tangent at a particular point at on the surface is found out tangent plane; u naught v naught k vector.

Similarly, tangent at the point u naught v naught, let me write v naught. So tangent at the point v naught is given like this, right? So tangent at the point u naught, so the tangent at the point u naught. So along the curve c at the point is all curves of the form u naught v. At v naught I am trying to find the tangent, so that is it.

So at the point u naught to the curve b, to the curve b (u) is given as phi of v is del x sorry phi of u, this was phi of v, del x del u u naught v naught i vector plus del x sorry del y del u x sorry u naught v naught at evaluated at u naught v naught j vector plus del z del u, u naught v naught at k vector, okay? So this is what it is. Now consider the curve, now consider n hat which is actually del phi del v cross del phi del u.

This I call the n hat vector. This is the n hat vector, the normal vector which is perpendicular to the plane containing phi u and phi v and that plane is the tangent plane, okay. So assume that n is not equal to zero. Then the tangent plane is a collection of all vectors which pass through the point x naught y naught z naught.

If I write, suppose I write x equal to x (u naught, v naught) as x naught y of sorry y (u naught, v naught) as y naught and z (u naught, v naught) as z naught teek hai. If I write this, then my tangent plane is the plane which is consisting of these two vectors and is a plane which has the n vector perpendicular at u naught v naught, right?

The plane where the n vector is perpendicular at x naught, n vector passes through the point x naught, y naught, z naught and is perpendicular to the plane which contains this vector. So the tangent plane then has to be such that if you take any point x the on the tangent plane any point x from x naught y from y naught z from z naught. If you take those three vectors in one vector and take a dot product with basically now it is like this.

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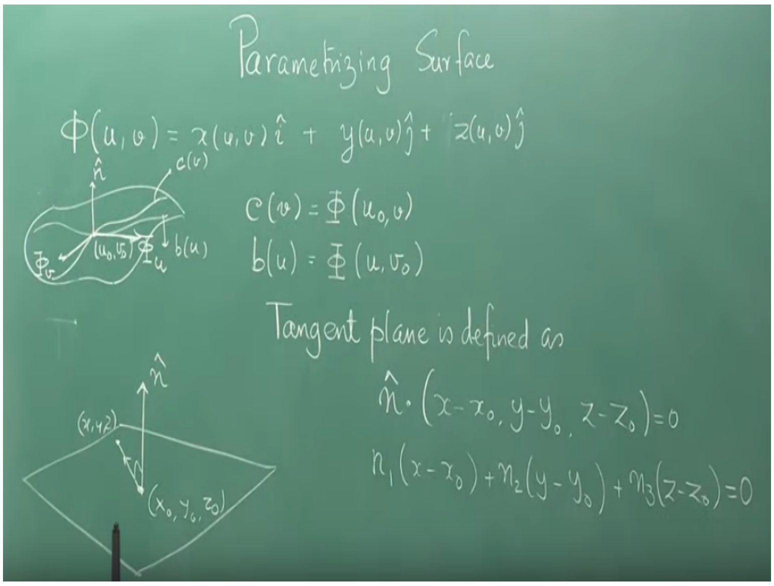

So here is n naught n hat and here is x naught, y naught, z naught. Now if you take any point x, y, z here and this vector should be also perpendicular to this vector to n vector because it is lying on the plane. That is how the plane would be defined. So that is how the plane would be defined and thus the tangent plane, a tangent plane is defined as n hat dot the vector x minus x naught y minus y naught z minus z naught equal to zero.

If the n 1 and n 2 and n 3 are the three components of the normal vector then n 1 into x minus x naught plus n 2 into y minus y naught plus n 3 into z minus z naught, that gives me the tangent plane. So let us try out. Let us try this out for how to compute at least for a function whose surface is given like z = f (x, y).

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$$z = f(x, y)$$

$$\Phi(u, v) = (u, v, f(u,v))$$

$$\Phi(u,v) = u\hat{i} + v\hat{j} + f(u,v)\hat{k}$$

$$\Phi_u = 1\hat{i} + 0\hat{j} + \frac{\partial f}{\partial u}(u_0,v_0)\hat{k}$$

$$\Phi_v = 0\hat{i} + 1\hat{j} + \frac{\partial f}{\partial v}(u_0,v_0)\hat{k}$$

$$\hat{n} = \Phi_u \times \Phi_v = -\frac{\partial f}{\partial u}(u_0,v_0)\hat{i} - \frac{\partial f}{\partial v}(u_0,v_0)\hat{j} + \hat{k}$$

$$(x-x_0,\ y-y_0,\ z-z_0)\cdot\left(-\frac{\partial f}{\partial u}(u_0,v_0),\ -\frac{\partial f}{\partial v}(u_0,v_0),\ 1\right) = 0$$

$$z = z_0 + \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x-x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y-y_0)$$

Suppose z = f (x, y) is given this function, so the graph is a surface and I know the parametrization. So your phi (u, v) the parametrization is (u, v, f (u, v)) where u and v are x and y. So basically what I have done, I have written phi (u, v) as u i vector v j vector and f (u, v) k vector. So now I first take with respect to v or u whatever you want to do, does not matter; v cross, u cross v, here v cross u whatever.

So here the definition they have given is del u cross del v. You can, it does not matter. You can take it in this way or you can take it in that way, it does not matter because ultimately if you just change it there will be a change of sign, n will go in this direction but the tangent plane should remain the same. We will try to go by the definition and draw it but it is u cross v okay. We will go by the definition.

I think this is the c u curve. Of course we have to check. I have done it little hurriedly. We will go by the book and put to this definition, u and v. See what happens is when you actually do an x, y, z curve, if you have an x, y, z thing here thing here then you will know what is happening. You will know in which direction these two are pointing. So whichever way, you can do it in the other way also.

Del phi u cross del, you can do it in the other way also, it does not matter. I am just going by the book so that if you read by the book read the book because I have defined it in that way. And if you read the book, you do not feel any problem because it depends on the actual orientation of the curves. Okay, so let us take here.

So here phi of u is i vector plus 0 1 into i vector, 0 into j vector and del f del u evaluated at u naught v naught say k vector and phi of v is 0 into i vector plus 1 into j vector plus del f del v into u naught v naught k vector. Now if I go by n naught n hat, n hat becomes phi u cross phi v. See if it is opposite they would have change in the sign minus sign. It would not have made a huge difference. It would look the same.

So anyway so that you can check out. So minus del f, so if you change it and do the thing, you see what the difference you have. So del f del u, u naught v naught i vector minus del f del v sorry del f del v evaluated at u naught v naught j vector plus k vector. So at that x naught y naught z naught point which is actually x naught y naught.

So my tangent plane should be given by dot product with minus del f del u, u naught v naught minus del f del v u naught v naught and one and that should give me zero, they should be orthogonal. And that would finally give me an expression like this. z is equal to z 0 plus z 0 is a function value f (x 0, y 0).

And because u is x and v is y, I will write is del f del x at x naught y naught because u naught is x naught v naught is y naught, this whole thing multiplied with x minus x naught plus del f del y because v is y. It does not matter if you have changed this, u cross v you can change it and see you will get the same result; u naught sorry x naught y naught because they are the same. I put v as y here and that into y minus y naught.

So if you look at this, what is this? This is nothing but the except the error part this is the first part of the first order Taylor expansion for function of two variables. This part is left out, error is left out and that is exactly the tangent value that is the curved surface can be approximated very near the point x naught y naught z naught by a plane. That is the meaning.

Of course, I do have some time to give another example which is given in the book which is a very good example and that is a example of a cone. A cone in this form.

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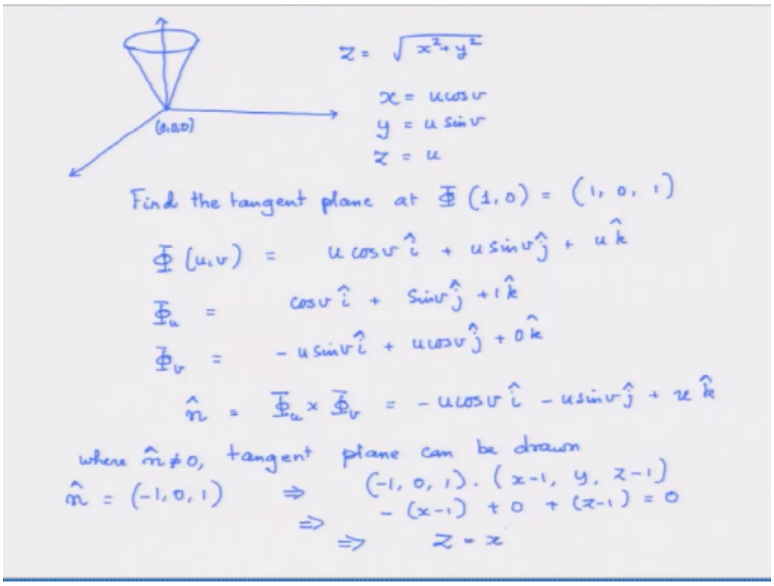

A cone you have seen, this ice cream cone or a funnel, in a oil shop or a funnel in a chemistry lab. So it is like this. This is a cone with vertex at 0, 0, 0. So at 0, 0, 0 we do not talk about any tangent, right? Because there can be many tangent planes. So at any other point other than 0, 0, 0 we can talk about tangents. So the surface here is given by z equal to root x square plus y square.

So how do you give? Basically what is happening? Your x square plus y square can be parametrized like a circle keeping z as a value of the radius, right. So if I put x is equal to u cos v and y equal to u sin v and z equal to u then x square plus y square is equal to u square which is z square. So I have to now find, suppose I find the tangent plane, find the tangent plane at phi (1, 0). What is phi (1, 0)?

Phi (1, 0) corresponds to u is 1 and v is 0. The cos 0 is 1. So it will corresponds to 1, 0, 1. So at this point it is asking you to find a tangent plane. So how do how have I represented phi (u, v) here? Phi (u, v) is represented as u cos v i vector plus u sin v j vector plus u k vector. In this case phi of u del phi del u, so phi of u this vector is now given as cos v i vector sin v j vector plus k vector.

Phi of v is given as minus u sin v i vector plus u cos v j vector, I am differentiating with respect to v and there is no v so 0 k vector. That is 1 into k vector. So now you take n. See at 0, 0 you cannot have, I can have more than one tangent planes, you can visualize that very simply. I can have more than planes passing through 0, 0, 0 without touching the body of the cone. So this visualization is important.

So n hat is phi u cross phi v. So if I define it like this, this turns out to be if I do the cross product and this turns out to be minus u cos v i vector minus u sin v j vector plus u k vector and this vector is 0 and u and v are non zero so if u is equal to 0 then everything is 0, 0, 0. Then there is no tangent plane. And it does not make sense. Because n hat is zero.

So and n hat is non zero right so we have to consider tangent planes only in those cases where n hat is non zero. Tangent planes can be drawn means at any other point. So basically we are looking at 1, 0 point. So n hat at 1, 0 is what, what is that vector? N hat at 1, 0 is, if you put u as 1 and this as 0, it is -1 this will be 0 and this will be 1. So it is, n hat becomes -1, 0, 1. And then if I want to find it at the point 1, 0, 1.

So I will have a dot product -1, 0, 1 dot product with (x -1, y, z -1). So what will be ultimately? It will come out to be $-(x-1) + 0 + (z-1) = 0$. So this will finally give me tangent plane is $z = x$. That is a tangent plane. So with this calculation, with this description of constructing tangent planes to surfaces, and how we can write down a surface in terms of two very parameters in terms of two real numbers.

We end our talk here with the six week ends, and the seventh week where the whole concentration will go into calculating surface integrals because they will take us to the main theorems of multivariable calculus and vector analysis. Thank you very much.