Calculus of Several Real Variables
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology-Kanpur

Lecture - 26
Applications of Integration to Mechanics

Welcome to the sixth week of the course. So if you look at the course plan, if I go by the course plan, then there is this whole change of variable formula was supposed to be the first lecture of the sixth week, but we have already spoken about change of variable formula in the last class. So I think I will not bore you on that again. Rather this week, because I have already exceeded as per the norms, I have exceeded the number of hours by quite a heavy margin.

I do not know whether it is good or bad, but I have done that. So I think the sixth week will have four lectures, which will also I believe, would exceed the given fixed time of two and a half hours, 30 minutes, five lectures 30 minutes each, but we will start with the second one and we will have four lectures in this session, in the sixth week session.

That is multiple integral and mechanics, which I wrote as application of integration to mechanics. Line integrals 1, which we will do and then line integrals 2. These are two important things which we need to know to do further things about Gauss and Stokes theorem, line integrals would come; Gauss theorem, Green's theorem.

Then we will talk about parametrized surfaces, means how can we look at surfaces which cannot be viewed as functions $z = f(x, y)$. How can we view them mathematically. Can functions be used to represent them. So in that case we need to parametrize and that is what we will do here in this sixth week. So we will start today with application of multiple integrals.

See one application of multiple integral I have already told you is that of finding average temperature. In the last class, class before that when we spoke about triple integrals, we spoke about why triple integrals are needed. For example, we spoke about finding the average temperature in a, say a rectangular room. Now if you go back to the notion of average, the notion of average plays a very fundamental role in mechanics.

The notion of average, for example, takes on important roles for determining things like center of mass, moments of inertia. So all these things has a notion of average. So average means summing up things and then dividing it by the number of things.

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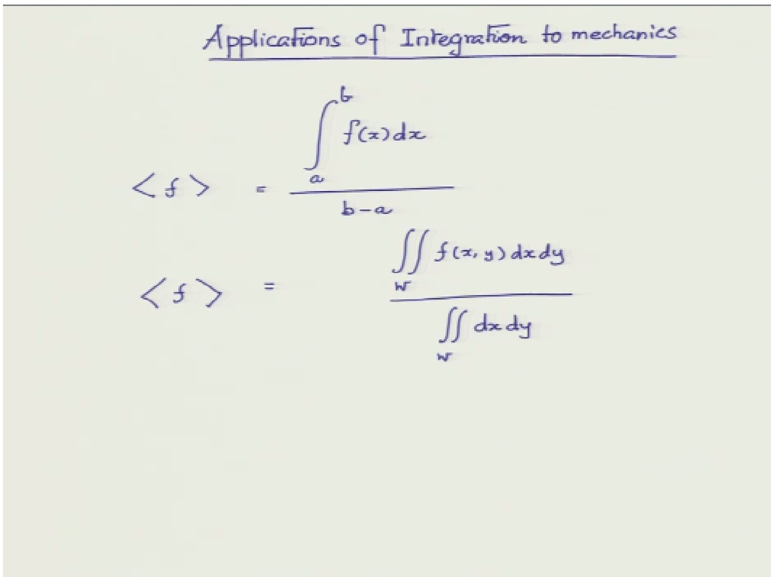

So in general if you have a continuous function, or even a bounded function and you want to integrate, then the average of a function, continuous function over a given interval, the integration represents a sum of the values divided by the total x's that you have, that is captured by the expression b – a. And this is often called the average of f and this symbol of average of f is something which I have borrowed from the physicist. Physicists use this sort of thing.

So what happens if I have a scenario where I have now an area to handle? That is two variable for when I am talking about functions of two variables, and I am talking about double integrals, then what I need, which way I can define the average and how can I generalize this idea that I already know. So again, I am just taking the same thing. Here my f is no longer single valued function, but a multiple valued function.

In that case, sorry is a function of two variables. And in that case if I am talking about a domain W and I am doing a double integral. So that is the sum of the function over the whole area divided by the whole area. So that is the average that you have when you have a function of more than one variable. So you the general this idea has now got generalized.

You can of course find the average, I am not going to keep on discussing this. I am not going to do computations here. Because why to unnecessarily take time doing simple computations.

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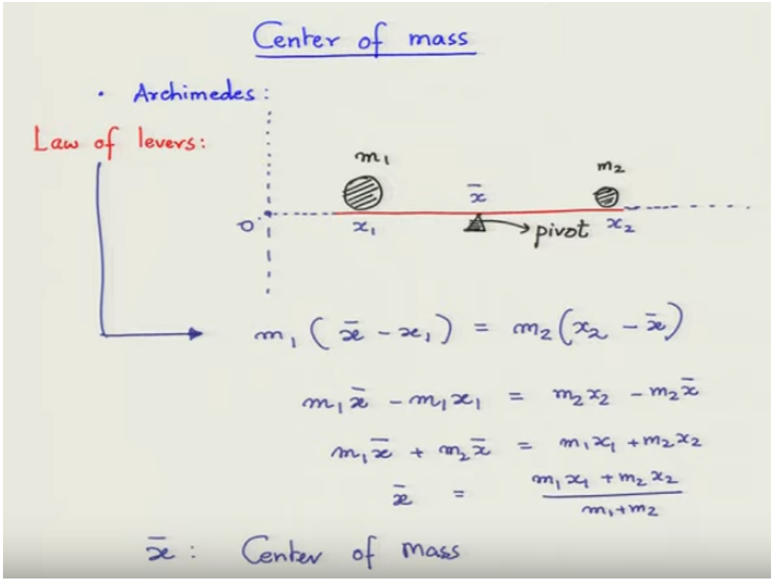

We will now discuss the notion of center of mass of a body or a collection of masses was already known to Archimedes; Archimedes that great Greek geometer and scientist. So what did Archimedes do? Archimedes had a seminal contribution to science. Not only he who found the area of circles, area of segments of parabola, but his key idea rested on something called the law of levers.

The law of levers is that okay, here is a lever. So we are talking about the law of levers. So here is a mass m 1 and here is a mass m 2. If I just put a, if I try to put my hands anywhere or put a support anywhere there is a high chance, if I put the support here more towards m 2 this m 1 mass will create a torque around that supporting point and pull the whole thing down, pull m 2 up and pull, put the m 1 down.

So this is a kind of seesaw type of thing. So Archimedes' question is, which way where should I place a pivot, this pivot so that this lever remains in equilibrium, it does not move. Where shall I place the pivot? So let me now look at it in a much more simpler way. So let me now put along the lever I draw the x axis. And suppose this is the origin.

From the origin this is at the point x 1, m 2 is at the point x 2 and here the pivoting is at the point x bar. What do I expect? What does the law of lever says? The law of lever says, as the distance of the pivot from m 1 multiplied by m 1 should be equal to the distance of the pivot from m 2 and multiplied by the mass m 2. If these two quantities are equal, so what I do? What is the distance of the pivot from m 1?

It is x bar minus x 1. This multiplied with m 1 must equal, so this is the law of levers. So this is finally the law of levers. This is what Archimedes did. Proving it is slightly difficult. There is a whole book by David Spivak, who also has a beautiful book on calculus. He has actually tried to explain why it is not so easy to prove the law of lever of Archimedes, right?

And that very simple mechanics problem can be very hard and there are logical issues. So, okay let us assume that we accept the law of levers. If I take the law of levers and because in practice that is what happens. Please understand, even the area of the circle that which Archimedes developed or the area of the segments of the parabola which was known to Archimedes was based on the idea of the law of levers.

So it says that if the lever need not move it has to remain at rest then it must have m 2 into the distance from the pivot that is x 2. So what do I have? I have m 1 x bar minus m 1 x 1 is equal to m 2 x 2 minus m 2 x bar. So this gives me m 1 x bar plus m 2 x bar because I transferred this here and take this part m 1 x 1 to the other side; m 1 x 1 plus m 2 x 2.

So x bar, the point that I want to really locate because x 1 and x 2 are known points. The point that I want to locate is m 1 x 1 + m 2 x 2 divided by m 1 + m 2. So it is as if this x bar is often called the center of mass; x bar is often called the center of mass as if the both the masses, the sum of both the masses is located at the point x bar. So this portion of this x bar can be obtained by the law of lever of Archimedes.

x bar is called the center of mass. So here I have made up a system of two discrete masses. What if I just look at a continuous body, a rigid body and then if I try to talk about a center of mass, what will happen, okay? If I then try to talk about a center of mass, a body which is lying along the x axis.

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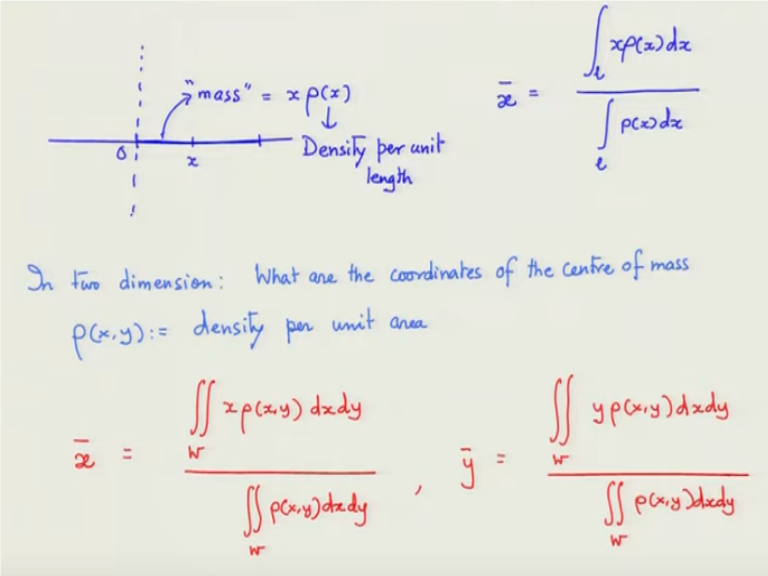

Suppose here is a body which lies along the x axis and I know that the density per unit length is rho x, okay. Then if I want to, so far a distance of x, the mass would be that distance into the density, right? So here volume is given by the length basically. So in that case, the upper part that summation m i x i , the m i is now x into that rho x, rho x is the density per unit length.

So if I come the distance x from the origin, so if I am here x from the origin okay and I am as you let me assume that the origin itself lies at one end of the rod. Then if I come a distance x on the origin, the mass is nothing but x into rho x, where rho x is the density per unit length. So in that case, your x bar is just trying to write down a continuous version of the thing that we obtained from the Archimedes' law of lever.

I then want to write a continuous version of x bar equal to m 1 x 1 + m 2 x 2 by m 1 + m 2. So if I had more masses it will be summation m i x i by summation m i. So that idea can be now progressed with the use of integral in this continuous mass. So here I have this x, this m i instead of m i the role is now played by m, m dx, mx dx, at every x what is the mass.

So mx rho x dx integration of that whole length basically the length l if I say an integral l of rho x dx is the total mass, summation m i that is the total mass. So now if I am in two dimension, see if I am in two dimension, what are the coordinates of the center of mass? In this case the density cannot be per unit length but has to be per unit area. So the density has to be like this.

So if you take a small unit area around x, y rho x, y is the density per unit area. Here similarly, the coordinates would be given in the following way; x bar has to deal with x part only. So could be a region W x rho (x, y) dx dy integral over the total area. So density per unit area. So the area of the, so the volume or the mass of dx dy the unit elemental areas the mass of the elemental area is actually rho (x, y) into dx dy.

And y bar is replacing the same thing, but x is replaced by y. That is what we are expecting. Of course, you can definitely talk about I am not going to do integrations here. I am just giving you the conceptual issues here because it is unnecessary to do same kind of integration. But I will show you some type of integration which will be useful. I will do one temperature thing. So you see here, what happens.

Can I take it to the notion of, can I take this whole thing to three dimensional system. So in a three dimensional body now, I have a point x bar y bar z bar which has to be the center of mass. So in a three dimensional body, so what is now three dimensional case, three dimensional body.

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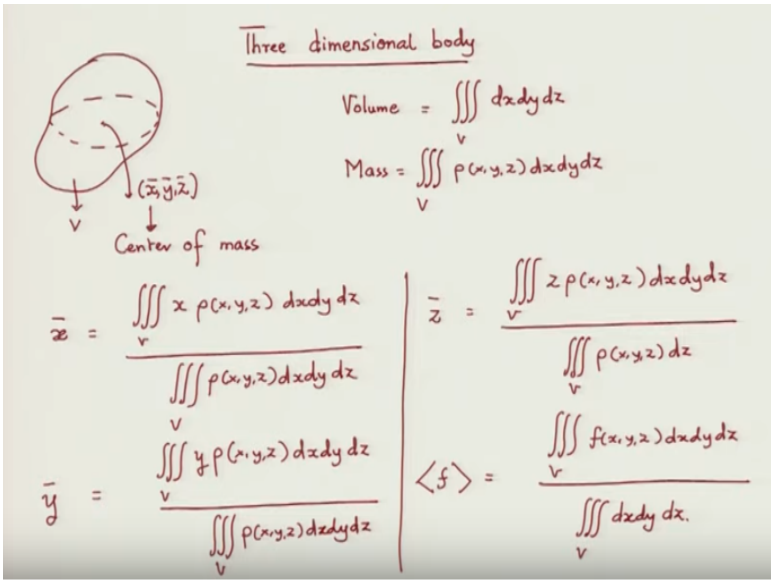

So let us take a body which is a closed bounded nice three dimensional body, body actually means a closed and bounded set like this. So we are trying to find a kind of point where all the mass would seem to be concentrated. So x bar y bar z bar. So what is the volume of this body? So we will call this body as V to differentiate everything. So volume integral, integral, integral V dx dy dz.

Now if for every unit volume rho x, y, z at a point x, y, z if the density is x, y, z. Then if you take a x, y, z point lies in a elemental volume dx dy dz and if the density does not change much. If you take the density remains constant over unit volume that is what is rho x, y, z. Then in a very small elemental volume the density would also remain as rho x, y, z. That will remain constant, do not vary.

So the mass in an elemental volume is rho x, y, z into dx dy dz so density into volume. So the total mass is to integrate over the volume, total mass of this body V of the density rho x, y, z. So the center of mass, which is called the center of mass, the center of mass x bar coordinates. The same story which goes on, do not worry is V x rho x, y, z dx dy dz divided by the mass.

If I have to write for y bar, so y bar will be the same story but x now replaced by y. So it is a repetitive writing which I do not want to do or for completeness sake, people might just ask me that why did you not write it in detail, I would complete the writing; rho x, y, z into dx dy dz. Of course, you can talk about z bar, z bar is the same story where we have z instead of x or y.

Obviously, got V and then we write down over V the total mass. Now if you want to find the average, what is f average in this case? f average is integrating over V f (x, y, z) dx dy dz and then dividing by the total volume. That is it. That is neatly summed up for the three dimensional body, for example. So let us go to the next page. And then here we start discussing a small problem.

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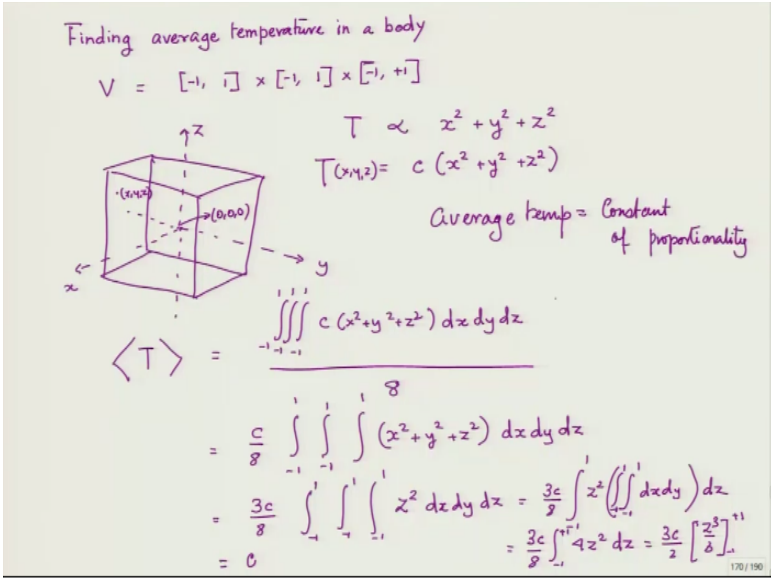

So let the average, so given any, finding average temperature in a body. In this case my body V is the cube, so unit cube with center at 000. So you can immediately visualize what kind of a body we are looking at. It is a kind of room and we are trying to get the temperature or the average temperature, okay? So let me draw, draw the, so it is a cube or other better things to draw cube.

It will be a cube of course, you can visualize it much better than me. Maybe I should draw a cube. This is also another way of drawing a cube. You know that 00 is at the center. So this could be the x axis, this is the y axis, this is the z axis. Now I am telling that okay at any point x, y, z in this cube the temperature T is proportional to x square plus y square plus z square.

So basically T is some c time some proportionality constant into x square plus y square plus z square, okay? Now I have to find the average temperature in the room. So how will I find the average temperature? So what I will do is exactly in the same way. T average is T is integral, integral, integral, where x, y, z are all varying between -1, +1, c into x square plus y square plus x square. Basically this is T.

So T I can write as a, now I can write it like a function T (x, y, z). Here also you can put T as T (x, y, z) it does not matter, into dx dy dz or dz dy dz it does not matter, whatever you want into the volume of the cube. And volume the cube is this length is 2 2 2, so it is 8. So 1 by 8, I think c should be also be taken out, of -1 -1 to +1, -1 to +1 x square plus y square plus z square dx dy dz.

You see here I have actually three separate integrals, x square dx dy dz plus y square dx dy dz plus z square dx dy dz, and all of them will have the same answer. Because we can iterate accordingly. All of them would have the same answer, all of the three. So instead, we can write this as so say at the same integrals 3c by 8 -1 +1 -1 +1, I just do the z, z thing. So and multiply, they are all the same.

So I can do one of them and multiply them with 3. So that is exactly what I am doing. -1 +1 -1 no not +1. I should write this 1. 1 -1 1 -1 1 z square dx dy dz okay. You immediately see what is happening, you immediately see. So basically what I can write here is 3c by 8 -1 z square. I can actually take this whole integral, you see how interesting it is. -1 here also -1 +1 dx dy.

Basically I am looking at the area of a square of side 2 which will be 4 into dz. So basically it is 3c 8 -1 to +1 4z square dz. So that will be 3c by 2 z square dz is z cube by 3. And I am now writing the answer. The answer to this everything is finally z cube means here one third minus two third. So basically it will be c, the answer would be c.

So the average the proportional so what see, by calculating the average we learn a very important physical fact that the average temperature is nothing but the constant of proportionality, average temperature. So what is the physical fact that we have learned? Average temperature is the constraint of proportionality.

See, so you have some information which the mathematics is giving you which by just looking at the proportional of, constant of proportionality you cannot say anything about, you may think that average temperature and constant of proportionality is different. But you see in this particular scenario, it is the same. It is amazing that here in this particular for this particular box, the average temperature is same as the constant the proportionality.

Of course if we change the box things will be different. But average temperature is same as constant of proportionality. So that is a physical fact that we derived from here. Well done. So here we end our discussion. We have been trying to maintain time, because the next class will take a slightly much more time. We will be talking about line integrals, and which we will start next. So maybe I should take a break.

And then I will talk to you about line integrals. Okay, so this is what you get that these are the some ways you can of course, you can talk on moment of inertia. The book also talks about how to use all these to find the gravitational potential and all those things. But we are not getting into so much of details about physics at this moment because of the variety of students that are there.

I am sure economists would be unhappy if I just keep on talking about physics or biologists might be unhappy. So we will go over to line integrals, which we will start in the next class. Thank you.