Calculus of Several Real Variables
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Lecture - 25
Multiple Integral

So we are here in the last lecture of the fifth week. So if you look back, it is time to take a look back and if you take a look back, you will realize that we have covered a lot of ground. All the lectures might not be in tremendous detail, but we have made considerable amount of progress. This section is about using the method of substitution that we use in your single variable calculus and integration.

Does that method have any role, that kind of methods have any role when you have more than one variable? Sometimes changing variables makes life easier. For example, the book gives a simple example which say which if I give to anyone, anybody will do it.

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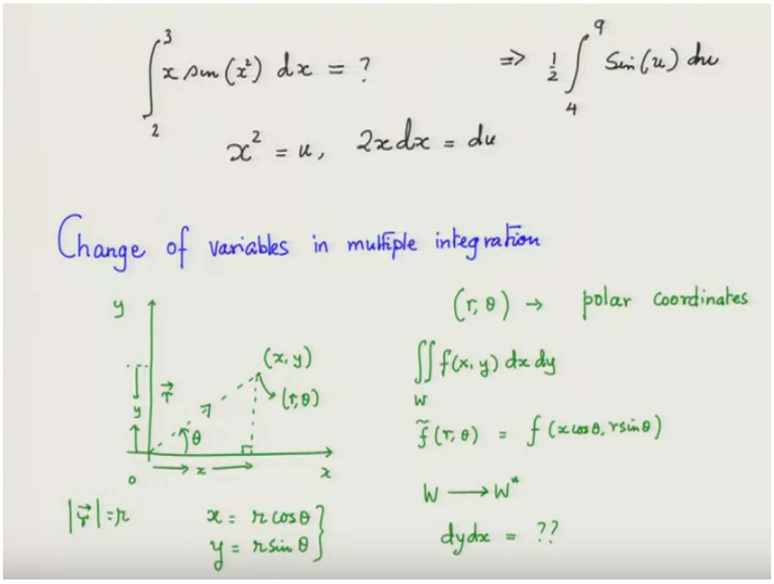

So if I have to evaluate it, this kind of integral, so we are going to talk about, so what are what am I supposed to do? How do I evaluate it? What I do is I will put x square is equal to u and I know to 2xdx where 2x is the derivative of this is du. So dudx is 2x, so 2xdx is du. So basically, what you do xdx is then half of du, so, the whole thing gets changed. So when x is 2 u is 4 and then when x is 3 u is 9.

So it is sin of u half du. And then you can integrate it. You know what to do, it is – cos u. So how will this substitution work when we have more than one variable. So if you have two variables x and y, how does this idea of substitution works. So this brings up us to this new chapter or new topic change of variables in multiple integration. There are two popular ways of identifying points on the plane r 2.

So these ways are following. Take a point which is given by the coordinate x, y. That this the Cartesian coordinate and if you observe this x, y this vector which we call r has a magnitude r that are r. Now this direction vector makes an angle theta with the x axis. So this r and theta can actually tell me where a point is located, because x can be, if you know basic trigonometry, you will know x can be expressed as r of cosine theta. This is your x and this part, this part is your y.

And y can be expressed as r of sin theta. So this r theta variables r theta, they are often referred to as polar coordinates. So I can represent this point by r theta called polar coordinates. Sometimes it is easier to integrate in polar coordinates than in the standard Cartesian coordinate system. So how will I integrate in terms of polar coordinates? First, suppose you have a region W and you are integrating.

So first you have to substitute in place of x and y f r and theta, x is r cos theta y is r sin theta. That is you have to construct a function f tilde r theta, which is f (x cos theta) f of x cos theta. In place of y it is r sin theta. And the domain W has to change to W some W star. Domain W has to change to some W star and then you integrate over the polar coordinates. Now here I have replaced for example, du is x dx is replaced with du.

Now here my elemental area dy dx or dx dy is replaced by what? What should it replace? Should be replaced by, what is that? What should replace dy dx in this case? So we have to now first look at what is the meaning of an elemental area when you were talking about polar coordinates.

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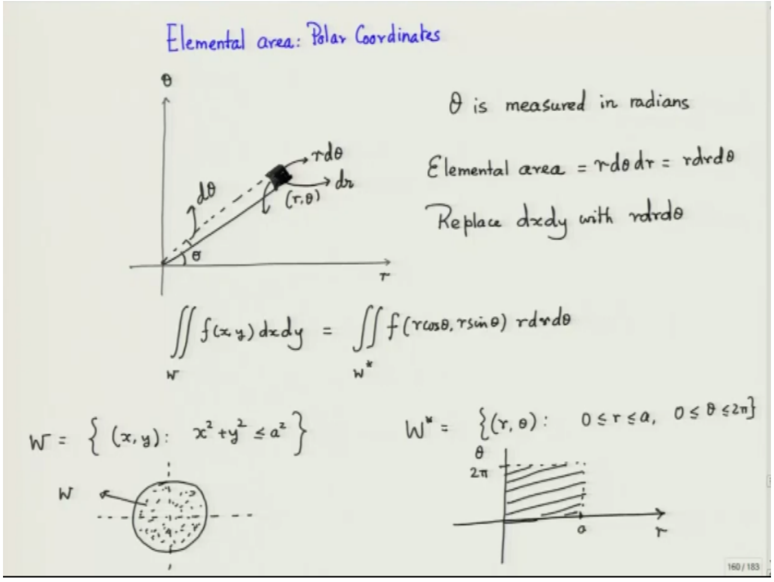

So elemental area polar coordinates, elemental area in terms of polar coordinates. So here I draw the x axis and I draw that y axis. Now let this be r, theta. Then you make a little bit of increment in r, this is dr. Now little increment in theta. So this was theta, so this is d theta.

So now my elemental area is this one and r is the radius. Since dr is very small I assume that this area this zone this line and this curve this the outside curve the outside arc means the inside arc means which is this one and outside arc which is this one they have they are almost of the same length, this kind of a rectangle though really it is not but we are assuming that it is.

What is the area what is the length of such a thing? That is r, s the rectangular the arc length is r into theta where theta is measured in radians. So if I measured theta in radians, theta is measured in radians. Theta is measured in radians. So if I am looking at this side or this side inner side, they are same. This is r, d theta and this little side is dr. So elemental area, elemental area is equal to rdrd theta.

So the idea is very simple. Replace dxdy with rdrd theta. So whatever was the elemental area there which was dxdy is replaced by the that elemental area should have a similar area which is rdrd theta. That is the elemental area, area of elemental area is calculated, both are da basically. They represent da. So what happens is, so in terms of the polar coordinates, so W f (x, y) dxdy is equal to W star f R cos theta r sin theta rdrd theta.

Elemental area is actually maybe I should write it much more is rd theta. This angle d theta into radius. That is the arc length then into dr. So which I can write as is preferably written as rdrd theta. So this is what I have where W star is the corresponding zone. For example, if I take W is a set of all x, y such that x square plus y square is equal to a square. So it is a circle with center at the origin.

This is your W. Well what is my W star? x square plus y square is r square. So r squares is r here is a is positive so r is less than equal to a. So r has to lie between zero and a, r is a positive r square is a positive number. In terms of x and y, so theta is varying from zero to 2 pi. Theta is free here. So my W star would consist of r theta because r is non-negative. Here r square is less than a square, r is less than a.

Where r must be here. And theta must be between zero to 2 pi. So basically if I draw the x axis and y axis by putting r in one r in this side, and theta in this side. So r and theta, so if this is 2 pi and this is a then it is a kind of rectangular box. So you see what is this transformation doing to this region, that a rectangular box has changed over to a rectangular box.

Sorry a circular region has changed over to a rectangular box and you know how to integrate over a rectangular box, the same kind of integration, that you have done. As an example, we use our knowledge here, what we have just done to understand how to compute.

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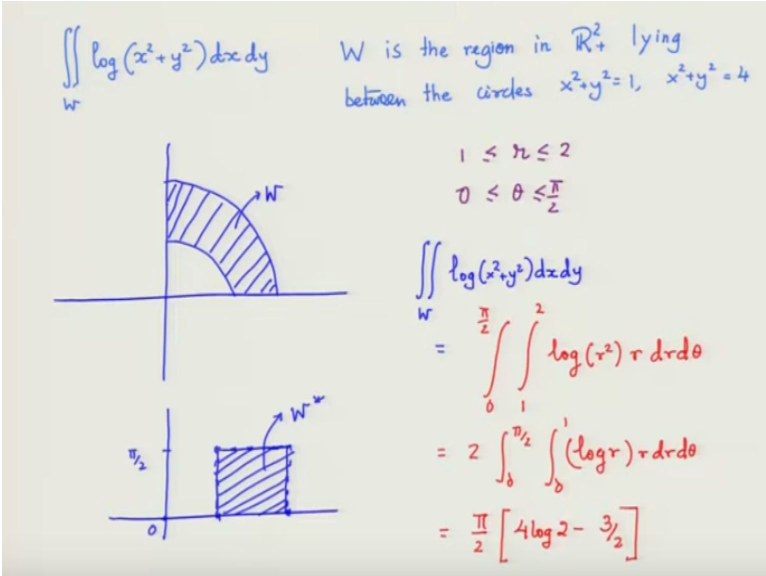

This is an example from the book. Please understand that we are continuing to use this book. Those who have not seen this book, again I want to show it. Please see that this is the book that I am using. It is called basic, Basic Multivariable Calculus by Jerrold Marsden, Anthony J. Tromba and Alan Weinstein, Basic Multivariable Calculus. This is exactly what I have been using throughout.

I maintained a single text, which is much more easier when these kind of subjects were, this particular subject where lot of practice is needed to get used to a method. So there goes a story of John von Neumann who is one of the world's greatest mathematicians of the 20th century. And John von Neumann has this story that there was a physics student in Princeton, and he got stuck with a PD, which his supervisor could not even solve, okay.

So then he told his student that why do you not go to John von Neumann and ask him. Finally he found John von Neumann in party dancing around and he showed the problem to John von Neumann and John von Neumann told him use the method of characteristics. So he goes back and tries to find what is the method of characteristics in solving the particular equation, partial differential equation.

And then he goes back and then meets Neumann and says that I have used that method of characteristic, and I have actually solved the problem, but I do not really understand what is the method of characteristic. So von Neumann told him, my son, you do not always understand things in mathematics, you just get used to them. So it is time that we also here in these kind of subjects for what we are doing now, we need to get used to things. So a single book and certain standard examples do help.

For example, this example where W is the region in R 2 plus and that is in the first quadrant lying between lying between the circles x square plus y square equal to 1 and x square plus y square equal to 4. So it is circles with radius 1 and radius 2. So if you draw the thing you just need to concentrate on the circle of radius 1 and circle of radius 2. So this is the zone that you are bothered with. So R is lying between 1 and 2.

In this case, if you look at the polar coordinates, this is radius here is 1 and radius is 2. So R is lying between 1 and 2 and theta is lying between 0, angle could be 0 or it could be pi by 2. So 0 to pi by 2. So you see again you have converted this so called domain W, this domain W had been converted into a, domain W has been converted into a domain W star which is rectangular in nature, where your iterated integration will work.

So your area is 0 to pi by 2, for example this is pi by 2. So this is now your zone, exists between r is between 0 to 1, 1 over x axis for example. The horizontal part is between 0 to 1. This part is between so this part is my W star. So now it has become a rectangular one. Now, you can easily use this transformation to compute out stuff. So let us take ourselves into the next step.

So the integral of W log of x square plus y square dx dy, so r is running from our r drd theta. So r is running from 1 to 2 and this is varying from 0 to pi by 2. And so you can use the iterated integral technique. x square plus y square is r square r drd theta. So you know that this is nothing but 2 0 to pi by 2, 0 to 1 log r r drd theta. So we integrate out this finally, I am not doing the integration completely.

The answer turns out to be pi by 2 into 4 log 2 minus 3 by 2. You see so this change of variable has made a slightly difficult looking domain transform into a much more easier looking domain and you can solve the problem by the method of iterated integration. The goal is to solve these by the method of iterated integration. You can come to triple integrals. In triple integrals, we are talking about cylindrical coordinates.

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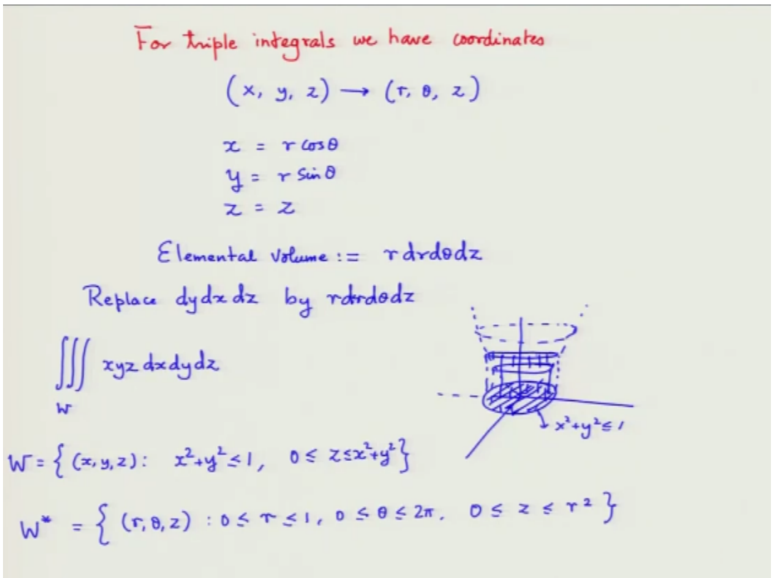

Just like polar coordinates, when it comes to triple integrals, we have cylindrical coordinates that is, for x and y we write it as a polar coordinate. Other parts we keep as z, cylindrical coordinates. That is we move from x, y, z to r theta z. We move from x, y, z to r theta z. So in that case here you write x is equal to r cos theta, y is equal to r sin theta and z becomes equals to z.

And here the elemental volume in this case is given as our r drd theta dz. So we have to replace dy dx dz replace by rdrd theta dz. So in the same way you can operate on this. For example, if you want to evaluate that this is an example here in the book. If you want to evaluate say, so here one has to understand, we can put a restrictions on x, y and also on z. So for example here we will integrate over W.

We will say what is it, xyz into dxdydz where the W domain is consisting of triplets of point x, y, z in r 3 such that x square plus y square is equal to less than equal to 1. So it is a circular region and z, so z is lying between 0 and x square plus y square. So what do we do? So, you first take x square plus y square less than equal to 1. That is the domain that is the domain what you have.

We are taking these points in r 2. This is circle x square plus y square less than equal to 1. Now, you are talking about the graph of x square plus y square. So here is the paraboloid. So take any point here, so this is my z point. So basically you are taking any point here till you come here to till the end. So this is the volume that you are trying to compute. So basically you have converted into a cylinder.

So it is not the volume you are trying to compute. You have converted the converted the zone into a cylinder. It looks like a cylinder. So let us see what W star consists of . So W star when I transform it, it consists of r theta z where r is lying between 1 and 0, theta is lying between 0 and 2 pi. And then x is lying between 0, z is lying between 0 and x square plus y square which is your r square, same as r square.

You take any xy, x square plus y square circle is equal to r square. So this length is exactly, if you take this is your x square plus y square value, this is exactly this length, this square type, so is a square. So z is lying between zero and x square plus r square. So now your domain has changed from x, y, z to r theta z. So that is a cylinder. That is a cylinder that has been made.

So now here you have converted into a cylinder. Now what will you do? You will again do sorry not the cylinder the cylinder coordinates what, I am making a mistake. So here what you see, you have a cylindrical thing. So when you convert the cylindrical, this is in x, y, z plane in x, y, z space you have the cylindrical domain. But cylindrical domain you cannot operate on.

By making a cylindrical transformation with cylindrical coordinates this W star has become now a rectangular parallelepiped. It is a kind of domain that you have already used, kind of domain that you have learnt in the triple integral. It is not rectangular parallelepiped, it is fixed with x and y. Only the z part is varying, right? x part and y part is fixed and the z part is varying.

And you know how to handle it using the things from the rectangle. You know how to handle it using iterated integral. That we already learnt in triple integral. So from this system of a cylindrical thing, we have come down to a one with much more simpler coordinates here.

One with r theta where they are fixed and this one is varying and you know how to handle the varying case because we have already learnt it that okay, this region can be put under a rectangle and this function can be defined to be zero outside this W star zone and so and so forth, which we have learnt.

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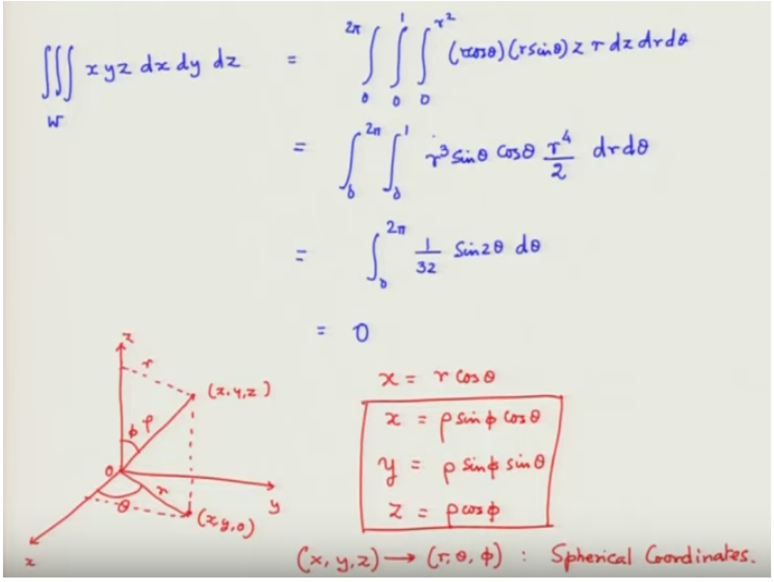

So in this particular case we see we write like this. First we integrate this with respect to z. So here you will have r cos theta r sin theta z, r cos theta r sin theta z, r dz then dr then d theta. So after this when I integrate out z it will be in terms of r. So then we integrate from r is from 0 to 1 and where we have just we will be left with functions of theta integrated from 0 to 2 pi. That is the way you should do it.

And if you integrate this so will have zr dz. Now you integrate with respect to r sorry z first, so you have that only one z. So r cos theta r sin theta remains constant and z you integrate out with so it will become z square by 2 and then you put r square r 4 and this r into r into this r will r cube, r cube sin theta r cube cos theta. So r 4 by 2 drd theta. Let me integrate the theta out.

So finally we will have 0 to 2 pi 1 by 32 sin of 2 theta. I will not go in the details of the computation. You can check in check the details yourself. That will give me finally the answer to be zero. So that is one way of it. There is something called spherical coordinates, which is also often used but we will just mention what is a spherical coordinate. Spherical coordinates come like this, this is how we describe spherical coordinates.

So if you have a point in x, y, z in space, so here is the distance that you know x, y, z. So angle so the angle of this line joining the origin, the lines have been joining the origin and x, y, z this angle it makes angle phi with the z axis and it makes an angle, suppose I put I drop a perpendicular from x, y, z to the x axis. So it gives me the point x, y, 0. Join that line with the origin and that new line makes an, which is r here makes an angle theta. So if you go at x, what is r, right. So what is x here?

So in this case x is equal to r cos theta. What is my r? r if you look at it is equal to this. This is your r also. This is your r and this is nothing but rho sin phi. So r equal to rho sin phi. So rho sin phi cos theta. Similarly, y can be written as rho sin phi sin theta. Because that r sin theta basically. And what is z? z is nothing but rho cos phi.

This is called the this three things this is the transformation from the Cartesian coordinates to r theta phi. This is called the spherical coordinates. So from x, y, z to r theta phi is called as spherical coordinates. So we had a pretty decent idea of what kind of transformation we use largely in natural sciences. But now, we want to give you a more general mathematical description of how to do a transformation.

So that brings us to a very basic idea. For example, if you had this, if you have looked at this transformation, what we had studied when we were doing that single integral when you had looked at the transformation u equal to x square.

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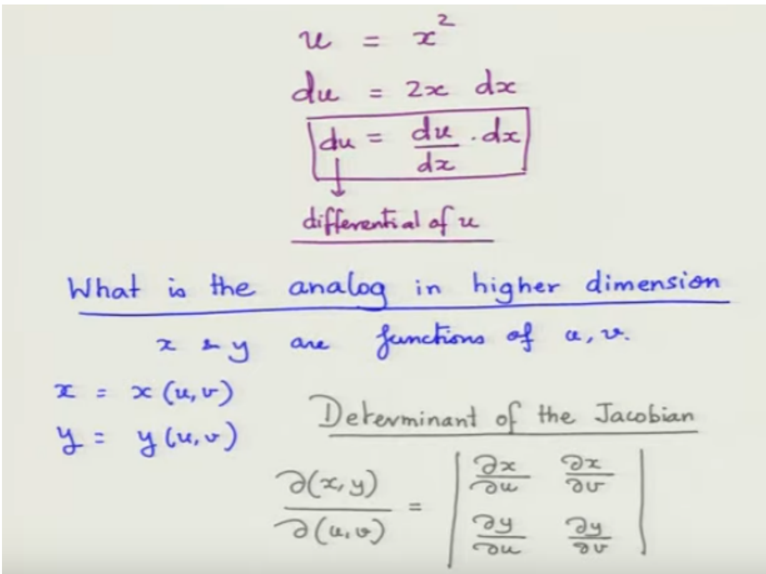

So what did you do? du is 2x dx. So what is 2x that is dud x. So that is the way transformation is done. du is nothing but du dx into dx. But then how do I go do this thing in case of how do what is this is actually called the differential of u. This is actually if you linearize the function, if you just take the first order Taylor's expansion this is nothing but this changes the quantifying the value of u as x changes, this term du dx into dx. That is all.

It actually quantifies the value when x is very small and hence u is very small, assuming they are all continuous because they are differentiable. So how do we speak about how do we do this kind of change of variables in higher dimensions? What is the analog in higher dimension? So what is the analog in higher dimension? So now suppose you have x, y are functions of u,v like x, y a function of r theta, x is r cos theta y is r sin theta.

So x, y suppose x and y are functions of u, v that x is x (u, v) that is x is kind of x (u, v). This is a one way of writing and y is y (u, v). We compute what is called the determinant of the Jacobian. So we shall compute what is called the determinant of the Jacobian, determinant of the Jacobian. This is symbolically written like this. This is a symbol, it does not have any significance.

It is a determinant, it represents a determinant; so determinant of the Jacobian matrix. So what is the Jacobian matrix? So x, y is a function of from r 2 to r 2, right? So when you have a function from r 2 to r 2 the derivative of that function is a matrix called the Jacobian. So what you do? You first find the gradient of this first function x (u, v) and put it as a rho vector.

Then you find what is the gradient of the second function y (u, v) and put it as a rho vector. So del x del u del x del v. So these are so this is the determinants. So del y del u del y del V. This is a determinant. So what we can actually show that the following thing occurs.

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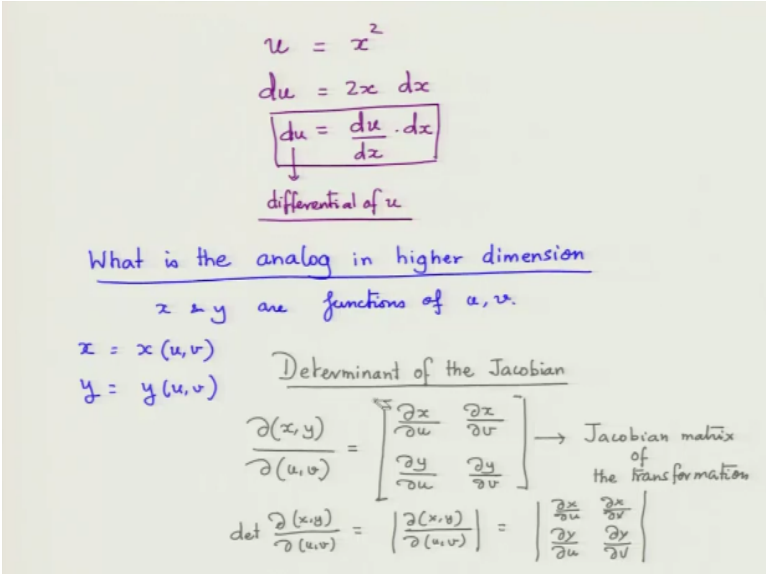

That we can show that integral some domain W. Same way you can go for three variables also, r 3 to r 3 but we are just concentrating on two variables. So now, you can have arbitrary, you need not have polar coordinates particular or you can have x is u + v y is u – v, this kind of transformations also. So if you have this kind of thing. So this integral can now be given us W star f instead of x I have x (u, v).

Instead of y I have the function of y as a function of u, v into the determinant del x y. So what we need is the determinant of so this is the Jacobian matrix. I first showed it as a determinant but let me just for not to make you uncomfortable, this is the Jacobean matrix.

First we compute, so this is the symbol a standardized symbol in multivariable calculus for the Jacobian when you are writing this transformation, Jacobean matrix of the transformation and determinant of del x, y del u, v is written as del x, y del u, v which is what we wrote earlier del x del u, del x del y, del y del u, del y del v, okay. I just made this change so that you do not get confused that I skipped it so much.

It does not matter even if I write this as the determinant, it does not matter. But people are much more concerned about particular signage and all those things. So determinant means something like this the matrix and then you compute the determinant.

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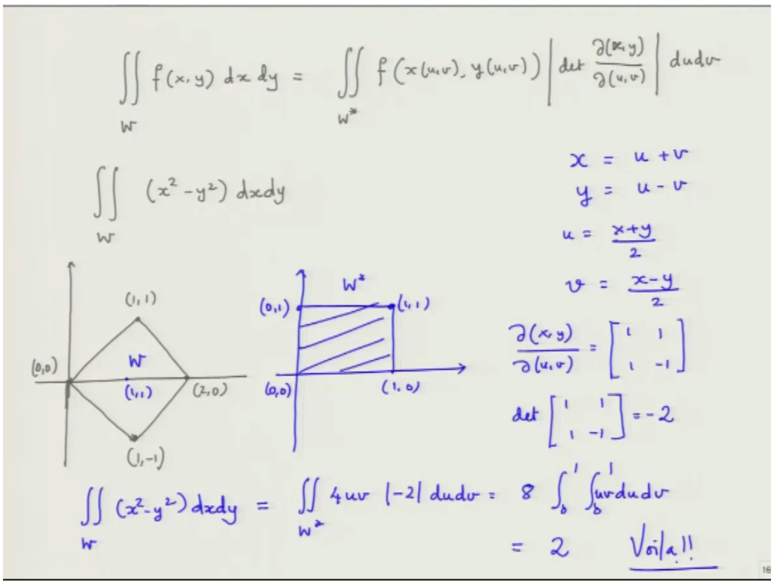

This into dudv. So you are replacing dxdy with this. This is the standard way of doing it. So you take a kind of transformation say you integrate W over x square minus y square dxdy. And let W be this given region. So it is a square with this vertices (0, 0) (1, 1) (2, 0) (1, -1). This is my W. Now let me look at W dash. So to compute W dash I need a transformation.

Let us put that the change of variables x is equal to u + v and y is equal to u – v. And from this transformation what happens? So you can actually calculate out u. So 2u here is x + y or u is x + y by 2. And v is x – y by 2. So you know when x is 0, 0 y is 0, 0. When x is 1, 1 x is 1 and y is 1. Then u is half, right? So when it is 2, 0 when x is 2 and y is 0. So when u is 0, 0 when x is 0, 0, that is my original thing.

Then I should have u as 0, 0. Now when x is 2 and y is 0 then u must become 1. And when x is 2 and y is 0, v also becomes 1. So 2, 0 gets shifted to 1, 1 right? So what is 1, 1? So anything so this 1, 1 was here in this zone that becomes 1, 0. That becomes a vortex here. So basically, this W star becomes this thing, W star becomes this rectangle with (0, 0) (1, 0) (1, 1) and (0, 1).

So similarly now we find we compute this. And you know, it is not a very difficult thing to compute. It is the jacobian is 111-1. And determinant of this thing 111-1 is equal to -2. So my integral of D, x square minus y square dxdy can be now written as x square minus y square is nothing but 4 uv, u + v whole square minus u – v whole square is my not D my W sorry, W to my W dash.

Where W dash is nothing but 0 to 1, 0 to 1 4 uv. So what we do here, does have some little problem in the writing. I just make a correction here. I forgot to write the determinant. Determinant of del x, y. I tell you take the positive part of absolute value into dudv. I forgot to write the determinant. So determinant is this. So you take the modulus of that minus 2 is 2 into dudv. So basically it is 8, 0 1, 0 1 dudv sorry uvdudv.

The answer turns out to be 2. So as a French man would say voila, we are done, we have covered a good amount of ground and now that everything needs is you need to have practice. Once you have practice, you get a hold of these things, and you will never forget how to do these things. Just a little bit of practice. That is it. For me also, it is very good to sometimes to teach calculus because it gets you back to your basics.

Maybe you also if you have to get into some integrate calculation one is to go back and look at the basic books. But it is sometimes always good to teach the basics because then you remain in touch with it. So thank you very much. So we will start our sixth week very soon. And we will get into more applications, we will get into gradually get into surface integrals, line integrals, the Green's theorem, Stokes' theorem and divergence theorem and end of the game.

The last lecture would be more general where it is little more on fields. It will be largely on applications to physical sciences.