Calculus of Several Real Variables
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Lecture - 24
Multiple Integral - IV

So today we are going to start on the fourth part of multiple integrals, multiple integrals IV, okay. So the fourth lecture on multiple integrals but this will actually cover triple integrals. By triple integrals I mean that I want to attach meaning to this expression.

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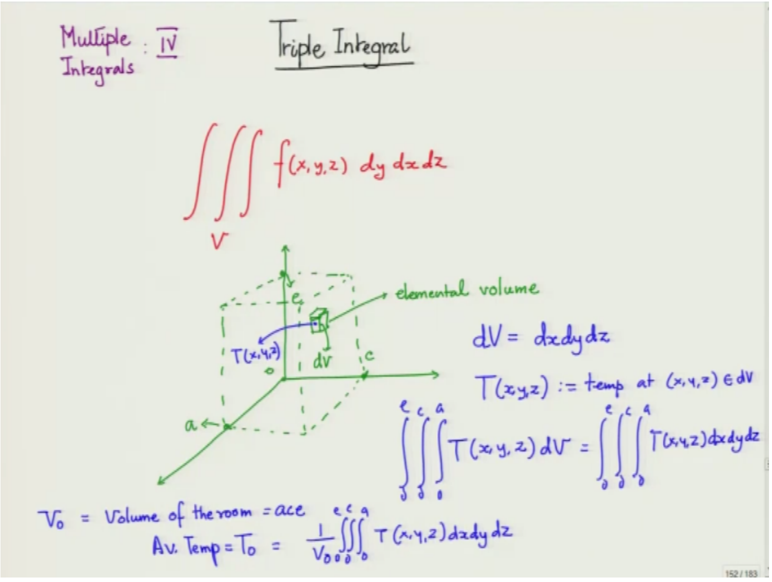

If you have a function f of three variables x, y, z. Then I want to attach meaning to this expression over some domain D. What is the meaning of this? I will not call it domain, I will rather call it kind of volume V, because V here means volume. We are in three dimension, so dy dx should actually represent a kind of volume. Let us look at a simplistic scenario. Consider this room and you know that at every point there is a different temperature. If you measure it there will be a different temperature.

T, temperature T is a function of x, y, z and I assume that they vary continuously, right. So now I want to know what is the average temperature of the room. So what I do? So the room is a kind of box. So here let me just try to draw the room. The room is a kind of box, which has a length say 0 to a and 0 to c and 0 to e. So it is a length of the, length, breadth, and height of the room.

And here I construct the picture of the room which is exactly a rectangular room, like the room in the studio where I am talking or any classroom, these are rectangular rooms. So a room is rectangular in the sense solid rectangles means rectangular parallelepiped. So this is my height e, this is my height e. So 0, a. This is your a, this is your c, this is your height e. This is my room.

And at every point there is a different temperature. But I want to know what is the average temperature of the room. So what do I do? So construct a very small elemental volume, a very small rectangular parallelepiped whose length, breadth, height are dx, dy, and dz. So this is what we will call elemental volume dV, elemental volume. So what is dV? dV is written as dx dy dx, length, breadth into height.

That is what dV is. Now, at a given point in a small point, in a very in a say, in an unit volume at one point suppose a point every point is viewed as unit volume, right. So in this if in the unit volume the temperature is T(x, y, z) at a point x, y, z. So at every point, suppose a point is there, where the temperature is T (x, y, z) a point inside. Temperature is T (x, y, x). So what I do?

The temperature over this whole elemental volume is so T (x, y, z) is temp at x, y temp at x, y, z. x, y, z belongs to dV. Now what happens is that if I want to know what, so I will assume that in that very small zone, the temperature does not change much. The change is so less that I can take T (x, y) that is the temperature around.

So what I will do, the temperature in that zone is T (x, y, z) dV and over the whole volume V is a rectangular volume is given by the triple integral, because they have to integrate over the whole volume. This integral from 0 to a, 0 to c, 0 to e dV which is same as writing 0 to e, 0 to c, 0 to a T (x, y, z) into dx dy dz. So, if I know that the total volume of that zone is V naught, then suppose this rectangular area now has a into c into e.

So V naught is the volume of the rectangular area, volume of the room say. Volume of the room is actually length, breadth into height, it is ace. So T naught the average temperature T naught is equal to 1 by V naught into integral this integral 0 to a 0 to c

0 to e T (x, y, z) dx dy and dz. So there is a meaning in computing this is a, in physics these kind of triple integrals come quite often.

Triple integrals, now once you know about triple integrals, you can talk about integrals in n variables. So there are four variables. What I will do?

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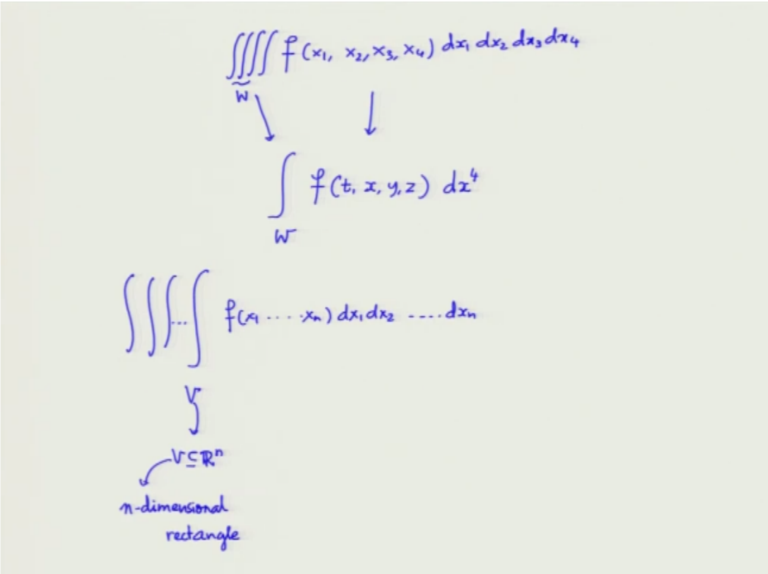

So you can, four variables you will have four over some four variable region which I say write W tilde. Okay, I am not telling it is a part of our four say. And you will write f (x, y, z) or now you have to give numbers rather, f (x 1, x 2, x 3, x 4) dx 1 dx 2 dx 3 dx 4. Please understand physicists use this very much, this kind of integration.

Because this kind of integration comes very much in physics because physics we talk about space time, where the first variable x 1 corresponds to the time and x 2, x 3 are the three space variables length, breadth and height. So physicists if you look at any book on special relativity, you or where you use kind of calculus variations, principle of least action to discuss relativity.

For example, this beautiful book by Leonard Susskind called Special Relativity and Classical Field Theory, this kind of integrals had been repeatedly used. Physicists of course, does not write these integrals in such a detailed way. This integral they can put it like this, just put one integral which actually combines all these three and will write the zone, W and then they will write f (t, x, y, z).

Sometimes they will just write a symbol called dx 4 which simply means there are four differentials that you will use. This is one way of writing, okay. So there are several ways of writing the same thing. So you can now generalize it over to n dimension. So we will come to n dimensions, there could be a volume V in n dimensions, V subset of R n or rectangular volume.

V is a rectangular n dimensional rectangle. So W is a four dimensional rectangle here and you can just write f of. So here you have n such integrals f (x 1 to x n) dx 1, dx 2, dx n. So these kind of integrals are actually useful and they are actually required and that is what we need to understand at the very first attempt that these are very, very useful things in physics. So here again we see the connection between maths and the physical sciences.

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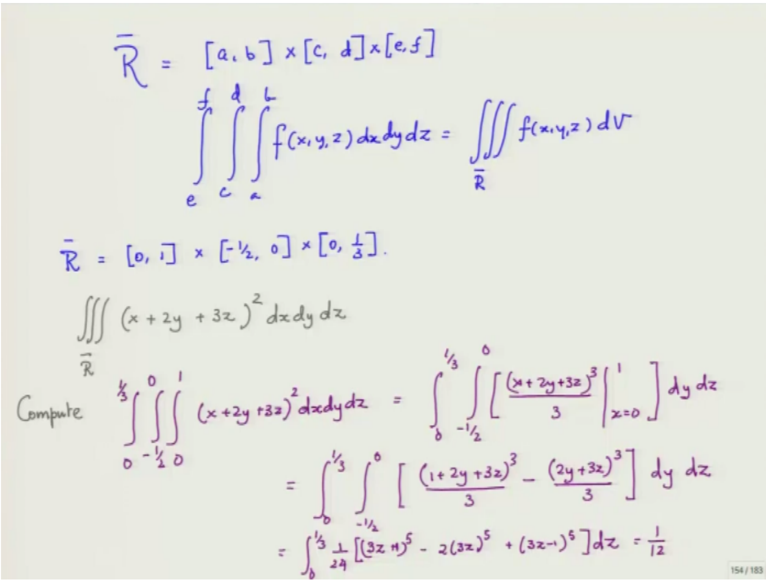

So very important thing to understand is that if an iterated integral exists for example, you take a rectangle R rectangular parallelepiped region. Let us put R bar and let it be a, b cross c, d cross e, f. So you can define and first you do it x, y and z. So you can define an integral suppose you want to integrate this suppose you are looking at iterated integral.

When you have rectangles you can talk about in terms of iterated integrals that is a to be first with x then c to d that is with y and then you integrate e to f that is with z and this actually is a triple integral of this rectangular volume f (x, y, z) dV. That is dxdydz. This is how. So once here there are six possibilities. There are three, three factorial is six or six possibilities of iterated integrals.

If one of them exists, then all of them exists and they are all equal to this. So simple example that I will compute for example, you just given here in the book is for example you take a box b, rectangular R bar zone, which is 0, 1 cross -1/2, 0 cross 0, 1/3 and my f in this case, so I have to evaluate over R bar the integral x + 2y + 3z whole square dx dy dz. So let us go by first with x then with y then with z.

So if one does that then, let me first compute the integral, so compute integral 0 to 1. So you write x + 2y. This is one of the best ways of writing. I learnt it in my undergraduate days that this is a very good way of writing. So you know a way to integrate put the function or the first variable you are integrating, put it here. Then the second variable that you are integrating is y in this case. So it is -1/2 to 0.

The third variable integrating is z in this case. Put dz it is zero to one-third. So if you do this integration, then of course, if you just keep these things on, then this let us see. I get x + 2y + 3z cube to the power 3, x is varying from 0 to 1 dy dz. If that is the case then this becomes 0 to one-third, -1/2 to 0. Now it will become function of y and z. So 1 + 2y. So one-third is common here, out, 2y + 3x by 3 cube minus okay I am not putting the 3 out. So let me keep it inside. That is how you go on.

Now you have to do it with respect to y and when you do it y, so your y is, so you will integrate this part. So you will actually do a substitution of this whole thing and then you will start working and then you will put back the whole thing. So if you work it out you it become, so you know how to integrate. I am not going into this integration business. If we integrate this whole thing out now you have to do in terms of dz. This is after you put the y thing. And the answer is 1 by 12.

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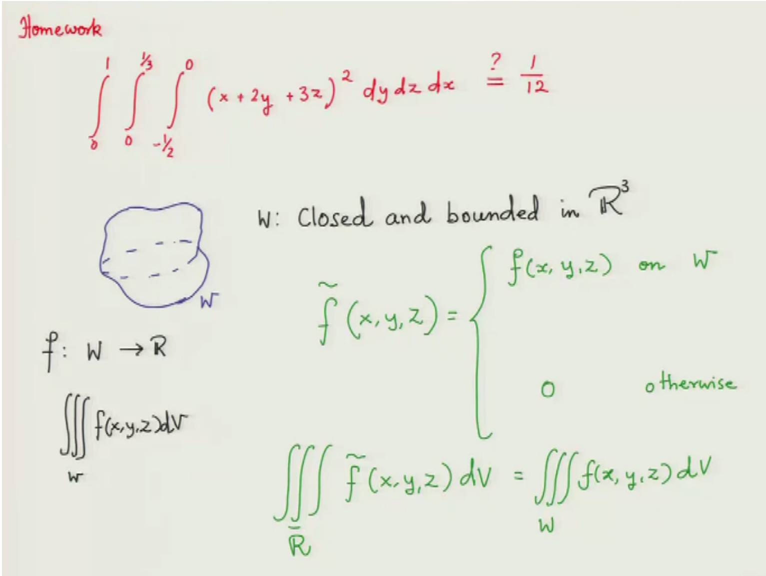

What I give you as a homework that instead of taking the integral with respect to x first, take y first. That is you compute the iterated integral x + 2y + 3z whole square, now I integrate first with y, then with z and then with x. So it is dy dx and dx. Now check whether this is also equal to 1 by 12. In fact, once one of them exists, all of them exist, and that will be the value of the integral.

So that is what you really have to check. And going forward what would happen if I am not talking about a region in space. Some region W in space, a closed and bounded region in space. So for example, W is a W is closed and bounded in R 3 in our standard three dimensional Euclidean space. How do you define integral? So what do you mean by suppose you are given, if you have a function f from W to R.

So how would you attach meaning to this integral? The idea what we have learned in the case of the double integral the same idea can be taken up. So what is idea. Idea is that you form a new function. Let me just write down. So what kind of function that I can form. So I form a new function f tilde (x, y, z) and this f tilde (x, y, z) agrees with f (x, y, z) on W and is equal to zero otherwise.

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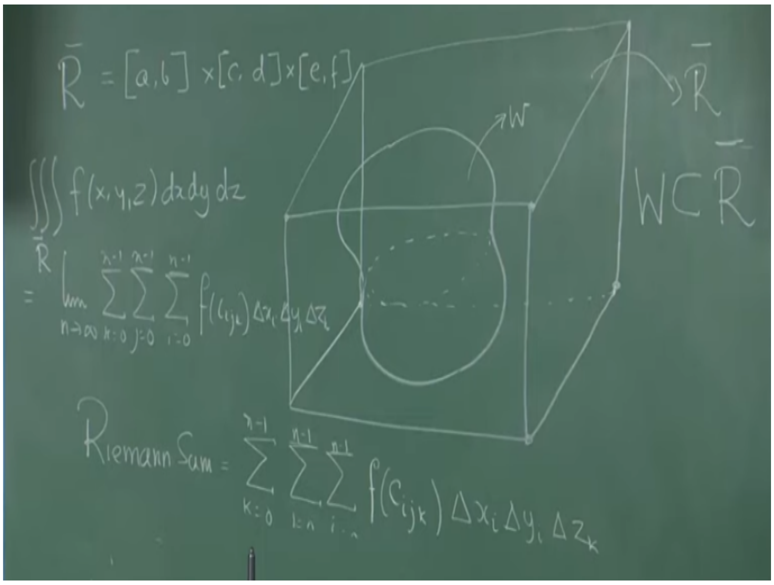

So what you do is that when you have this kind of this kind of space, this kind of zone, a closed arbitrary closed and convex zone, a closed not convex, closed and bounded region W and then you need to integrate over it and what you do is you put a box around it that create a box and put that zone in that put that thing in the box. Basically, isko box mey dalo, that is idea.

Because outside that zone on the box the function f tilde is zero. So we put this damn thing in the box. So you get a rectangle R, R bar and W blue is now properly contained inside R bar. You know now, we have just defined you know how to integrate over R bar. We have not spoken, we have just spoken in terms of iterated integrals or what do you how do we integrate our R bar? You know how to do it through iterated integration.

Now what is this R bar. If I really go by my formula for integrals, the one using Riemann sum, then if I have a region R bar a, b cross c, d cross e, f and I make n partitions, put n -1 points and make n interval partitions on all the of the same on all the three axis and hence I make a grid of small rectangular parallelepiped.

So I have the rectangular parallelepiped and I make small grids full of rectangular parallelepiped just like we make a grid of small rectangles in a given rectangular domain. Then, so call any point in the rectangular parallelepiped i, j, k; i, j, kth rectangular parallelepiped because we have length, breadth and height as c, i, j, k. So what do you do? You construct the Riemann sum.

The Riemann sum in this case looks like this. So here I would have three variables. So the Riemann sum must be summed over the three variables F (c ijk) delta x i that is x i plus x i to x i plus 1 that interval, the length of that interval which is and all the intervals are the same length; delta y j delta z k. This is the Riemann sum and in sum i going from 0 to n – 1, j going from 0 to n – 1, k going from 0 to n – 1.

Now my integral as n tends to infinity the integral over this over any rectangular region R bar of a given function f (x, y, z) is limit n tends to infinity of this Riemann sum that we had written. Of course, we are not going to really compute Riemann sums, it will be very complicated. So that is why iterated integral comes to your help. So you all we know about real variable calculus that is a major thing, and we build on that, everything has been built on that. So maybe I, this will guard it.

So I will write it here on the top. So maybe that is better. So the Riemann integral, ultimately is the limit of the Riemann sum. So the integral R bar f (x, y, z) dxdydz or dV whatever you want to write is same as lim n tends to infinity, summation, summation, summation i is equal to 0 to n – 1, j is equal to 0 to n – 1, k is equal to 0 to n – 1 f (c ijk) grad x i grad y j grad zk. That is the definition.

But that is that looks pretty complicated in it is slightly intimidating and you would not like to get into all this. But what is the idea behind this is that once you take f tilde and you know by iterated integrals, you can actually do this job of integrating f tilde over a given region R bar which contains the region W, then the key idea to observe here is that integral integral integral over R bar f tilde (x, y, z) dV is same as the integral of W of f (x, y, z) over dV.

Because I can decompose this rectangular region R which contains W bar into two parts one is W bar one is the compliment of W bar in R in and in this rectangular region R bar and then on those two parts I integrate. So I integrate them separately and on that first part, it is zero that integral goes away and the next part the function f tilde value is f (x, y, z). So that is what remains.

Now what sort of regions we are interested in? We will just like in the two dimensional case, we will not be interested in every sort of region but a sort of region which has a kind of well-defined boundary. And that is what we are going to now talk about, right.

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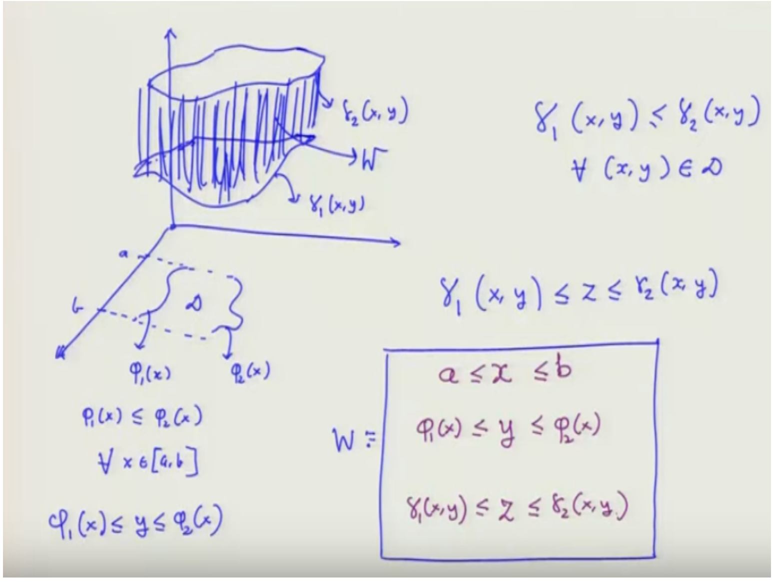

The sort of region that we are interested in for example, I construct this region where I first construct a region D in the xy plane, and then I have two functions, gamma 1 and gamma 2 of x, y. So I consider the region which is bounded by that. So I have x suppose I have x here given on a, b and I know that y my y varies from say phi 1 x to phi 2 x, phi 1 x to phi 2 x. So this is phi 1 x and phi 2 x.

Of course you know that phi 1 x of course is less than equal to phi 2 x for all x that you choose in the interval a, b. Then you have two well-defined on this domain D, you have two well-defined functional values and their graphs. Another is like this. Do not take my drawing seriously. So this is gamma 1 (x, y) and this is gamma 2 (x, y) where so what we now do is you see that I have drawn the surface the graphs of two functions phi 1 and phi 2 with gamma 1 and gamma 2 which satisfy this.

Do not take my drawing too seriously, it might look very strange that he just mind this guy is doing something and he is drawing them out of my mind. This is true for all x, y now in D. And you know that D is this region. Only x values are fixed a, b and this for y values are given through this functional values. That is y here is lying between so x so we are now concerned with those z's which are lying between the so this is the zone that I am considered about.

I am only considered about this zone and that is the volume over which I want to now integrate. This is my W. So basically I am now bothered about gamma 1, x, y less than z less than gamma 2. So we are only considering those x, y, z which is lying between these two. So we have now defined the region as follows. So finally, this region W is defined as follows and I can put it in a box.

This region W has x a to b has y between phi 1 x and phi 2 x. But that is not the only way we can have regions here. Please understand, this is one of the ways. You could have had this domain on yz axis yz plane also and looked at the volume in this way, this form. That you would define a function of in terms of you could have fixed up your y and defined two functions as a function of y and z lying between them and define a function of y, z two functions of y, z and x would lie between them.

You could have done like that also. But this is one way of doing it, one type of region basically. And then z is lying between gamma 1 x, y and gamma 2 x, y. Let us see an example of how this can be done.

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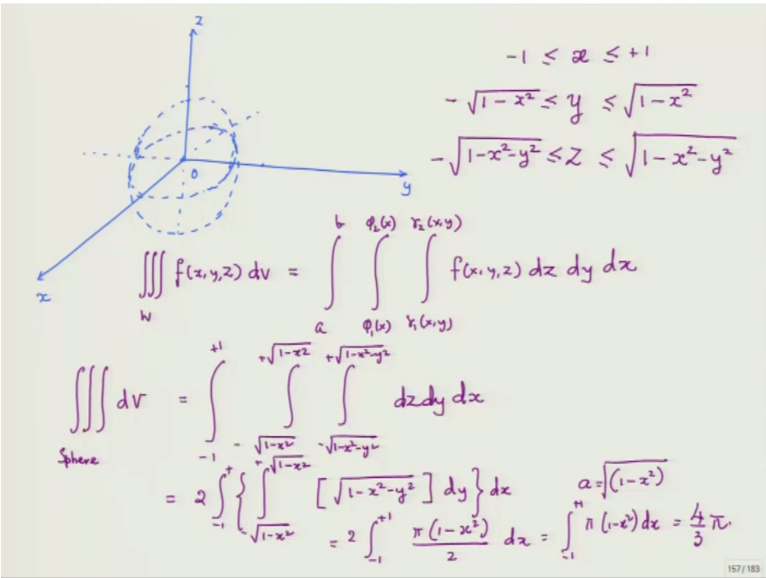

For example, you would look at a unit ball unit sphere, center at zero. This is x. So here is the cuts. So you draw first the unit circle. It is lying on the xy plane which means now y is varying given an x, y is varying between these two parts. So x now is fixed between -1 sorry +1 and -1. Now y varies between the two parts of x because given an x, x is plus minus root over y square.

So it is so it y varies between so y can take any, given an x y can take any value between this part and this part if I am considering this whole region as a domain D. So this is my domain D this circular zone that you have, not just a circle. So y varies between, so this is my phi 2 x. While z is the so on the top there are two parts. Given any x, y you get two and you take or draw a perpendicular through it you hit two parts of the sphere.

So one part the circular the top spherical part is given by root over one plus this part is the positive part, this part is the negative part. So the upper part is given by root over z equal to root over 1 – x square minus y square. The lower part is given by minus that is minus root over 1 minus x square minus y square. So z in this case is lying between okay now, let us use this idea.

So once we have this idea then so what we what we can do? Suppose we have a scenario like this. So how do I write the integral. So integral of W f (x, y, z) dV. Now so, what we have done? We have how we have put z, how we have put y, then we have put x. In this case the way we have just done it. In this case we will first put the z case.

That is we will first integrate out with respect to z and have a function of x, y, then we will integrate out with respect to y function of x and then we integrate out x. So gamma 1 x, y to gamma 2 x, y. So I am writing down the iterated forms dz phi 1 x, phi 2 x dy and then a to b dx. So that is the way to compute it. For example now I take the unit sphere, right? And I compute the volume, volume of the unit sphere.

What is my function value here is one because volume the total volume is computed through this. So sphere dV. So here I know how I have written it down. So I will use this iterated integral scheme. So integral minus root over one minus x square minus y square to plus root over one minus x square minus y square dx. Again minus one minus root over one minus x square plus root over one minus x square and minus one to plus one. So I will write here dz. So first I do with z just away here dz dy dx.

And if I integrate it out, if I do the first integration, I get -1 to +1 minus root 1 – x square to plus root 1 – x square and then if I integrate this out, what will I get? z so I will get what I will get? I will just get a z with these limits. So that will turn out to be 2, 2 will come out 2 into root over 1 – x square minus y square. So it now becomes a function of x y z goes away dy. And this has to be integrated out and then dx.

So again we will have dy and this will come in. So how do I integrate this part? So this is nothing but a circular zone. If I fix x and then I put say a equal to 1 – x square to the power half root over so this whole thin can be written as a square minus y square. And you know this is nothing but a circle, line between this and this, between –a and +a. I am putting a equal to this. So integral between –a and +a.

So if you integrate it, it will be nothing but half of the circle between –a and +a. Just the positive part. It will be pi a square by 2. So this integral would be 2 will go cancel out and finally this part will become -1 to +1 pi a square. So pi into 1 – x square. So that is what I have to integrate out, pi a square by 2 dx. So this is evaluated very cleverly. That is taking root over 1 – x square as a, then this is a square.

So a square minus y square –a to +a. This is nothing but integrating on the upper part of the circle between –a and +a, so it is half the area. It is a area of the semi circular zone, which gives you pi a square by 2. So a square is 1 - x square, so pi a square by 2 into dx. Of course, you can actually put in all these things and get the same answer in the normal way, but any way.

Two into because here you have to observe that there is a y here. So you have to really integrate. Here carefully, very cleverly, take away the tough part of the integration. You do not need to put the formula a square minus x square and all those things. So -a to +a you do it and that is what you get pi a square by 2. Now 2, 2 cancels and you have integral -1 to +1 pi 1 – x square dx.

And the answer here comes out to be pi goes out, 4 by 3 pi. Lo, the radius was one. So 4 by 3 pi r cube. That is the volume of the 4 by 3 pi is the volume of the unit sphere. And that you could find so nicely by integration but using this kind of ideas. Please understand that this is not the only way to break up a zone, right. There could be any other ways to break up a zone and you can try out your own problems. For example, you can break the zone as follows.

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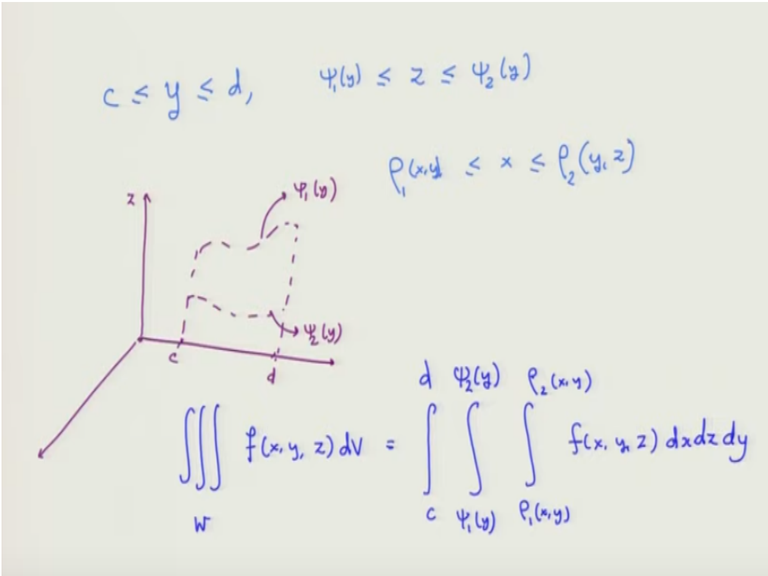

You can have fixed y between c and d, that is what I told you. Now you can vary z between two functions of y. So z will be between psi 2 y and psi 1 y. So basically here you are considering y between c and d. And then you are considering the domain D to be consisting of two functions. So this is z. So it is a function of., so z is lying between two functions. This is your psi 1 y and this is your psi 2 y.

And now here just which where everything comes out of the plane of the this board this electronic board, I can have two surfaces, one which is always lying above the other or to or more towards the positive side of x axis then the other is I can now talk about a function say rho. So now this domain is on the y, z plane. So you have a function on y, z and x can lie between them.

So in this case when you integrate, in this case when you integrate so what is your integration policy? So if you want to integrate out in this and everything is in this particular format, you want to integrate out W is the thing lying between these two zone, the x, y, z between these two zones. And I want to integrate out this. Now if you run an iterated integral, first you have to take off the x.

Then you take off the z, z goes from phi 1 y to phi 2 y. And then you have to integrate out the c to d. That is the way. So you have a probably, it depends on what way you are looking at the region, which way you are looking at the region, whatever be the way you look at the region, there is another way also. You can take the x, z plane and do the other things. So you can try out your own problems.

There are many ways to do things. So the several problems you can try it from the net, etc. But this is the way to go about it. So we have made quite a discussion of the triple integrals and the next class we are going to see the how change of variable or substitution technique can work in this multiple integral setting. Thank you.