Calculus of Several Real Variables
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Lecture - 23
Multiple Integral - III

So welcome to the fifth week, we saw multiple integrals. So multiple integrals three as per the course plan.

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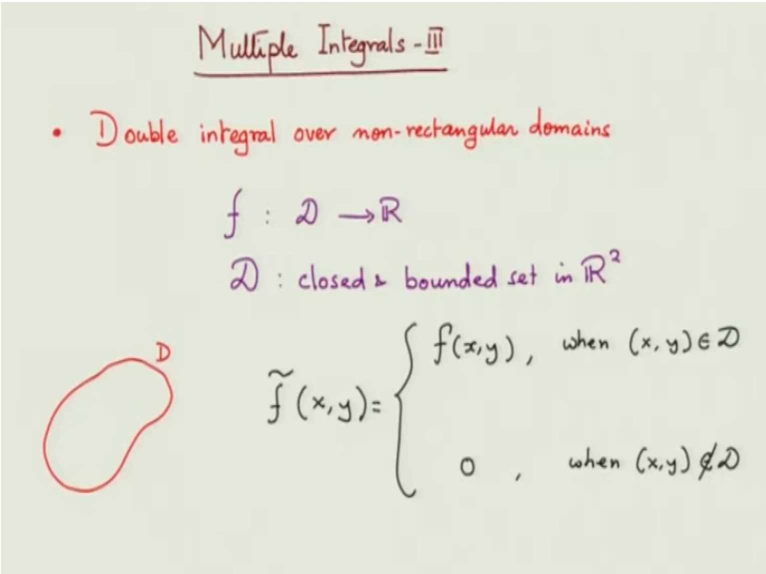

So here today our focus would lie on double integrals over any given domain, double integrals over non-rectangular domains, over non-rectangular domains. Suppose I have a function f defined over at domain d to R where the domain means always it is a closed convex sorry closed and bounded set in R 2, okay?

Now if you have this given domain D how do you integrate? How do you make the grids and all those things that you could do for the rectangular case. The idea is to make an extension of the function. So how do you extend the function? We would say construct a function f tilde x, y such that this function coincides with f (x, y) when x, y is in the domain D and is equal to zero when x, y is not in the domain D.

Not in the domain D. This little idea of creating another auxiliary function which coincides with the value of f when x, y is in D and is zero otherwise, goes a long way in taking the definition that we know for the rectangular case to that of the general domain. See, what happens here is very simple.

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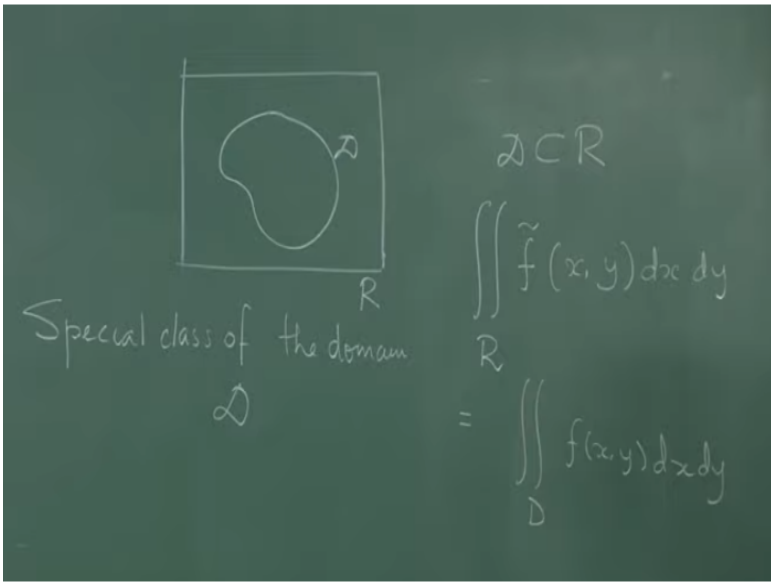

Is that, if I am having a domain here in R 2 which is non-rectangular. And if I define the function like that, then take a rectangular domain which contains D and this is D. Here R is a superset of D. R contains D. That is a bad way of writing of course. I should rather writer as D is contained in R. So D is contained in R.

So what I am, what we are going to harp on this very simple idea that if I integrate this function f tilde over R, R which contains this D and this is exactly same as integrating over D the function f (x, y). Because, at the end when I am integrating this f tilde apart from points in D, this function value is zero. So the integral has no effect, the integral becomes zero in that zone.

The integral actually becomes zero in that zone. There are many ways to look at it, this is one of the ways to look at it. You can figure out other ways too which please understand there are many things about multiple integrals and many applications. Everything cannot be shown in this little course that we are doing. So it is really not possible to do each and everything.

So this is exactly the idea that we are going to harp on that if I can get a rectangular zone like this then integral if I integrate this over this is the same as this. But in our case, we are going to talk about we are going to talk about very special classes of the domain D. We are not going to take any arbitrary domain D, the domain D for us must have a structure and what kind of structure they would have.

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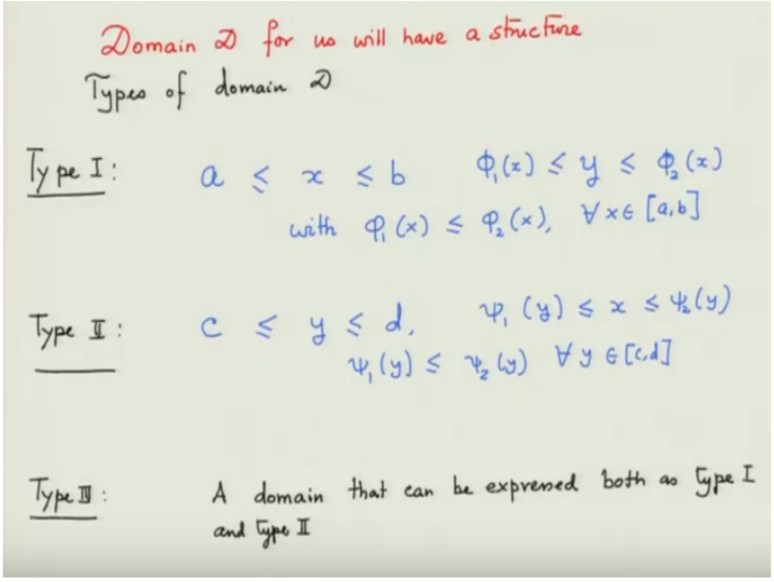

So these domain D's, we will talk about the domain D, for us will have a structure and the book calls it the elementary region, right? So we will have three kinds of domain D's. One is type one, one is type two, one is type three. So let me write down what are the three types of domains. So we will have type I domain, type II domain, type III domain, which we will draw on the board what these things mean.

So here we have types of domain D, okay. Type I domain is when I have given you what the domain where x lies and I say that y lies between two functions of x when x is defined between a and b, y lies between phi to x and phi 1 x. So let us draw what this domain D means in this particular setting. This type I domain, what does this type I domain actually mean?

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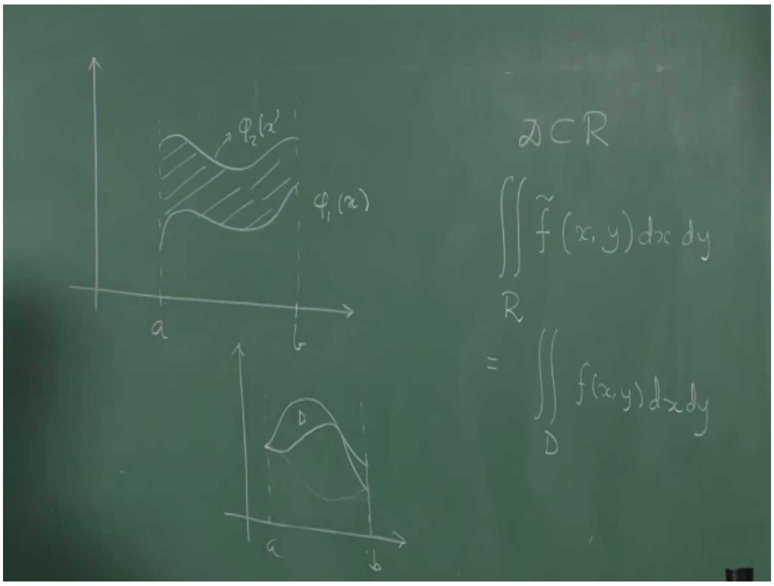

The type I domain means, so here is your x which is from a to b. And so you have a function phi 1 x defined like this. You can have phi 2 x defined like this, but that is kind of domains what we are not expecting. Here you will have an important thing to remember with phi 1 x always less than phi 2 x for all x in a, b. This is a restriction that we have to maintain.

That is phi 1 value can never go above phi 2 value and that is how you create the region. It could be like this. So this is phi 2 x. So and that creates the domain D. There are many other ways also. For example, it could be, it could be like this. Of course, it could be as bad as like this. So that is also domain D, domain D of the type I.

Domain D of type II is of course, you have y given to you from c to d and x will vary from psi 1 x, psi 1 y to psi 2 y, the x will lie between them where psi they will psi 1 y is always less than or equal to psi 2 y for all y in c, d. And type III domain is something that can be expressed both as type I and type II. A domain which can be expressed both as type I and type II.

Every type I domain cannot be expressed as type II domain. A domain that can, many of the domains are type III domains of course, can be expressed both as type I and type II. For example, I can draw a type II domain which cannot be expressed as type I.

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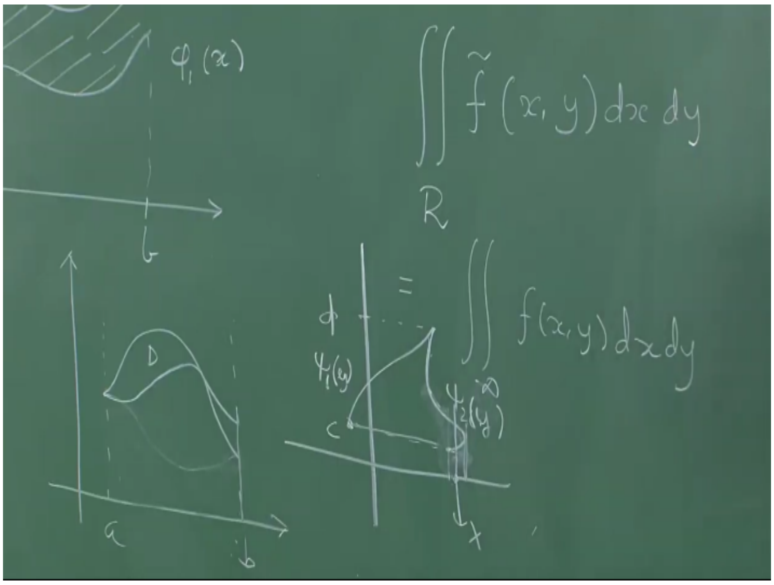

For example, I can have a type II domain. So here is your d and here is your c. So it is something like this. So these two are functions or y. This is psi 1 y and this is psi 2 y. Now note I cannot express this as a type II because I cannot express these curves as function of x because if I take an x here in between this zone. So this is my a, b zone and I can take an x here.

Okay, I am just wait wait I will just try to make it a bit like this. For example, it is like this. So now if I taken an x here, so it comes up to here, is the maximum value. So if I take an x here, you see here corresponding to this particular x there are two values of y. So this curve cannot be expressed as a function of x. So it cannot be made into a type I domain. For a type I domain it has to be expressed as a function of x.

We will come and show with some examples for type I and type II, how can we handle both these domains through examples. That is much better. What is important now is to show that this iterated integral business can be still carried on in this sort of setting. Now how do we start those iterated integral business. The idea is very simple.

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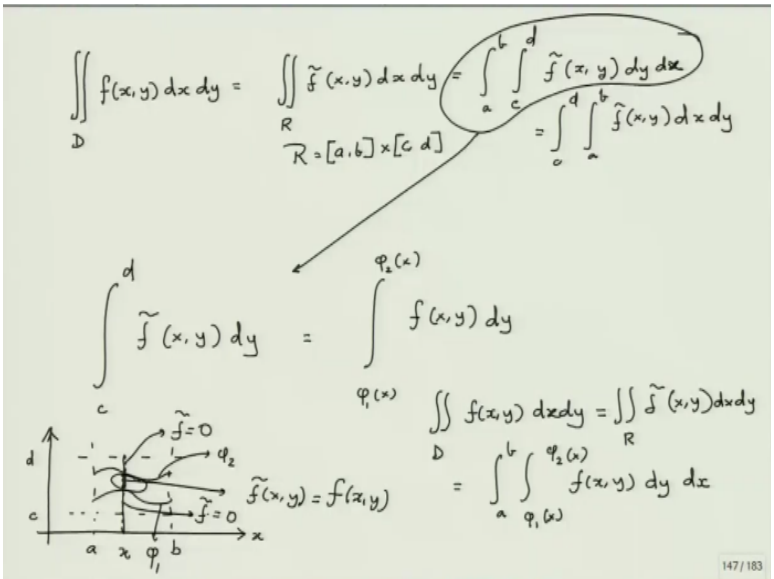

So we are looking at this very simple definition what we wrote down that the integral D of f (x, y) dx dy is nothing but the integral over R. Let the rectangular region be a, b and c, d. So it can be written as say a, b. So let us now take R to be of the form, so whenever I am writing these kind of things, you just take R of the form a, b, c, d; a, b cross c, d. So we will always write it like this f tilde.

So that is, now this again this integral is again equal to integral c to d integral a to sorry this is dy dx. Because here first I integrate with respect to y. So it is dx and then dy, okay. Now let me try to compute such an iterated integral. In this iterated integral in this particular iterated integral, let me try to compute because if I have to compute this, I have to compute this. Let me compute integral c to d, f tilde (x, y) dy.

Observe that now it has just become a function of y because I fixed my x. So when I fixed my x, when I fixed my x, so what I have done is the following. So here is my zone. So let me consider a type I type of zone. Here is a rectangular zone. This is a, b and this is c, d. Now what I have done, this is suppose this is my phi 1 x and this is my phi 2 x. This is my phi 1 and this is my phi 2. So I have taken a particular x.

So I have taken a particular x. Now, I am trying to integrate the function on this thing as if it is a function of one variable. Now in this case, you know that the function value will coincide with f tilde will coincide with f (x, y) only in this part, the part which I am now marking. So it will be only coinciding with f (x, y) in this part. In this part f tilde (x, y) will coincide with f (x, y).

Remaining part, this part f tilde is zero. On this part also f tilde is zero. So that is very important. You have to understand that d here is bigger than or equal to phi x because the rectangle has to contain the whole region d. So d value has to bigger than the maximum value of phi x and c value has to be lesser than equal to the minimum value of, it has to be lesser than the minimum value of psi x of phi 2.

So d has to be bigger than the maximum value of phi 2 x and c has to be lesser than the minimum value for phi 1 x. Then only it is containing everything. Then only the whole domain is getting contained in that rectangle and because that is what we want. So see if I integrate here I know that only on this part, the part which is cutting through the domain that is for those y values which are lying between phi 1 x and phi 2 x, the function value of f tilde (x, y) coincides with f (x, y).

Otherwise, the function values are zero. So this integral is nothing but phi 1 x to phi 2 x f (x, y) dy. And once you know that, then integral D now because this is same as R dy is integral R of f tilde (x, y) dx dy. That this now would be same as integral a to b integral phi 1 x because I have computed the integral part the c to d f tilde (x, y) I have computed and that is nothing but phi 2 x f (x, y) dy dx.

That can be done in the other way too. If you can write it as a type II zone I could switch it you write this as a to b, integral c to d, integral psi 1 y to psi 2 y f (x, y) dy dx. That can also be done. So if you write it as a type II zone, we will do it in the different way. We will use the iterated integral in the different way, right? Why you should try both the ways?

If you get a, suppose you have a type III region where you can express it both as type I and type II what happens sometimes that if you take it as a type II region, type I region it is difficult to compute the integral. But if you take it as a type I region it becomes a type II region it becomes easier to compute the integral. That is the kind of thing that might just happen.

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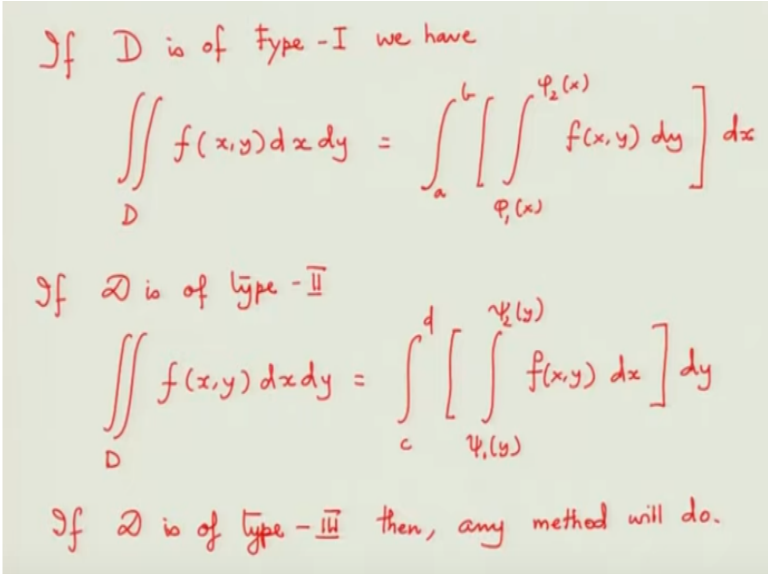

So summarizing what we have we write if D the domain D is of type I, we have integral D. See this iterated integral story is been actually exploited very much here. So it is of type I integral a to b, integral phi 1 x, phi 2 x, f (x, y) dy dx. If D is of type II, so if D is of type II then integral D f (x, y) dx dy must be of this form. Now you have to change, you have to change the role of x and y.

So it will become c to d integral psi 1 y to psi 2 y, not phi 2 but psi 2 y f (x, y) dx dy. This is the way you compute it. Now if D is of type III whatever or anything is applicable, you have to choose the way you want which is easier. If D is of type III then any method is applicable, okay. So this is how we have defined the integral over the domain when you have this very specialized form of the domain, okay.

So let us look at a case where the region is actually of both types. So first example will be at a case where the region is of both types.

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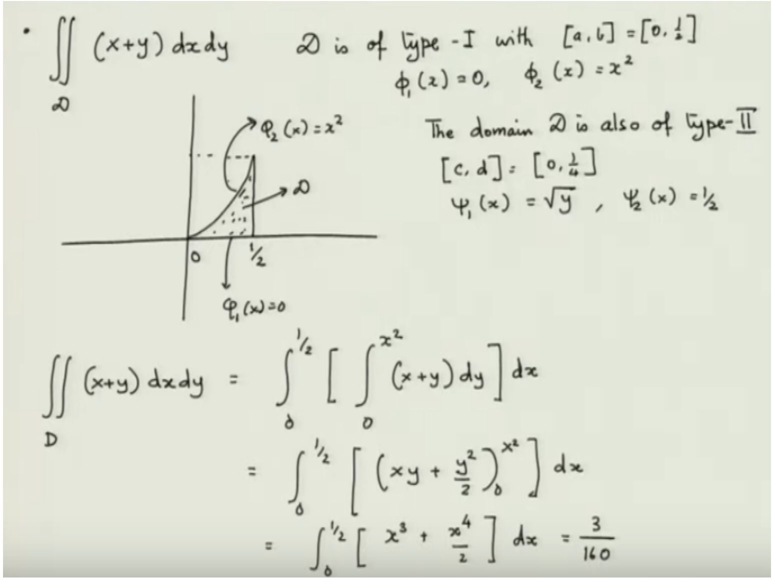

So for this example computation let us first we will just use the black pen. So in math black is pretty interesting. So integral find this. So D is of type I with a, b because x has to be x zone has to be specified 0 to half and phi 1 x is zero and phi 2 x is x square. So if you look at the zone on which you are integrating; so look at the zone on which you are integrating zero to half.

And phi 1 x is zero phi 1 x is zero and this is phi 2 x equal to x square. Of course, we are maintaining that phi 2 x is always bigger than phi 1 x. This x square is anyway greater than equal to zero. So this is my domain D. This is a type I domain but later on, we will see that this can also be expressed as a type II domain and you can check out the answer.

So because it is a type I domain, I have a very straightforward thing to do what we have learned in the last slide. So it is zero to half integral zero to x square x + y. So you see here, this is very important. Now what is the, this is a type I domain. So my y values are lying between phi 2 x and phi 1 x. So for every x, the y value, so I have fixed my x, so y is lying between 0 and that particular x square, that particular free states. So you evaluate that and keep on doing it.

So do it for all the y basically and do it for all the y. That is the idea and do it for all the x, right? So this will give me zero to half. This works out as x y plus y square by 2 zero to x square dx. When it is 0, so when I put y is 0, it is 0 x square. So it will become x cube here. And zero part is gone. So I will just have to bother about putting x square here, x cube plus y square. So it will become x 4, x 4 y 2 dx.

So if we integrate it out this will simply become 3 by 160 as answer. So is that is the domain D is also of type II? The domain D in this particular case is also of type II. The domain D in this case is also of type II because I can write y because when x is equal to half phi 2 half is one forth. So I can write c, d this domain of y is between zero and one-fourth. Phi 1 x I can write as root y.

This function is a inverse in this part is the inverse of x square. So root y is equal to x, positive root. So it is root y and phi 2 x is half, this is half. And psi 2 x is half. So you can try out this particular thing for this particular problem. And then we can think in bit more detail. Now we want to show that sometimes when you have type II, so for example, you have this case, let us draw a particular zone.

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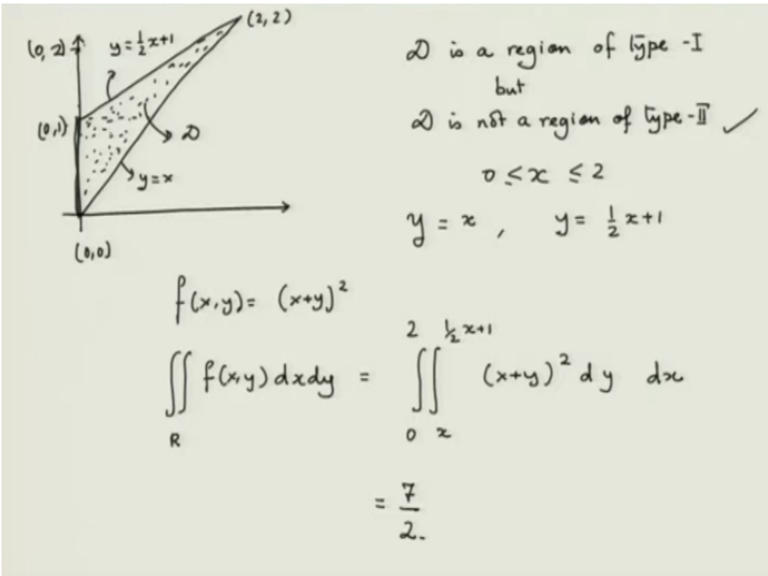

I am drawing a particular zone. The region we are going to now draw is of type I, but not of type II. You will think why. So in the x axis here is 0, 0 here is 0,1 and here is 2, 2. So x is varying from 0 to 2 and this is the region we want to integrate on. This, so y is varying also from 0 to 2. Observe here that this region D, this region D is a region of type I but D is not a region of type II.

D is not a region of type II, why? So I have said it is not a region of type II. Why it is not a region of type II? It is a region of type I, it is clear because here you have a fixed x is from 0 to 2, that is clear. And y is lying between in this particular case the y domains are y is moving from y equal to x which is this line. And also this line is y is equal to half x plus 1.

But if you look at it from the perspective of y, when now you take y as your main axis, so between 0 to 2 and you look at the D. Then you do not find two functions binding this zone, domain D. There are should I take these two only icon because y can take value up to 2. So y will also have value from 0 to 2. So can I take just these two curves and talk about the domain?

No, because I have to consider also this part of the curve. I have to also consider this part. So there is a third entity coming in, not just two entities here. Not just these two lines, two functions, which I can write express or express them as so y is equal to x to y equal to half x plus 1 which I can write as y 2y -2; x is 2y – 2. So this for me, it has become x is 2y – 2 to x = y. But what about this part?

How will I take into consideration this part. If I come from here to here I can take I will take I will consider up to this part. So if I am y if I am in the yth segment if I am coming from here to here 0 to 2 to so this is my 2; 0, 2. When I am coming from here to here, up to here, I am considering it this domain, this domain can be thought about in terms of these two curves.

But when I am coming from here to here, the domain is now I have to thought of in terms of x equal to zero and this curve. So there are two different integration, two different regions that has to be posed. It can be done in terms of those thing also. First integrate over this part and then integrate over that part.

So if you just go do if you want to do it in one shot, if you want to look it as a type II domain it is not a type II domain because you have to bring it bring the whole thing into two different parts. So it is a composition of two type II domains. Is not just one type II domain. So when a type II domain, so it cannot be taken as a region of type II, whereas we can do the whole integration in one shot.

So here if you want to do the whole integration in one shot, you have to use the type I game. So suppose you have an integral whatever integral say in this example, it is f (x, y) is equal to x + y whole square. Then if you want to integrate then you have to maintain that order of integration. So here you see I have to maintain that thing, because I know it is not a type II domain, I cannot do anything.

Because in type, this is actually composed of two different type II domains. So I have to separate the integrate, you can try out that. But here if I just want to integrate in one shot, I have to use the fact that it is a type I domain. So I will have to go from x to half x + 1. That is the way I have to write it, x + y whole square. So dy first integrating over y and then dx okay. So if you integrate it out, the answer would be 7 by 2.

So here we have a simple idea of how, you will see there are several cases where the region is of type II and type I, that is region of type III. So it is both. So sometimes, if you use the type I approach, it is hard to compute. If you use the type II approach, it will be easy to compute. For example, I will just mention this and end my talk.

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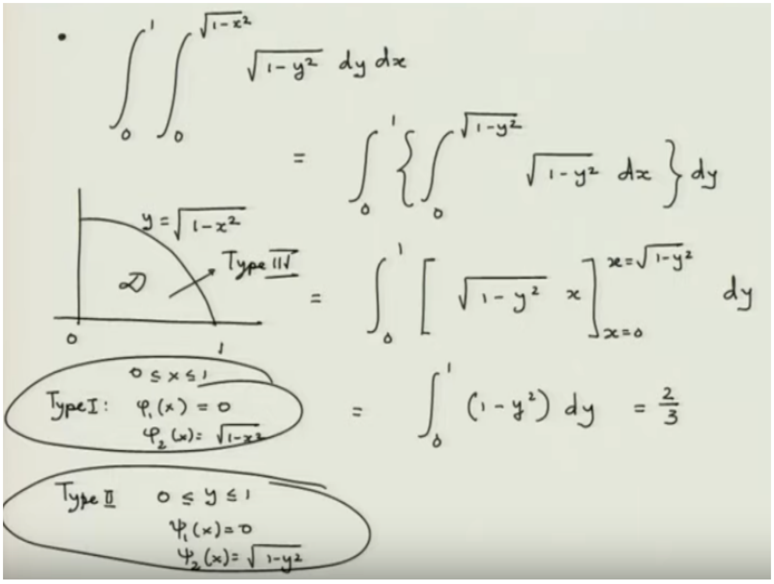

For example, if you take this integral. I want you to so this is and you are integrating in the type I approach. This is called a type I approach where y is viewed as a function of x. If you want to integrate this, root over 1 – y square dy. So basically, it is a kind of circle and but if you want to write it in that form, it will become very tough to do it. You will f (x, y) is root over 1 - y square. Then the whole integration becomes much more complicated, right.

But if you want to do it much more simplistically, much more simplistically, you can write it as a type II integral because in this region that we are talking about is a circular zone, 0 to 1 and this is y is root 1 – x square. So similarly, you can, this is this can be thought about, this is a type I domain where phi 1 is zero and phi 2 is root over 1 – x square.

And also type II domain where psi 1 is 0 and psi 2 is root over 1 – y square. So it can also be written as 0 to 1, 0 to root over 1 – y square, root over 1 – y square dx dy. So if I had done just this, root over 1 – y square dy into root over 1 – y square. So this root over 1 – y square would have given me in a 0 to root over 1 – x square this one.

So root over 1 – y square would have given me the area of the circle for the fixed x. So area of the area of a circle. So this is the function of, this is area of the circle of radius 1. So pi of R square but here I am not taking up to 1, I am taking up to root over 1 – x square. So you cannot just simply use the area of the circle to easily integrate it. Here we have to use the formula and then put the limits of y.

You have to use the formula for the y and put the limits and that is a very complex integral and complex writing. So what do you, do you change it because it is also type II zone. So it is type I. It is phi 1 x, x is lying between 1 and 0 and phi 1 x is 0, phi 2 x is, it is also type II with y lying between 1 and 0. Psi 1 x is 0 and psi 2 x is. So if when I am using the type II thing it becomes very easy. So it is in terms of dx .

So this is constant. So 0 to 1 root over 1 – y square x, x = 0 to x is equal to root over 1 – y square. This is not the case, if I use this formulation, if I just integrate over y, try to integrate this function in terms of y. This whole thing now into dy. So this is just giving me 0 to 1. This is also 1 – y square. So it becomes as 1 – y square dy and this integral is very simple. I do not have to tell you the answer is 2 by 3.

So you see that a region which is both of type I and type II, this region D, this region D the one we have drawn, this is both type I. So these are type III basically, both type I and type II. So type I because of this, type II because even we presented it as type I in this form and type II in this form. So what happens is that if I take it in the type II form I can do the integral much more easily.

So when you have type III region you have to be very careful about which integral can be easily done. So with this, I end my talk here and in the next class we are going to talk about triple integrals, okay? Thank you very much and see you tomorrow.