Calculus of Several Real Variables
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology-Kanpur

Lecture - 22
Multiple Integration - II

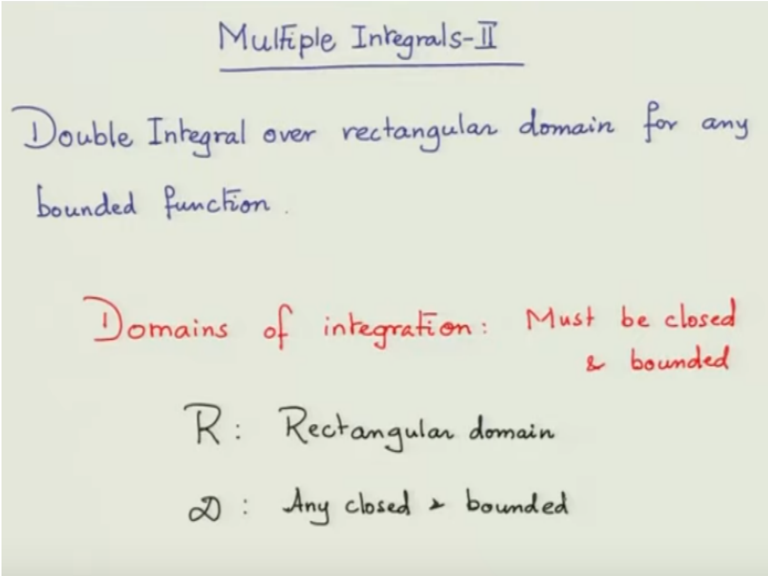
So welcome to the second lecture of the fifth week. It is on multiple integrals. It is very important that I tell you certain things right at the beginning. That if you have seen me drawing a domain D, if you have seen me drawing a rectangular domain, please understand that we are always integrating or finding double integrals over a closed and bounded set. A set which has something inside and it has boundaries.

In mathematical parlance, those who knows some bit more of geometry or topology, would understand that these sets are called compact sets. Any closed or bounded sets in R 2 are referred to as compact sets. Our domain has to be a compact set. It cannot be that okay, it has a boundary or it does not have a boundary or it just goes open in one direction that I can be in the set and remain in that set forever.

We would be looking at those sets so that I cannot start from a point in the set and move in any direction and remain in the set forever. For example, if you look at the first quadrant of in R 2 called R 2 plus, that is the set of all coordinates x, y, where x is bigger than or equal to zero and y is bigger than equal to 0. Start from any point 1,1 and you move along the line y equal to x equal to 2,2 3,3 4,4 and you keep on moving.

You will still continue to remain in the first quadrant. So that so such type of things, such type of sets, we will not consider where we are learning, multiple integration now. We will those are improper integrations though we will not consider it here at this moment at your level. They are not going to get in delve into such issues right away here.

(Refer Slide Time: 02:24)



So I want to say that whenever we are talking about domains, domains of integration, the domains of integration has to be closed and bounded. Now we have tried to understand in the last class double integral, we interpret a non-negative function over a given domain D as a volume under the graph with one the domain D as one of its end surfaces. We had done some integration on the rectangular domains.

And of course, we figured out that we can interpret that also as a volume. So we have started looking as we moved in the last class if you have observed, we had started coming into more and more rectangular domains so it is much more easier to handle. Now this allows us this would force us to make two different significant notational changes. One is that when I am using rectangular domain I will use the symbol R.

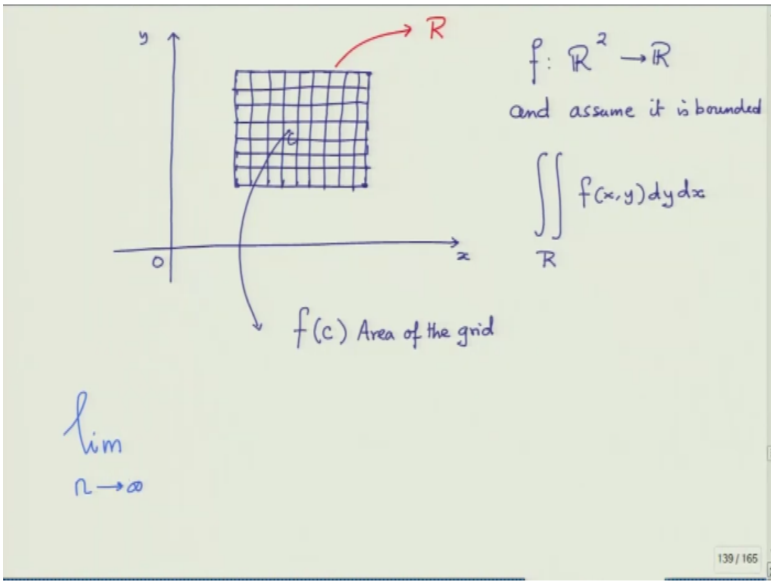
When I am using a symbol R means I am talking about rectangular domain that is it is a Cartesian product of two closed and closed intervals in R and D any closed and bounded domain. So this is very important to keep in mind, these two symbolically stuff that I am writing now, because that is what I will follow through.

Yesterday even for the rectangular domain, I wrote D because we were talking on general domains and then we said okay, you can have if the scenario is rectangular, you can write it as iterated integrals and all sorts of things. We have not told you how to handle the double integral over general domain using iterated integration. That we will come next.

So first what I will do is that knowing what we mean by fx, mean by the integral of a non-negative function over a rectangular domain, which is a volume contained under it, volume of the solid contained under it, we go on to define now the definition of any bounded function. Any function f (x, y), which is bounded on a given rectangular domain R, how would I go ahead and define the integration, okay.

How will I define an integral? So my scenario is, I will just not use the board that much today.

(Refer Slide Time: 05:50)



So my scenario is the following. I am trying to not talk about the which Riemann would do but in, so we are talking about Riemann integration in two dimensions. So here is my rectangular domain. I am drawing it like this but it has all its boundaries given. There is the x axis. This is and this is my rectangular domain R. Now how do I start integrating on it?

So now I will consider a function f from R 2 to R and assume it is bounded. I do not bother much whether this function is bounded, whether this function is non-negative or not. And I seek to find the definition of the integral over a given rectangular region of f (x, y) dydx. I seek to find a meaning to this integral. What does this integral mean? That is the thing that I want to seek and find.

Can I do some kind of a partitioning. You might say okay, it is not a big issue. How will I do that partitioning. So let me do one thing, I divide this, I will draw some lines parallel to the y axis within that rectangular zone. Of course, you can say take the function f to be bounded only in the rectangle or zone, that would also do, but we are just making life simpler and simpler.

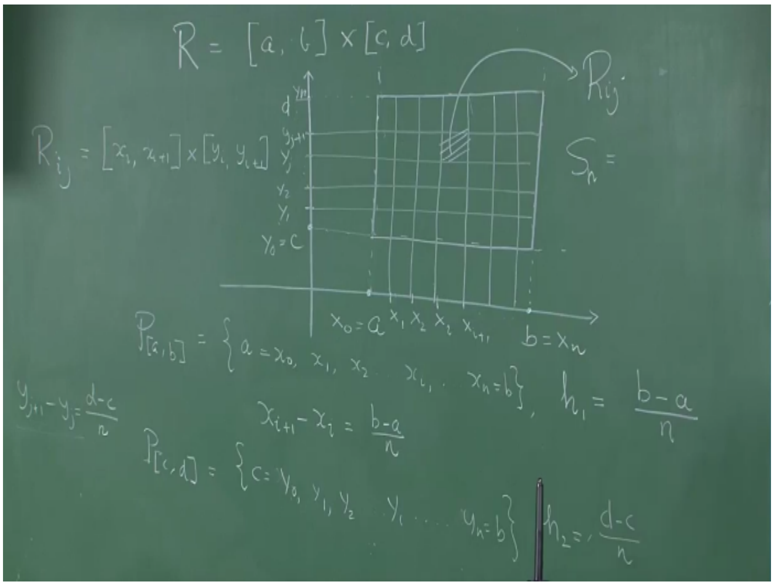
And then draw some lines parallel to the x axis. So what we have formed is a rectangular grid. And in that rectangular grid, I would know the area of each rectangular grid. So take, for example take this rectangular grid and take any point c from here, c. So I will look at f (c) into the area of the grid. So function value into the area of the grid.

That is what I will do, kind of constructing us kind of cylinder, or a parallelepiped. And that is trying to construct such a thing. So this I will do for just one of them. And actually, I will do for all of them. I will take some arbitrary point c and then I do it for all of them and get my and then sum that and that is an approximation of this integral. Then I take much more and more. I see here there are a number of cells, right?

For example, here you can count what the number of cells are. Then I can keep on increasing. I am basically I the partitioning that I have here on this side of the rectangle, rectangular domain and this side can increase. And the number of cells would increase and hence the sum would become more and more and more refined. It will be more and more near to the integral and in the limiting sense, it will we can define that limit as the integral itself.

It is exactly the partitioning process, what we see for Riemann integration. And that partitioning process, we will explain it. Now we will be much more in more detail. So now, what we can do is to consider R rectangular zone is expressed in terms of the Cartesian product of two intervals.

(Refer Slide Time: 10:36)



Which anyway this rectangular zone that is c drawn here on the screen is also of the same type, this one you see in the screen. And here, what I can do so what would happen is that you now partition a, b and you partition the side c, d in equally spaced interval. So here is my x axis, I am keeping everything on the x, on a positive side. It does not matter it is all may not be positive. So here is my a, b, a and b and here is my c, d.

So this is my rectangular zone. So what I do instead is I partition a, b with starting x 0, x 1, x 2, x i, x i+1 and to this is x n. So a, b is the partition of a, b. Partition a, b consists of points a = x 0, x 1, x 2, x i, x n = b. And now they are equally spaced. If there are n such points, right there are n + 1 such points so there are n such gaps between them, n such intervals.

So if they are equally spaced, then the length must be divided by the number of intervals you want. So your interval length h, say h 1 because I am taking it on a, b; h 1 the interval length is given by for any x i +1 minus x i by sorry interval length is given by the total length b – a divided by n. Basically what is happening is that you take any interval x i + 1 minus x i is equal to b – a by n.

A similar partitioning you can do of y with starting your y naught at c and ending with y n at d. So we start with this as y naught and then put y 1, y 2, dot dot y j, y n. So in that case the partitioning of c, d is equal to c equal to y naught, y 1, y 2, y i, y n = b.

With h 2 the gap is nothing but d – c by n. That is you can always write here that y j + 1 minus y j. So j is equal to 0 to n – 1 actually.

Or maybe I can write this as j and put that as k that does not matter. I put i and that is j, so okay. So here this is this distance is nothing but d – c by n. So this what I am, what I have done. So what I do now is corresponding this through x, y I draw a line perpendicular to the x axis x 1 which will be become parallel to the y axis. Through x 2, through x i, through x i + 1 and so on and so forth.

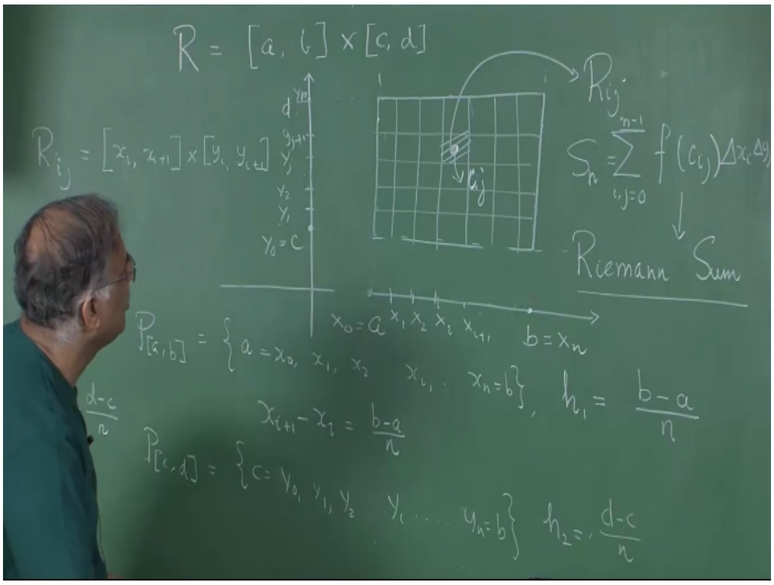
Basically and the same thing, okay. We have assumed that it is bounded. So R jk sorry R ij is the rectangular zone formed of the Cartesian product of x i, x i + 1 plus y i, y i + 1. This gives me x i, x i + 1, this one with say some y j, y j + 1, this zone, this one. This is y j + 1. So this one, this gives my rectangular zone R ij. This gives my rectangular zone R ij, okay? I think that is clear, that is visible.

I have no clue on the sign of the function. So I am constructing a sum, the nth sum. So I am taking any point c ij from R ij. So for each such i and j, so i and j starting from 0 to n – 1, f (c ij). So c ij is some point here. So this is your c ij. It is like a matrix formation that is all, c ij. So c ij, so f (c ij) is into, this is called the delta x i, the gap between these two into, so f (c ij) into this area. That is delta x i into delta y j.

So x i into delta y j. What we are looking at is the following. What we are looking at is the limit is the limit. So this is called the Riemann sum. So here of course, f is lying, you can obviously talk about an upper approximation or lower approximation because f is bounded by say capital M and has a lower bound small m. So this thing this whole thing is called a Riemann sum.

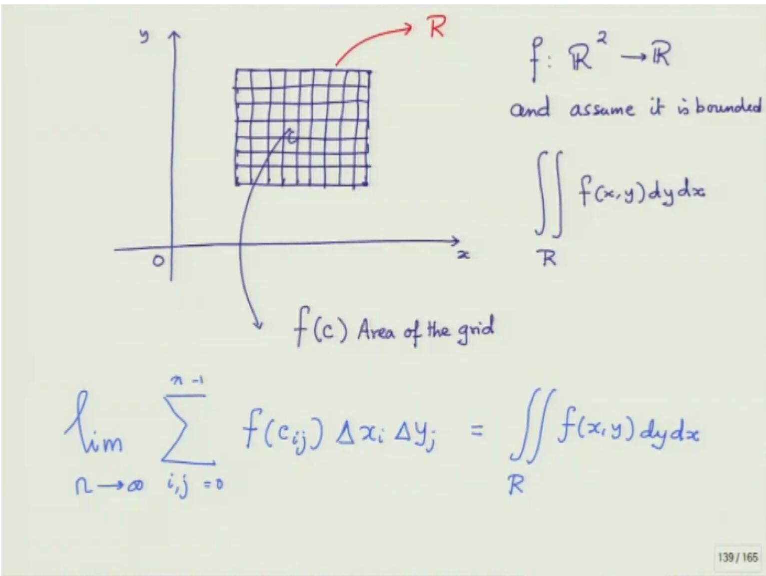
So it is a Riemann sum in the two dimensional setting, not in the one dimensional setting. It is Riemann sum to define a double integral of a bounded function, okay. So here what we are seeking is that the limit, that the limit as n goes to infinity that you keep on increasing the number of partition points, you make finer and finer and finer grids. This is called a grid.

(Refer Slide Time: 19:55)



If we just take the rectangular region and take off this just rub off this part, this part of the drawing and what you have formed is the grid. So if you have more and more partitioning points, your grids would increase the number of grids would become finer and finer grids. So limit n tends to infinity.

(Refer Slide Time: 20:18)



If this Riemann sum, if the limit exists, f(c ij) into del x i into d yj. If this limit exists and if it exists, this is called the integral over the rectangular region f (x, y) dA or dx dy or dy dx whatever you want to call. Or dy dx if you want to say dy dx you call dy dx. It does not matter whatever you want to. It is a symbol. This is just a symbol, this limit. So you will understand calculus far better if you know at the very outset that every continuous function did not have a derivative.

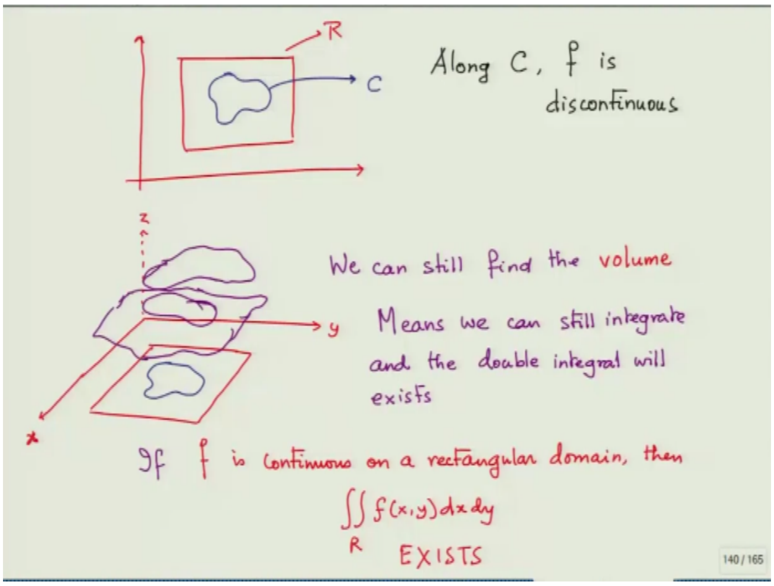
This is a very famous statement of Benoit B. Mandelbrot the inventor of fractals in one of the interviews given to Mathematical Association of America. And I would also say, taking a clue from such a great mathematician that you would also understand calculus far better if you know at the very outset that integrals and derivatives are names given to special kinds of limits.

So integral is nothing but the limit of the sum. This is what Steven Strogatz calls as the infinity principle, an idea which was already known to Archimedes. That you, the problem is big. You break it up into congregate problem, break it up into smaller parts, look at each smaller part and sum them up and go to the limit and keep on making the smaller parts smaller and smaller and smaller.

And then in the limit as you keep on doing it wherever you are heading towards is your answer. So that is the infinity principle. That is the guiding principle of calculus according to Steven Strogatz and I kind of like to agree with him, because that is, what happens all the time. Whole world actually runs on this thing. So we have no idea how much this human civilization is indebted to this one subject called calculus.

Now what has happened is that this is what we will call the double integral. This is a definition for a general bounded function. You do not have to go into the geometry and to details of if you have a function of if this is greater than equal to zero you can start defining things and all those things will come. One thing you have to understand just like Riemann integrals, if in the rectangular domain.

(Refer Slide Time: 23:09)



Suppose here you have a rectangular domain, here you have a rectangular domain. But along this closed curve say c in a rectangular domain R, along this closed curve c f is discontinuous. That is f could have a kind of a strange looking graph. So basically, it could be like this. For example, if this is your R, so along c suppose you have a graph if you have the rectangular domain.

I am just writing it and trying to this is the x axis. This is 0. This is y, z axis. Suppose here is your c along which, so the graph could be like this, that okay, the some part of the graph, so here is the graph. But here it is not so easy to draw it. Let me just draw it up and then I will put the x axis. Somewhere here, there is a gap. And then there could be other parts. So there is a gap and then there from here it picks up like this.

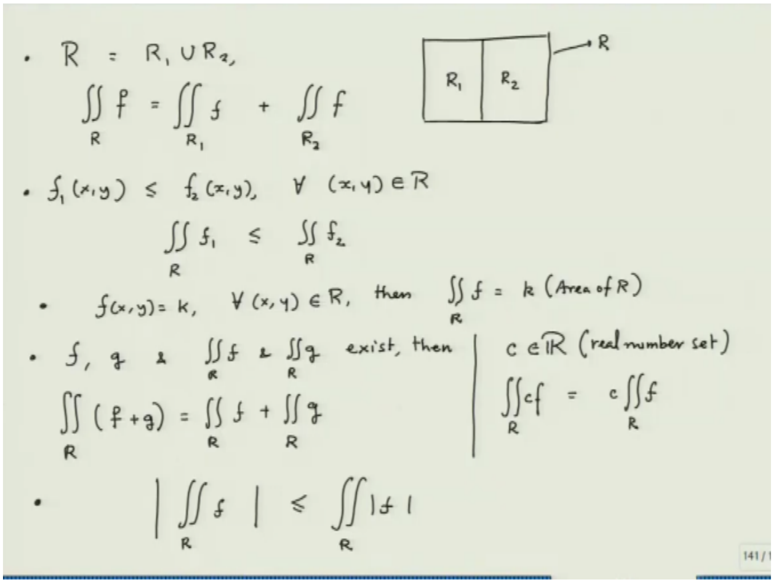
Okay, so there is a gap here. So here there is kind of discontinuity, only here in this zone. Otherwise it is all, so here the functional jumps up. So let me I think the drawing is not become, so here some of the function value along this curve seems to along this domain seems to have jumped up. Can you find the volume still? So along this side the domain part should be, this drawing is taking a little bit of time.

It is slightly harder. Suppose it has jumped up like this. So can we still find the volume? What the Riemann's theory would say that yes even in this case, you can find the volume, we can still find the volume, find the volume. We have taken positive functions. From here I should draw the z axis at the back. Means we can integrate, means we can still we can still integrate and the double integral has a meaning.

So which means if you have a continuous function on a rectangular zone, you will the integral would always exist. If f is continuous on a rectangular zone or rectangular domain then this integral f (x, y) dx dy exists, finitely actually. Exists means exists finitely. So that is the kind of idea that we are going to look at in a much larger scale. So let us now go back and see that if this is my situation where I have this kind of thing. Can I use an iterated integral idea to compute the integral?

That is the question. But before we go so we will write down certain properties of the integrals, which are so obvious that I will not even give a proof of that. If you want to make a proof of that you make a proof of that yourself, but I am not going to personally come around and give you a proof of such things. So here, of course, I promise you certain things which will come up right in the portal and they will come up. Those promises will be kept. So let us just write down the properties.

(Refer Slide Time: 29:17)



So you have a rectangular region R which is broken up into two rectangles equal to I cannot say phi and equal to a single line that is R 1 union R 2 and maybe they share just one boundary as an intersection. If that is the case, then you can write that double integral R of f is doubling, I am not going to write dy dx. I am just making a shorthand this time on this page. Double integral R so 1 f plus double integral R 2 f.

This is just a shorthand I have put. You should write to be fully precisely you have to write f (x, y) dx dy. I am just not wasting the time. Similarly, is f 1 and f 2, f 1 (x, y) is less than equal to f 2 (x, y) for all x, y; x, y in the rectangular domain. This rectangular domain is not the real number line. Please understand real number line there is a double stand.

So, what I do is now this would imply integral R f 1 is always less than integral R f 2. So these are quite simple consequences which even if you think in terms of volume and all you can figure out. But it is true for any kind of function. f (x, y) is equal to k for all x, y in R. Please understand that this k could be negative. Then the integral of f is nothing but k into area of R. Now you have two functions f 1, f and g.

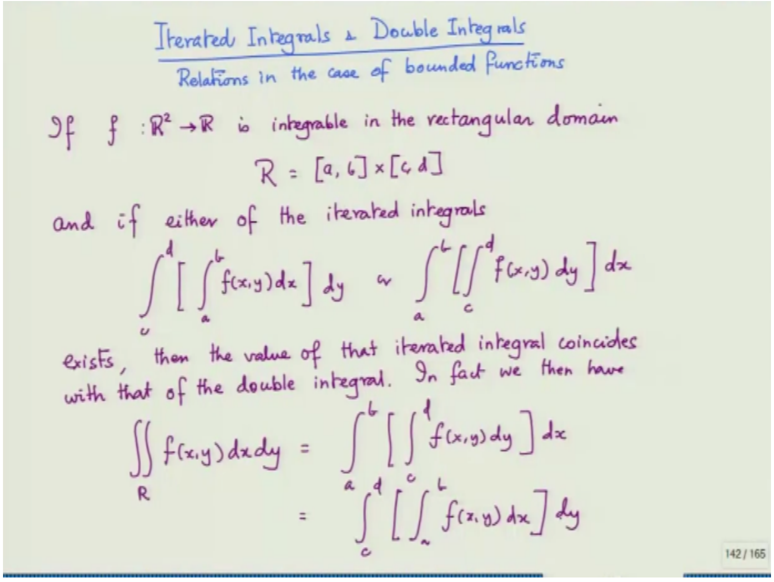
And both has their integrals finite over R. And integral R f and integral R g exists then of course you have f plus g this integral R f plus integral R g. All this shorthands are because I want to put in a lot of stuff in this page. Of course there is another thing which I am just giving on a side if c is a real number, right c is any R, c is in R. Now here R means a real number, real number set.

Then integral R c of f if you multiply f g value of f with c is same as multiplying the integral and then integrating then same as multiplying the integral of f with c. And the last one that the absolute value of the integral must be less than the integral of the absolute value which is a very basic result. That is what is the meaning. Of course, you can write x, y and all this.

You can fill in the gap and make it look much more good. So here what it means let me just draw, some people might get confused by what I am meaning by R 1 and R 2. So this is R 1, this is R 2 and this is the whole R. So their union is R but you know what that their intersection need not be empty. Their intersection is this line. So now we are going to this question of using iterated integrals.

This idea of iterated integrals that came from computing volumes using the Cavalieri's principle for non-negative function can though the idea of iterated integrals be extended to the case when we are talking about just bounded functions, which need not be having non-negative values okay. So how do I do that? Can I evaluate it? So let us write down these very important results about iterated integrals.

(Refer Slide Time: 34:57)



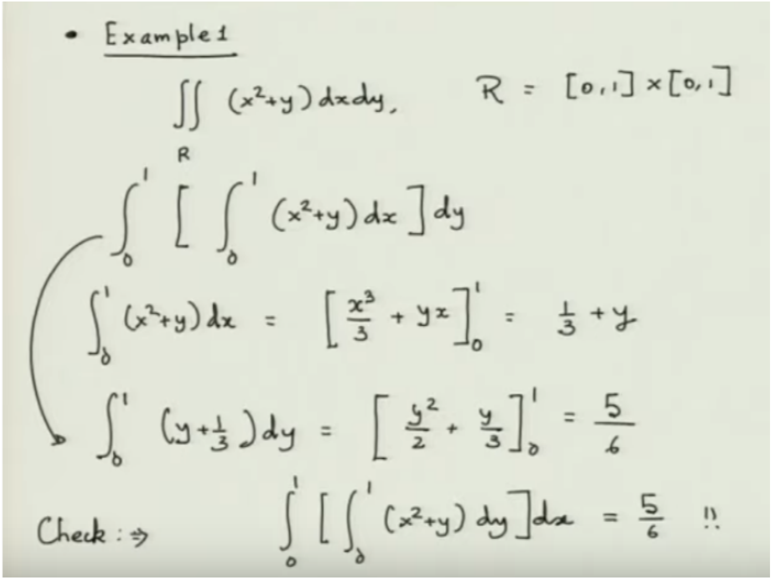
So iterated integrals and double integrals for general case, iterated integrals and double integrals relations in the case of bounded functions, bounded functions. If f from R 2 to R is integrable in the rectangular domain, integrable means that the limit exists and limit of the Riemann sum exists in the rectangular domain, domain R equal to a cross b into c cross d.

And if either of the iterated integrals, we are not going to get in the details of proof etc., we are just going to give an example after we state this and you can try for the things or integral a to b integral c to d f (x, y) dy. Either of these iterated integrals exists if either of the iterated integrals and if either of the iterated integral exists then the value of that iterated integral when sides with that of the double integral.

In fact we have, in fact we then have f (x, y) dx dy is integral a to b is integral c to d f (x, y) dy dx. Integral same as integral c to d integral a to b f (x, y) dx dy. So once you know this, you first have to check whether iterated integral exists. If somebody asks you to compute a double integral just check whether the iterated over rectangular zone, just check whether the iterated integral exists.

If the iterated integral exists, then just compute that and equate it to the double integral. That is the story, that is it. You have to keep, keep an observation that the function is bounded.

(Refer Slide Time: 39:54)



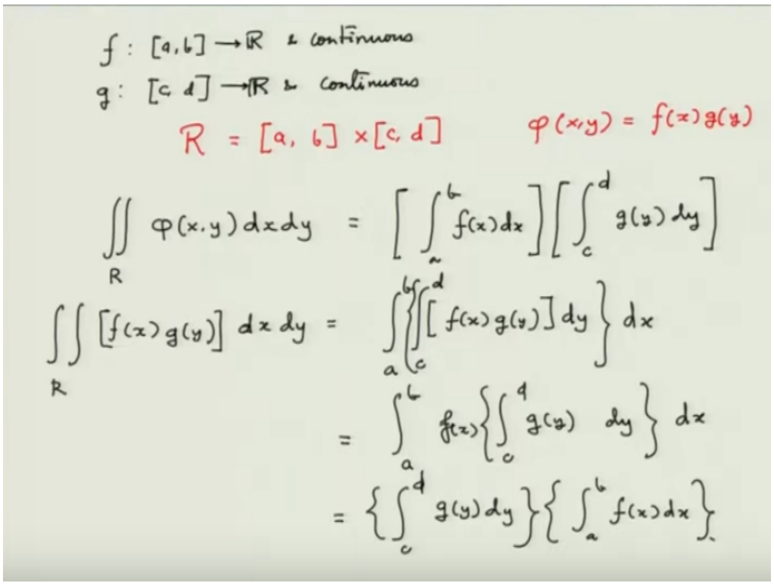
For example I am taking it from a book, from the book which we are using so that I also do not commit mistakes, but I will take one example from the exercise also. So compute integral R x square plus y dx dy because a rectangular region is 0, 1 cross 0, 1. So let me do this. 0, 1 so can we compute the iterated integral. 0, 1 x square plus y dx into dy. So let us see what happens to this one 0, 1 x square plus y dx.

It becomes x cube by 3 plus yx from 0, 1. When you put 0, it is 0. It is basically one third plus y. You can try out the other form of other iterated integral. I will not try it, you try it out for you yourself. And then you see then I do 0, 1 back to this part. So I will have 0, 1 y plus one third dy which when I compute finally will give me y square by 2 plus y by 3 zero and 1. If I calculate it the answer is 5, 6.

You can take the reverse one. You can for example check. So for homework, not homework if you wish to check I hope nowadays people have checked the following. You would have some interest to check it up for yourself. They are just fun, just play around, nothing else. To play around and see that mathematics is actually working what has been told as of in the result actually holds true.

So if these iterated integral exists so that iterated integral also should exist basically. So 0, 1, so do this. It is a very simple example I am computing and show that it is 5 by 6 okay. That is the kind of thing. Another example I am using black pen because I am trying to do just as if I am doing some homework calculation.

(Refer Slide Time: 42:49)



Take two functions f from a, b to R, these are real valued function and continuous, so it is integrable. Continuous means it is by Weierstrass theorem, it means there is a bound and the bounds are achieved. That is this functions are bounded by Weierstrass theorem. It has a upper bound and a lower bound. So both are continuous. I apply Weierstrass theorem to know that they are bounded.

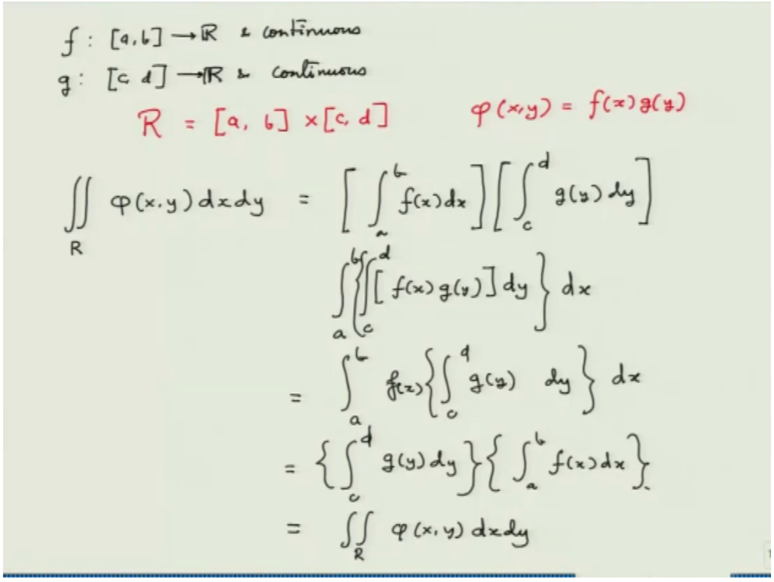
This is basic result in one variable calculus, see my course also. So if I take the rectangular region R as a, b plus c, d. So there is exercise in the book trying to ask you to show that integral R and define phi x, y as fx into gy okay. So that is telling so that this is nothing but integral a to b fx dx into integral c to d gy dy. That is, but it is again a very simple situation.

So here so if this integral exist these integrals exist we are going to use the iterated integral technique. So integral integral R f x into gy dx dy a, b into fx into gy integral c, d dy and this into dx. Now here fx should go out of the integration. So a to b fx into integral, this will become integral c, d gy dy and this whole thing dx. Now this if you look at this sorry relative to gy dy. So fx should go out of the integration.

So fx because fx is constant. Yes, it goes out of integration. So this is now c to d gy dy integral because it exists, it is finite. It will exist the dy integral will exist. It is a continuous function, so the dy integral exist. So it is finite. So now this is just a number here, I can take it out. So it will become integral c to d gy dy into integral a to b fx dx. So that is it.

So this is the use of the iterated integral technique that if this integral exists, and these iterated integral exists because these iterated integral is this. This iterated integral is where we are just writing equal to. So however we just do not bother about it. First we just write these iterated integral. So let us not bother about it.

(Refer Slide Time: 47:06)



Let us just compute these iterated integral. If I write this iterated integral, this iterated integral is the product of these two integrals which exist, and hence by and hence by definition, or by sorry, and hence by what we had written down here, we know that this simply means that this iterated integral, which is now equal to this is actually equal to the double integral R phi x, y.

Basically first you just compute the iterated integral. If it exists, that is equal to the double integral. That is it. Here we show it exists because these two integrals exist. Because these are just integrals of real valued functions and any continuous function over a closed interval, the integrals exist. And with that we stop the talk. And next we are going to talk about double integrals, but over general domains. But the general domain D, but the general domain will have a particular structure. And because those are the structures that come up in application, and we will talk about those things.