Calculus of Several Real Variables
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Lecture – 02
Inner Product and Distance

Welcome back to the second lecture of our course from calculus of several variables. So, let us look at the title of the lecture inner product and distance. So, this notion of inner product and distance is going to play a fundamental role as we move along. So, what is an inner product so, inner product has a following definition, which I will just give for two dimensional vectors first. So, you know that in two dimensions, if I draw the picture,

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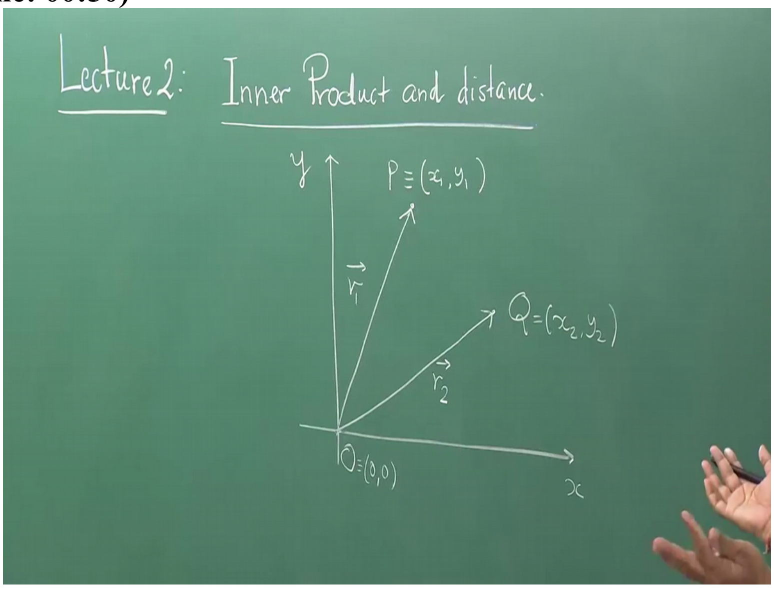

This is a two dimensional plane with x axis on as horizontal axis and y axis of the vertical axis. I have two points P was coordinates are x1 and y1 and some other point. Q whose coordinate is x2 and y2 it could be anywhere in this plane it will be here does not matter. So I am just drawing for my own convenience and here is the origin which is the point you know 00.

So, the tip of this vector, OP is the point P ok. So, this vector OP is actually represented by this set of two numbers, which actually decide the direction as well as the magnitude as we will see. And here is another factor OQ can call this is 1 Vector OP and call this is art 2 Victor. I hope this drawing is clear. And so, now once this drawing is clear, please have a look at this drawing.

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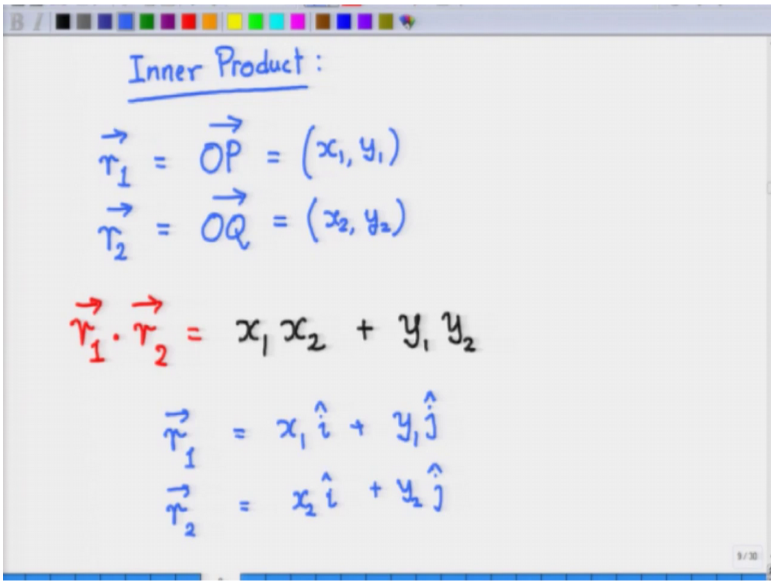

Once this drawing is clear, and I define the following so your r1 Victor is the OP vector and which consists of points x1, y1, I am writing these as road things rose as a horizontal warriors for space otherwise people usually read it as columns r2 vector is the OQ Victor has with us Q as coordinates X to Y two and now. This inner product is also called the dot product, the inner product is defined as follows r1 inner product r2 is you take the first two.

So, I have told you that there must be a kind of way to talk about multiplication in higher dimensions in what you do real numbers you know how to multiply. So, if you have to coordinate you know how to add them that is what we have learned in the last class and how to stretch them, pool them and all sort of things and how to multiply the what is the meaning of multiplication.

There could be several meanings of course there are but the most natural ways to look at each of the components first component of this and first component of the other one and then multiply these two numbers and then keep on adding what we get and do component wise So, here I take x1 multiply with x2 and I take Y1 multiply with Y2 of course you should not forget that r1 and r2 physicists would like r1 and r2 be represented as x1 into I vector plus Y1 into j Vector.

And you know these are vectors 01910 and 01 and are two can also be represented as x2 into I vector plus Y2 into j Vector. So, this is the definition of inner product. Now, once you know what two dimensions you can talk about three dimensions and how can I Go to n dimensions. That is the question. So, in a dot product in n dimensions. So how do I have so suppose.

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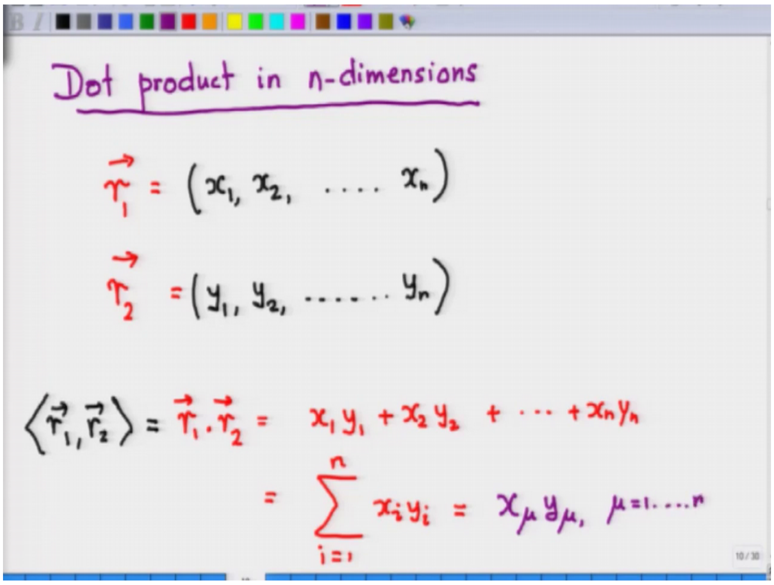

I have two vectors now, one is r1 Victor we just coordinates x1, x2 up to XN, there is RN and then I have r2 vector just coordinates y1,y2 y n and the dot product see the dot product is sometimes you written like this r1 r2 is called when it is written like this it is called the inner product, but any way we can you can call it dot product or in your product whichever one pleases you or suits you, can call even you any name scalar product.

Some people call it scalar product. So, you can choose you can give your own name also. Now, this is nothing but the same story will go on it will become x1 y1 plus x2 into y2 plus XN into YN which in a more shorthand form, and we return as summation, I is equal to 1 to N xi yi or just to make physicists happy here, those who are in physics is can be also written in kind of Einstein summation convention that it can be written as x mew, Y mew.

Were mew is running from one to end. So, you once the indices are repeated you basically these a sum that is all. So, now with the basic definitions in we will now focus all our energy in looking at vectors in three dimension because beyond three dimension, it becomes largely algebra and visualization becomes slightly difficult. So, you will have to view it through more through an algebraic angle than a pure geometry angle where you draw figures.

So, let me now draw the diagram of, Victor in three dimension and I would like to figure out what is the distance of a given point in space from the origin. So here is my three dimensional space,

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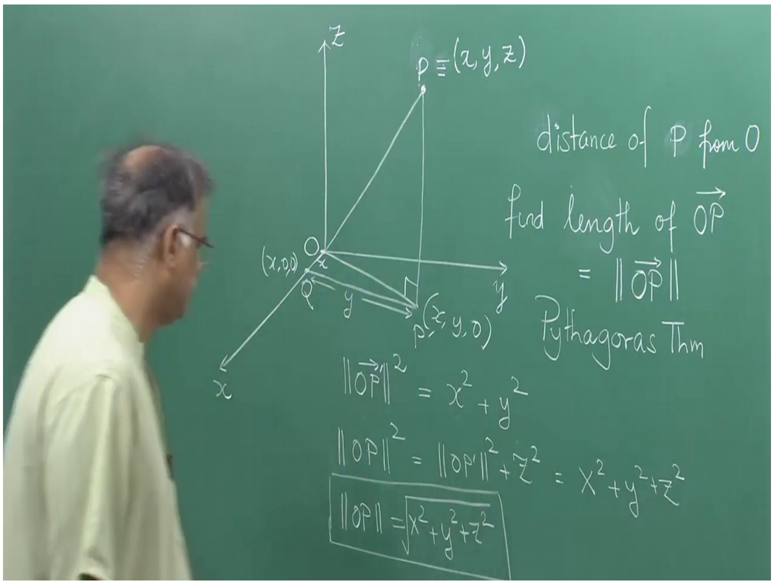

x axis, y axis, z axis. Now here is a point P. So it is given by the coordinates x, y and Z. We should put the point P here. So, this is the point P given by the coordinates x y Z. I should write this point P is here and this is xyz. Now the question is what is the distance of O from P? So, we want to find now, the distance of O or P rather P from all.

So, basically, I want to find the length of the vector op, find length of OP will be this length of the OP vector. It is sometimes called a normal OP. So length of OP sometimes denoted as a symbol. Now, how do I find this length? The clue of finding this length Pythagoras theorem.

Let us see how I can apply the Pythagoras theorem to find the length. The key idea is that you drop a perpendicular from the point P straight onto the plane XY and see where it meets it will meet exactly at the point whose coordinates are x y and 0 and call this point as p dash. So, let us first see what the length of this is. If I dropped once i drop perpendicular did they this OPP dash the former right angle triangle?

And now if I know the length of OP dash and then I know that this length is nothing but Z. I can use Pythagoras theorem to find OP so, what is OP dash? So, what do again draw a perpendicular x axis from OP dash to come to a point would you call Q. So, this point is x 00 Okay. So, you know that these distance PQ this distance, the length of this distance is nothing but the value Y.

Here we are writing it in a positive way we are in the positive orphan it is we are writing it as Y otherwise it is more of Y absolute value Y. Here it is XOQ is X. So OP dash square if you take this vector and take the square of the link which I can write as normal length digit as normal. It is nothing but the length of OQ Square of OQ plus PQ square. So, this is nothing but x squared plus y squared to offer OP dash square is x squared plus y squared.

And so, what is OP Square again by using Pythagoras theorem, OP square is actually OP dash squared plus Z Square is that is the length of PP dash and here we are in a non-negative or 10. If it is not it is model mod Z square. So, this is nothing but mod Z square and Z square is the same thing. X squared plus y squared plus z squared. This is called a you cleans norm or Euclidean norm.

This is OP is equal to x squared plus y squared plus z squared. So, here you have square root of this non negative square root of course, with us we are talking about length and length is never negative. So, what do I see from this expression? What do I see from here? What do I get from this thing, I get a very interesting feature. This idea of finding the length through Pythagoras theorem. Allows me to actually connect to the motion of length and Nina product and let us see how it is done. So, if I call OP vector,

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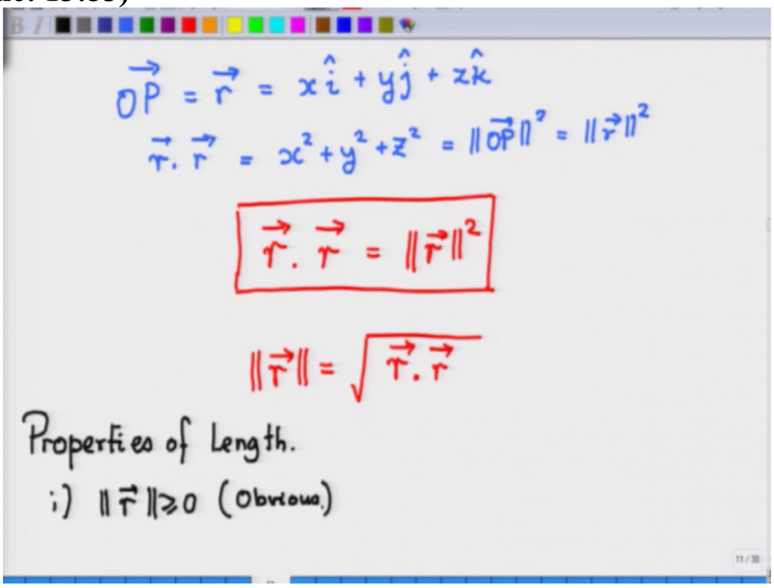

As, r Victor then r Victor can be written as x of I Victor plus Y of J Victor plus Z of k vector. This is I clear to you. Now, once this is known, you know let us see what is the dot product r we r dot product is nothing but x squared plus y squared plus z squared, which is nothing but the norm of the Victor OP Square.

Since OP is nothing but odd this is so what I have a very interesting relationship here, relationship is like this r. r is square of the length r this is an important relationship. Once this relationship is known, you know what is r. So, our Victor is equal to route over r dot r this is what we have. So, you see we have now linked these two things, what does essentially this op is talking about these the length of a vector appoint from the origin.

So, whenever I am talking about the length of the vector, it is a distance between the origin and the point which the vector represents. So, these vectors of op vector is called a position vector of the point P because it is establishing the position of p in space you take some other vector that we position vector that particular take another point position vector all the particular point Q.

So it is kind of vectors allow you to positioning the position the coordinates. And by that, you see that because Vector is two positions are different coordinate two different coordinates the vector have to have different directions and different magnitude different lengths. And as a result of which, these points the these strides here, this this coordinates 333 coordinates length, but hide, they can also be identified with a vector.

So, these top three real number can be identified with a vector because of this motion of position Vector, because here is the point P, so here is a vector, he is so different and here is the point QC, the reaction has changed and the length is change. So the two points this point P here and Q here have different directions and different lengths from the origin. So with respect to the origin, these can be thought of as Vectors and that is exactly what.

It is and so, you we are not just during that the whenever we are more than we are whenever we have some real numbers accumulated in JNXYZW will call them vectors we call them vectors because of the geometrical fact because you can position them you can identify their position in space by using vectors by using geometrical vectors okay. So, let us now see certain important properties of the vector of the notion of a length.

So, properties of length so, property 1 length has to be always greater than equal to zero. Obvious what is the spelling obvious, okay? So this is an obvious property which we all understand. We cannot say something as minus two length. 2 property,

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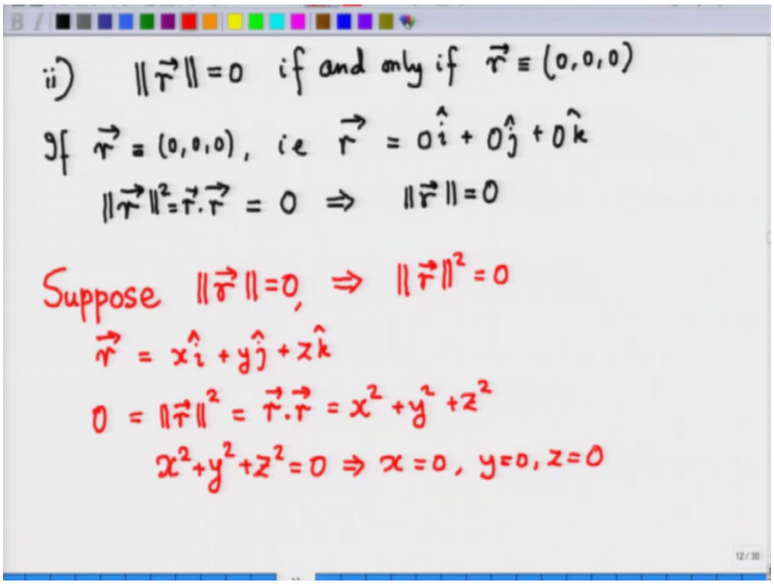

Which I admire and it is a very interesting property very simple but very interesting. Is that the length of a vector if it is equal to zero if and only if, r Victor actually identified with the zero vector r is identified with origin, so how do we do that? So if r vector is identified with origin. That is r Vector is 0 I vector 0 j Vector 0 k Vector and what happens then what is the length square of the length.

So, squared is equal to r. r and that is nothing but zero because zero plus zero plus 00 squared plus zero squared plus zero squared. Now suppose I have so these would imply square or something is that length continuous is zero it is a natural. So you suppose mod r equals to zero then it implies that model r squared is also zero. So if RR war r square is X I Vector Y J Vector and Z K Vector then implies that zero is equal to its.

Here it means that r is identified with the coordinates this. So RR sins square of the length is zero, this is nothing but r dot r. so r dot r is x squared plus y squared plus Z square and this is equal to zero. So what we get from here is that x square plus y square plus z square is equal to zero. So now these are non-negative numbers x square y squared z Square. So non negative numbers adding up to zero means all of them can be has to be 01 of them is on the zero.

And the sum would be non-zero, which means that your x squared equal to zero, y squared equal to zero, and Z squared equal to zero which also means that x is equal to zero y is equal to zero and z is equal to zero and hence we can identify the vector are with the zero vector and hence this rule holds.

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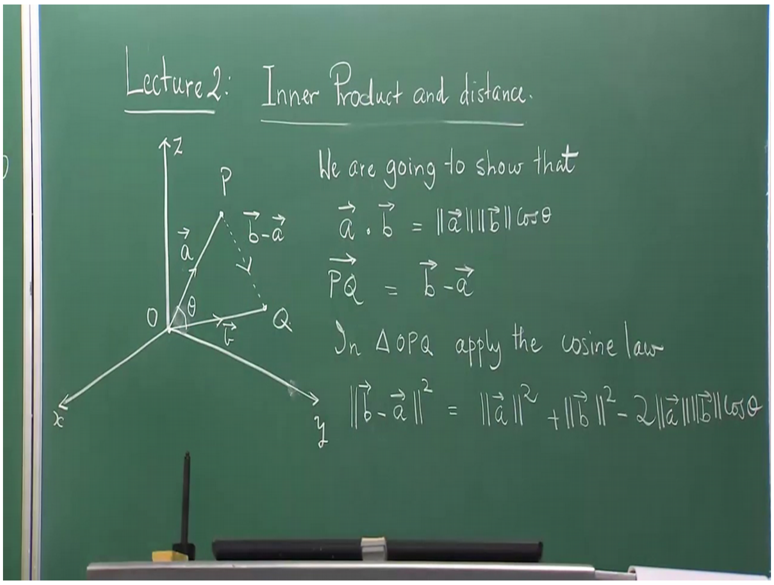

Now there is another interesting formula to calculate the dot product of two vectors. In three dimensional space and that makes use of the angle between the two vectors. So let me use the board now. So this going for between the backend board and the electronic board electronic slate is much better because math is always traditionally in a subject of the board and it needs to go back to the board sometimes it is much more fun to see things to worked out.

So, here I will have a scenario have two x y Z. x Rector Will increase the length of the sport. Y Sorry x axis y axis z axis with the origin and I have already eliminated here I will have two vectors A and B. So, this is a vector. This is the b Victor and this is the angle tita between the two vectors. They are representing to different points at P and Q. You are not writing the coordinates at this point.

So, you observed that OPQ is a triangle. What we were going to show we are going to show that a vector dot b vector is equal to the normal way vector into the normal b vector length of the vector into the cosine of the angle between them. Those who have some idea or liking for Euclid geometry would immediately understand that some kind of cosine law has to be used.

So, this vector PQ if you look at this letter PQ those with a basic idea of geometrical vectors would immediately know that this is the present this is representing the vector b hot minus a b minus a virus. So, PQ is presenting the vector. If you look at the triangle OPQ in triangle OPQ apply the cosine law. So, once I know this angle here, I can actually use the cosine of this angle.

And the length of these two sides to find the length of this side that is the cosine law which you know apply the cosine law. So how do I do it? So, I have to find the length of this and I know the cosine laws are you know, I am just writing on the this square is this is equal to this square plus this square minus twice this into this cos tita. So b minus a whole squared is equal to a squared plus b square minus to norm AB cos Tita.

This is what you know. Okay now how do I go about doing this, I really have to play with this side, I have done something here already this part has appeared this part has appeared. So some game has to be played here. So, let us see how the game is played. And once we try to play that game, we will see we will learn some law about how to take how to handle in our products.

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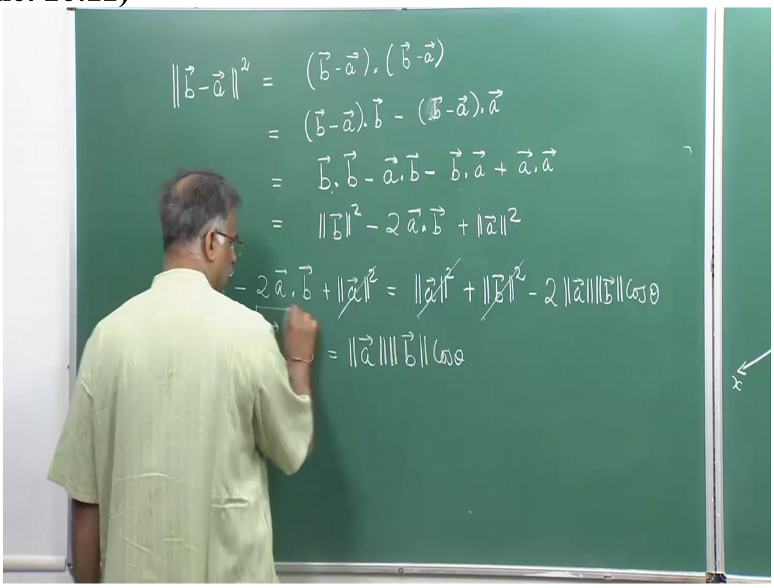

So norm b hat minus a hat whole square is equal to the dot product of b minus A in to be minus a b minus a vector dot product into b minus a vector. Now, how would I actually do this in a product these do these dot product. Just think as if you are really multiplying numbers. So, you can actually do this you can do b hat minus a hat, dot product of b hat minus of B hat minus of B hat minus of a hat into a hat.

So you see you are playing just as if you are multiplying numbers, again to the same game here. So remember the thumb rule, multiply, take the dot product as if you are really taking dot products of numbers, ordinary numbers, but instead the multiplication will put the dot product. It is that easy to remember. So, of course, I can prove them and of course I can do things prove that what I am doing is really true.

The dot product of these two is exactly this. But I am just telling the humble rule sometimes. As john one woman said, you do not understand things in mathematics. You just get used to them. So sometimes you go with a tumble and later on even check for yourself with simple examples, that is indeed working you can actually prove it in detail. So, let is just do this. So, I love b dot b minus a. b here I have minus b dot A minus plus a. a so what do I get?

I get this by definition it is square of the length, here I am minus two, a. b was a. b and b. a are the same thing, because ultimately, these are nothing but one multiplication numbers and summed up and because standard multiplication of real number is community x into y Same as y into x and hence, you have this one and we clubbed them together. Plus norm so this is what. So this side is this.

So taking in what I have here, I will plug it into here. So what do I get? I get, I hope this is not blocking, this is blocking, I will just go to this site. So I get now norm b squared minus two, a dot b plus norm a square is equal to norm a square. Just this is equal to the story I am writing on the right hand side. So this is nothing but this so the right hand side is equal to that.

We had squared minus two Euclidean geometry is that action here? Now, what do you do here is you know that I can cancel this with this, the negative science can be cancelled and so two two can be cancelled and what you have is at the end this implies that A dot B is nothing but Wala as a French would save olla means, that is it done. So, we have done this.

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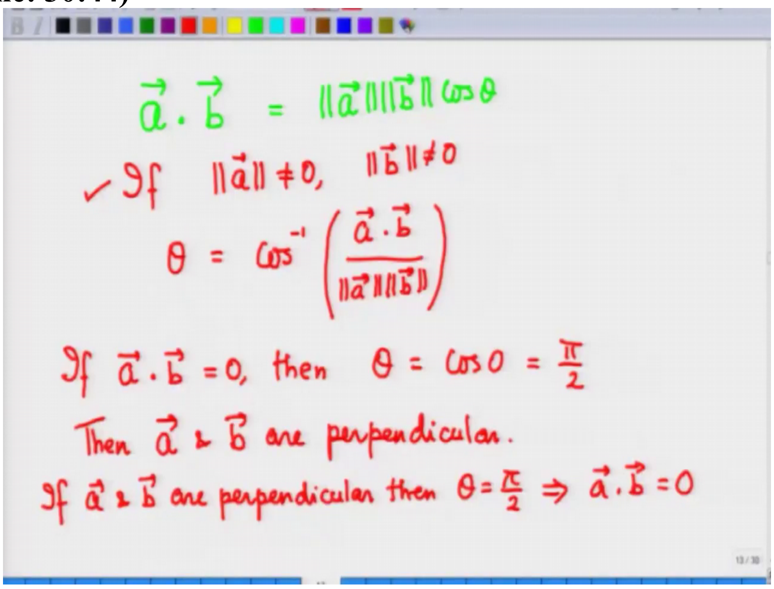

So, once you know that a dot b is equal to mod norm main to norm been to cosine of the angle between them. Then you can definitely right. If norm A is not equal to zero means A is not equal to zero and norm b is not equal to zero then you can obviously find Tita as inverse cos inverse of A dot B by okay. Suppose we have this scene suppose we have two vectors and none of them are zero and suppose a dot b is zero.

So, if now in this case if, this whole that is this is not zero, and if a dot b is zero. Then from this hour inequality then Tita equal to cos of zero which is buy by to 90 degrees. So, if you think of cos inverses of function cos inverses dealing or function is a multi-function because for one particular value it can have many titas corresponding to it, but if you just take the acute angle values.

Then it becomes a function the principal values, then cause zero is buy by two, which means if a dot b zero, then the vectors a and b are perpendicular. In fact, if a and b on the opposite side if a and b are perpendicular then Tita equal to pi by two and it immediately implies that a dot b is equal to zero. So, you see this, the fact that a. b is equal to zero gives me makes the two vectors perpendicular. The Tita because 90 degree 71 is here and other is here.

This is possible because of this formula, this, this, this alternative formulation of a. b. So, because of that you could now do such a thing. Try to figure out some mix of vectors yourself play around a bit and see those we are also constrained by time. And so, what we are going to do, we are going to talk about

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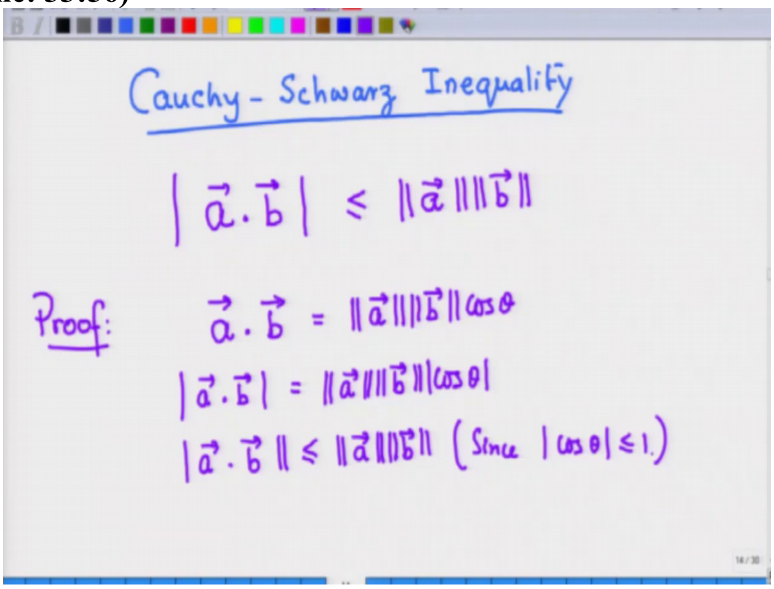

An interesting thing called Cauchy Schwarz inequality. So what is Cauchy short inequality? Actually Russian phone like just to call it Cauchy Schwarz inequality they would call it coffee short boondoggle. Riskiest inequality. So there are a lot of credits giving in mathematics that okay, Russians are okay we are done it sometimes without knowing what you guys were doing in the West and they west would always turn out to say no, no, of course he is a great man. His name has to be there.

But it seems several people already also knew it. In fact, as I know that la grunge if I am not wrong, la brands possibly knew about this idea. So, but some stories remain and some stories do not and so we call it Cauchy Schwarz inequality. Inequality actually are the hallmarks of mathematics. What does it say? What does Cauchy Schwarz inequality says that if you take the dot product or two vectors is a real number?

It could be positive negative anything and you take its modulus value absolute value, then this absolute value is less than the product of the length of these sectors. That is what the Cauchy Schwarz inequality says. So, let us prove this Cauchy Schwarz inequality, and you will see that Cauchy Schwarz inequality is nothing but a consequence of this formula. Why? Let me tell you so you see proof that we prove like a mathematician.

I am truly not one but I am posing as, I am. And then I am writing this word proof and trying to prove that this happened source that the Cauchy Schwarz inequality. So I know this formula, which we have deduced there on the blackboard. Here it is, here is the formula. So what I do, I take the modules of both sides. And you know that if I take the modules of a real number of a positive non negative number.

So the non-negative numbers will just come out, they will, it will remain the modules is the same modules of two is two or modules of minus two is two us. So we not a negative number, the modules does not change. So it is just so that take the model as it this into cosine Tita modules of cosine Tita but you know the cosine value lies between plus one and minus one so more absolute value of cosine Tita has to be always less than one.

So it simply means a. b is less than equal to how beautiful the proof is very simple as a consequence of that idea that that derivation. This is the beauty of mathematics I am thing I think this is a beautiful thing. So, here is my proof of Cauchy shorts inequality. Okay, now I come to the third property of length. Third property of length is another way of telling a very standard geometrical fact.

Which you know that given a triangle in plane the sum of two sides of a triangle is greater than the third. So, if you have a triangle in space basically you can think of the triangle as a part of a plane, it is flying in a plane essentially, though it is a in a three dimensional space you know we are talking about but still For example, here this triangle OPQ you can think of one that is flying the plane that is passing through these three points. So, again, that idea is built into what is called a triangle inequality.

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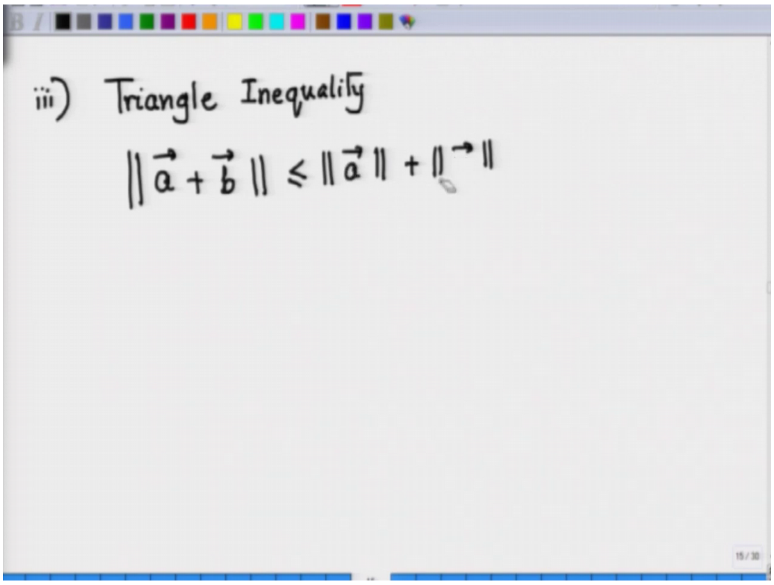

So, it does not matter whether you are in two dimensions three dimensional and n dimension, this is true. It says that this two vectors, say x and y then x plus y is less than mod x plus mod y even that a b also we are uniting a b it does not matter what ever. So what I do the same thing I come if you want.

I can write this as a. b that is a better thing because it will be easier for you to follow because the Cauchy Schwarz inequality is written in terms of a and b though it does it matter. These are dummy things ARX is same when you know the meaning, but I just want to confuse you on the notation. So let me just do the job here. So now I consider

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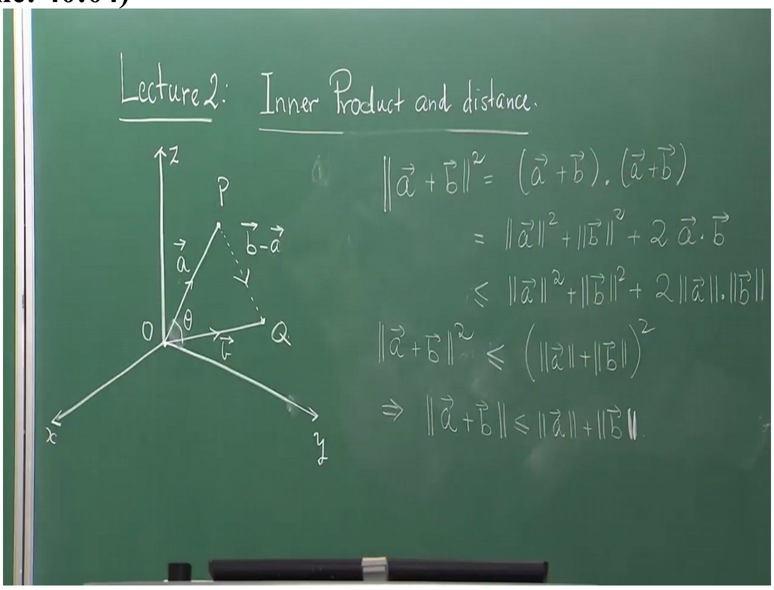

a plus b whole squares the same game. This is the name of the game squaring up and then writing this as a plus b dot A plus B. If you do it work it out then it will become a square plus b squared plus two a. b Okay and once you know this now, here So, I can apply the Cauchy

Schwarz inequality and right this is less than equal to square plus mod of normal b squared plus this part is less than two times normal a normal into normal b, this is the Cauchy Schwarz inequality.

This is where I have applied the Cauchy Schwarz inequality. This is something you have to keep in mind that this is where I have applied the Cauchy Schwarz inequality. And hence, once I have done it, it is very simple. So I have got a plus b whole squares. Now right this is nothing but mod a plus mod b whole square. So I cancel I know two non-negative quantities whose was square one is done the square others.

This simply means and, again, Wala, we have done it. And this is a very key inequality. I think I am stretching this talk a bit, but this is a very, very important part. This is the first. You are you are being equipped with tools. I hope this chalk is being clear. I hope this is so much clearer chalk. I hope you will be able to see this. I do not know maybe I should go near the board and I guess at least can be seen.

Maybe I would right to make it more brighter.. I do not know whether I should do it. I hope you did understand it. In case you do not. I just repeated a proof here. So that you the main idea that what you do is to take A plus B whole squared and this would give you nothing but a plus b dot A plus B and that leads to norm a square plus norm b square twice a. b applying the formulas of computing. In our products and now applying Cauchy Schwarz inequality

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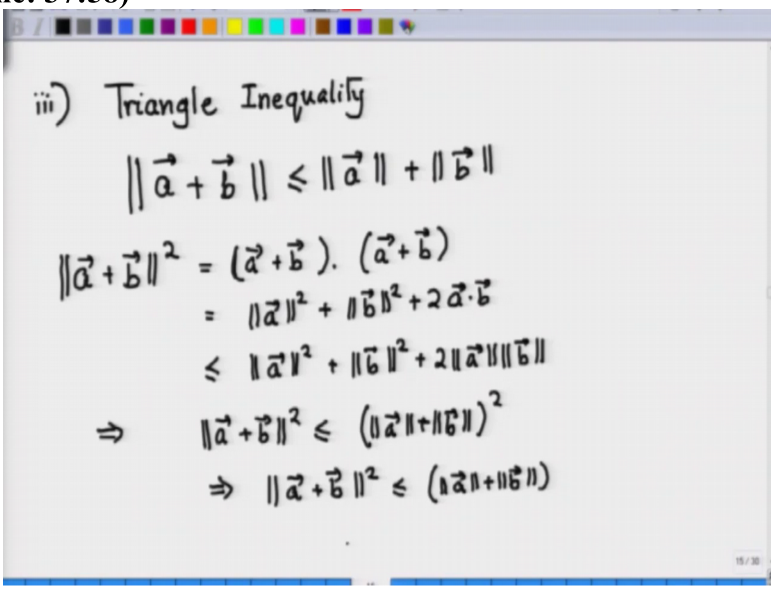

You will get this and then that simply shows that a A plus B some rewarding the proof here in case The thing that you do not see the whole thing clearly on the board compared to that riding that was much more deep cleaner. So and that is it once you know that then this is nothing but norm plus norm be whole square. So this implies because your tune numbers which are triangle inequalities done?

So, with this I end today is lecture there is something called projection formula. The projection formula would be given an assignment and I am trying to have some notes written down for the first few chapters which are very basic. And this I will hand over to the entitles and staff here so that they can actually scan them up and put them on the course website. So you have some basic idea now how to do things and that is very important to have a basic idea.

So, I hope you understood it there. If you have not feel you please feel free to ask questions that can you can anyone ask any fundamental questions which we might have forgotten to tell you or I might have skipped for some reason. Or it might be I myself do not know. So please feel free to ask questions because at the end, mathematics cannot be learned without discussions. Thank you.