**Calculus of Several Real Variables**

**Prof. Joydeep Dutta**
**Department of Economic Sciences**
**Indian Institute of Technology – Kanpur**

**Lecture - 16**
**Second Derivative Test for Maximum & Minimum**

So in the last class, I made an announcement, that okay the second order test for deciding whether a point, critical point that is the one which satisfies gradient of f is equal to 0, such a point x0 y0, whether it is maximum or minimum or local maximum or local minimum or neither of them or saddle point, how do I decide? So I said, okay, we do not have time, so we will push it in the next class and all these sort of things.

But I realized that as you grow old, you possibly forget it your own well planned stuff, then I went back and looked at the lecture series that I had planned and I found that actually second order condition had been shifted, there is a separate lecture for second order condition. So as we start the fourth week, we are actually, our lectures would remain as planned. There will be no change, which is a good stuff, of course. So here we are going to discuss second derivative test for maximum and minimum.

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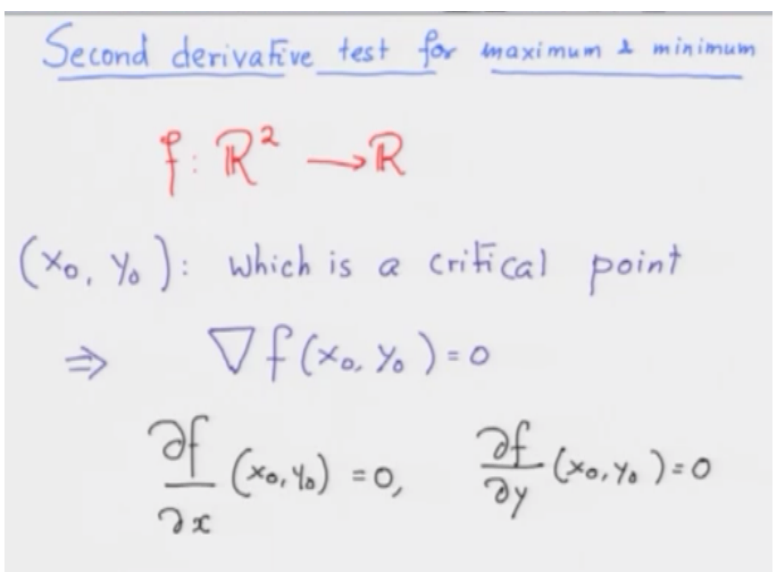


Second order or second derivative tests, I am not writing second order, I am just writing second derivative test. Now here as always, we will consider a function f from R2 to R. This taking of just 2 variables is a very important marker in a sense that it gives an idea what things are happening, but going to higher dimension, whatever you can do in this particular case, in optimization, in 2 dimensions, such things may0 be carried so easily to the third dimension, may be done for higher dimensions.

So two dimensions has a specialty where you can handle things much more simpler and have look at things much more simply, then what you can do in higher dimensions. So we will start by considering a point x0 y0, which is a critical point. So what is a critical point? You know what a critical point, means something which satisfies. So this would imply that the gradient. Critical point is something, which satisfies that the gradient of f at x0 y0 must vanish.

That is a gradient of f at x0 y0 must vanish, which put in much more simpler form means that first derivative of f with respect to x, first partial derivative evaluated at x0 y0 is 0 and the second partial, and the first partial derivative is evaluated with respect to y at x0 y0 is equal to 0. We have shown that any point satisfying this unless you have a function of concavity or convexity, you would not have them to be a minimizer.

Now assuming that x0 y0 is a critical point, that you know that x0 y0 is a critical point, how do I think of moving ahead, right. So the way to move forward is the following. So knowing that del f del x at x0 y0 is 0, and del f del y at x0 y0 is 0, let us write down the second order Taylor's theorem. So Taylor's theorem will have a key issue here.

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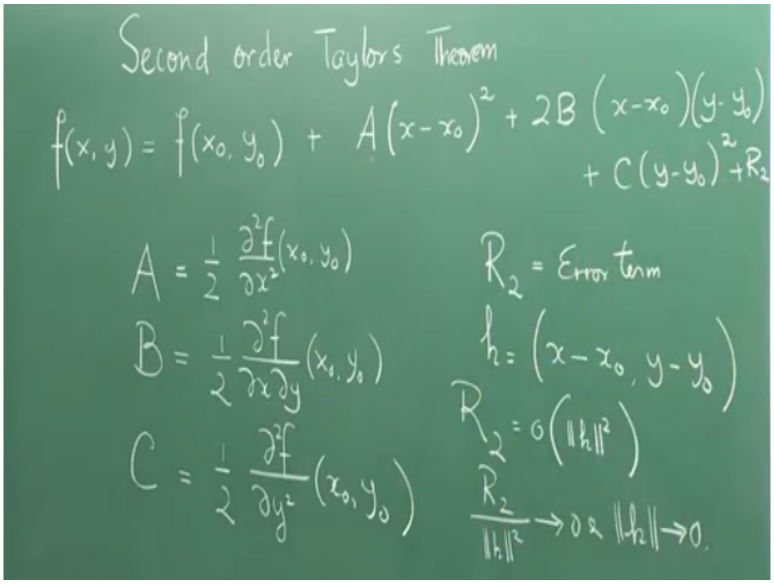


Second order Taylor’s theorem, so here that any term del f del x and del f del y at x0 y0 will not appear. So we are looking at a point x0 y0 and we are looking at the expansion of f of x, y, which is now second order term, we can write it like this. We will write down what is A and B and all these things as we go on, because these terms will become important. Of course a lot can be done in terms of inner product, but okay, we will not get into such details at this moment.

Plus R2, so where A is half of second derivative of f evaluated at x when del 2f del x2 evaluated at the point x0 y0. So we just, for brevity of the writing we wrote, B is half of del 2f, these partial derivatives and we are assuming that everything is continuous and nice because unless we assume like this, Taylor’s expansion is not really possible and so this evaluated at x0 y0 the second derivative of y, del 2f del y2, so these are what is there and R2 is there, a term.

R2 is a term, so that it vanishes as x0 y0 approaches, xy approaches x0 y0, these term vanishes much faster than xy approaches x0 y0, which means if I write h as a vector x – x0 and y as a vector y – y0 R2 is actually small o of norm h whole square, which means R2 by norm of h square goes to 0 as norm of h goes to 0 or h goes to 0, whichever, whatever you want to write. So the error term, what it says that R2 must vanish faster than h vanishing.

That is and in the second order term means R2 the way the quadratic phenomena is the norm square of h, cannot vanish faster than R2. R2 must vanish faster than the square of the norm, which is a very, very important thing to observe. So now instead of taking any arbitrary critical point x0 y0, for simplicity we will set x0 y0 to be 0.

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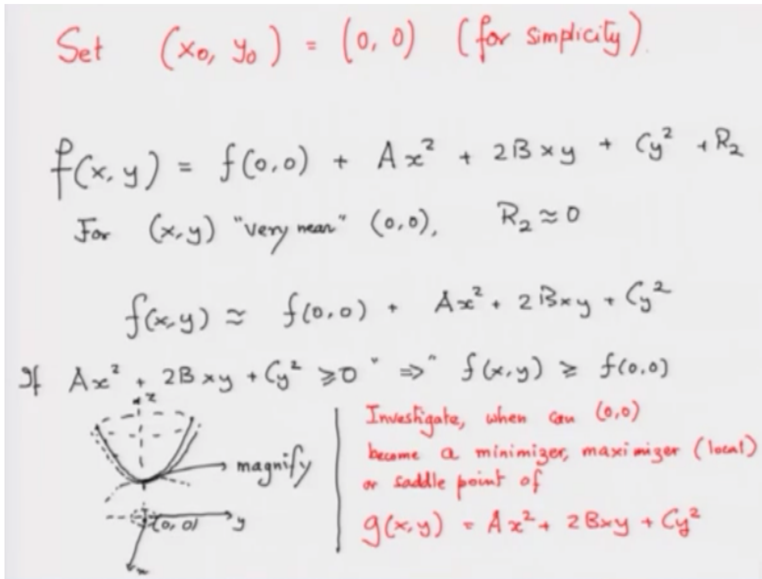


X0 y0 will be the origin 00. Now if I put 00, now I put 0 0 here x0 y0 equal to 00. So let us see what the expression will become. So this expression will become f of xy. So xy, suppose is a point very near 00, so it will become f of x0 y0, which is now 00. So I will write this as f00. So it will become f00 + Ax square, x0 is 0 and y0 is 0, so it will become Ax square + 2Bxy + Cy square + R2.

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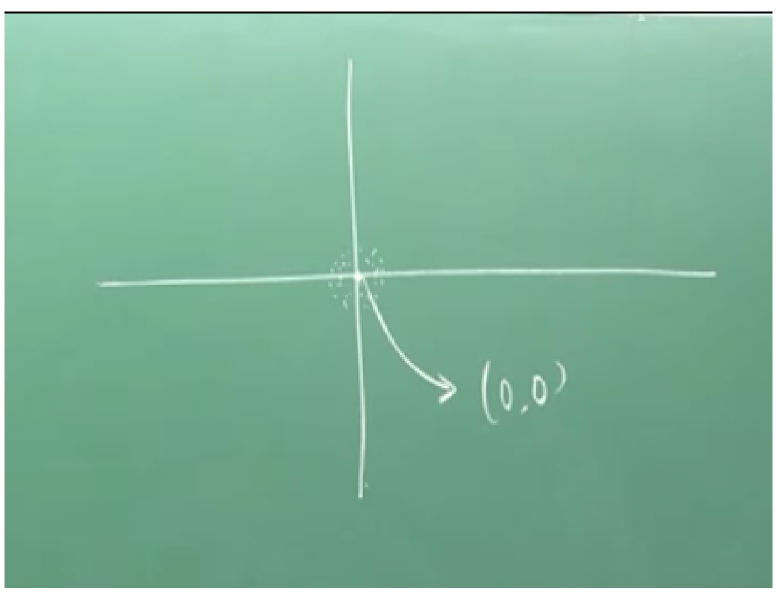

When xy is very near 00, in a very neighbourhood around 0. So if I take the point 00, this is the origin, right 00 and I take a very small disk around all points inside this circular disk except the one in the boundary. Now as you make this radius shorter and shorter, you go near and near into 0. The more near you go to 0, the R2 value diminishes very fast. It goes to 0 very fast. So for xy very near 00 R2 for all practical purposes is almost 0. So this is a very crude way of writing.

So we are also making a crude description, a crude way of writing and then what I would have that fxy can have an approximation. They are almost same. So when xy is very near 00, the functional value is very well evaluated by its quadratic approximation. So this point, this part is called quadratic approximation of f around 00, so now what is happening that if, suppose look at the function Ax square + 2Bxy + Cy square.

Now for a given choice of xy for example, if you can do that, we can prove that this is 00 if for some xy very near 00, if I can prove this, this simply implies that fy, fxy is in some sense greater than f00. So f00 becomes the minimizer. So this is a very crude sense. You cannot say this implies this, but in a very crude sense, this is the meaning. So essentially I have to determine a kind of sign on this. I have to see whether what happens.

So basically it tells me that the graph of f, the function f near 00 looks almost like the graph of Ax square + 2Bxy + Cy square translated by the amount f00. So here is the quadratic function basically this looks like some kind of a quadratic function. This is say a function of 2 variables, so graph would be something like this, function of 2 variables and basically now this is your x0 y0. So around x0 y0, the function values almost match the function values.

So around x0 y0, here if you magnify this zone, if you magnify, you will see the function values default by a very less margin. So when x0 y0 xy is very near x0 y0 means, suppose in this zone, then what can happen is the following. Then in that particular zone, the graph of this function Ax square + 2Bxy + Cy square almost coincide with the function value and at 00, it coincides with f00.

Because fxy if it is, if you write it like this, we consider this, this is nothing but just take this part, then when you put x equal to 0 and y equal to 0, it is f00 at x equal to 0, y equal to 0. At x00, this thing matches with the function value or at other places, there is a very small gap, so minute that you really for practical purpose you do not bother. I am making a mistake. Maybe this should be my point 00 actually. This is 00, so this should be the origin, not this one.

I am making a mistake. So I am making it as it is for an orbit point x0 y0. So this is the x axis, y axis, z axis. So this is the kind of quadratic function, so around this point 00, where it points is this, at 00 the function value of fxy is f00 of this function, of the quadratic function value. The quadratic function matches with f00 at 00. Apart from that, the quadratic function does a very good, whichever way this way or that way, it could be like this also.

The matching could be like this. That is dependent on whether this is greater than 0 or lesser than 0 where it is kind of maximization, minimization sort of thing. So this is a very crude writing, do not take it very seriously. This is a kind of intuition you try to build up. So you want to say that, okay, very near 00 the graph of the function f looks almost like the graph of Ax square + 2Bxy + C. So what we should now do?

Investigate when can 00 become a maximizer, a minimizer, maximizer or saddle point. I am talking about local, of course in this local sense. Global is also local, but local is not global. So local or saddle point. Saddle point of this function gxy is equal to Ax square + 2Bxy + Cy square. Please understand that we have now transferred the problem from that of the function f to that of the function g, which is the quadratic approximation and we want to know whether 00 is a very 00 minimizer or a maximizer.

Saddle point, when you say 00 is a saddle point that if in the neighbourhood or xy, this is not the saddle point that we really speak about in modern optimization theory, but in calculus that is what is viewed as a saddle point. So if you take a point 00, say and take a small neighbourhood around that point on the disk, right, you take a neighbourhood. So if at one point the function value becomes positive, at another point the function value becomes negative, then we say that the function value.

And the function value is suppose 00 at f00 and one value it becomes negative and for another value, it becomes positive, then we say that 00 is neither a point of maxima or minima, it is a kind of saddle point. This is what is the current definition of a saddle point used in calculus when you are talking about maxima and minima of more than one variable, right. Now let us investigate this function.

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$$g(x,y) = Ax^2 + 2Bxy + Cy^2$$

$$g(0,0) = 0, \quad \frac{\partial f}{\partial x} = 2Ax + 2By$$

$$\frac{\partial f}{\partial y} = 2Bx + 2Cy$$

At $(0,0)$, $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$ ( $(0,0)$ is a critical point)

Let $(x', y')$ be another critical point

$$\frac{\partial f}{\partial x}(x', y') = 2Ax' + 2By' = 0 \longrightarrow (1)$$

$$\frac{\partial f}{\partial y}(x', y') = 2Bx' + 2Cy' = 0 \longrightarrow (2)$$

$$(AC - B^2)x' = 0$$

If $AC - B^2 \neq 0 \Rightarrow x' = 0$ & from a similar analysis we will have $y' = 0$

Let us investigate gxy. So first thing that you observe is g of 00 is 00 is 0. Let us now observe what is the value of del f del x, del f del x is 2Ax + 2By and del f del y is equal to 2Bx + 2Cy. Now it is clear that at 00, del f del x at 00 is 0, and del f del y at 00 is 0. So which means that 00 is a critical point, which means that 00 is a critical point. Once you know this fact, we have to do some more work.

I have not told you that 00 is the only critical point, but it is good that if I have 00 to be the only critical point. That makes life much more easier. So what is the condition under which 00 cannot, 00 is the only critical point. Now let x dash y dash be another critical point. That is the partial derivative vanish at x dash y dash, del f del x at x dash y dash must be 0. So 2Ax dash + 2By dash is 0 and del f del y at x dash y dash must be 0, so 2Bx dash + 2Cy dash is equal to 0.

So these 2 equations, equation 1 and 2, so multiply the top equation by C and multiply 1 by C and 2 by B and subtract them to get an equation of the form AC – B square into x dash is 0. So you multiply the first equation with C and the second equation with B and then subtract that from 1 from the 2. So you will get this condition. Now if AC – B square is not equal to 0, it implies x dash equal to 0 and from a similar argument to the opposite thing.

So multiply the top equation by B and next equation by A and then subtract them to get the similar sort of thing. So from a similar analysis, from a similar discussion basically similar demonstration, we will also have y dash equal to 0. So if AC – B square is not equal to 0, then whatever be the critical point, it does not matter x dash is equal to 0 and y dash is equal to 0. That is the key idea. So what happens if I do not have this condition.

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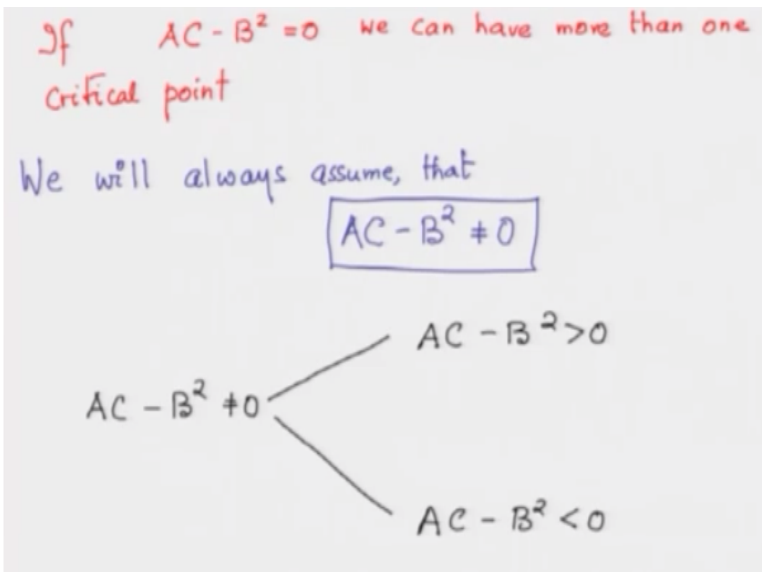


So my condition would be, so what happens if AC – B square is equal to 0. Then, there can be situation where there are more than one critical point and that is exactly what I am going to demonstrate here through an example on the board.

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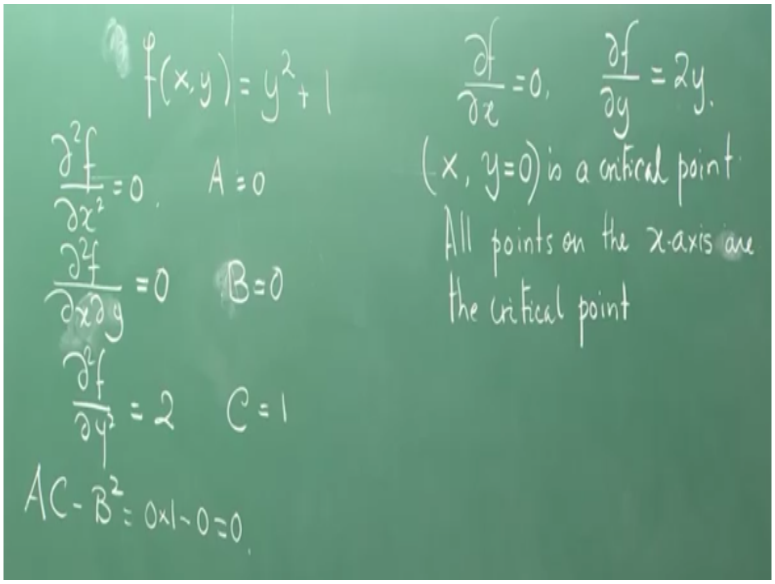


So consider, so I will write fxy is equal to y square + 1. So you can evaluate the partial derivative del of del x is 0 and del f del y is 2y. So which means y is equal to 0, any y equal to 0 will make del f del y equal to, so whenever y is equal to 0 whatever x you take it does not matter. This del f del x is anyway always 0. So it does not matter whatever your x, does not matter. So y equal to 0 is, any x and y equal to 0 is a critical point.

The x axis, all points on the x axis is the critical point. So what about this AC – B square? So we will check out that part also, what is AC – B square. So let us do what is del 2f del x2? Del 2f del x2 is 0, so A is equal to 0, because A is half of this. What del 2f del y del x, sorry del 2f del x del y is 0. So B is 0 because half of this is B and what is del 2f del 2f del y2? It is 2, so C is half of this. So C is 1. So AC – B square is 0 into 1-0, which is 0.

So here the situation where AC – B square is 0 and hence we have more than one critical point. We can have more than one critical point, but we will be happy just to keep one critical point that is 00. We will be working with that and then testing whether it is maximum or minimum or whatever, local minimum or local maximum. There will be global maximum and global minima and global maxima. So here we will always work with the premise.

We will always assume that AC – B square this is a holy grail. This is what we have to assume as we go along. So when so 00 is the unique critical point, this is something out. Our assumption now would be AC – B square is equal to 0. See, we will only really look at this issue when is 00, which is actually a critical point of Ax square + 2Bx square + Cy square is actually a maximum or minimum or a saddle point and once we know that, we can talk about the other things.

Our own function, that we have just seen. So, now when you have this condition that AC – B square is not 0, it implies two situations, either AC – B square is greater than 0 or you have AC – B square less than 0. There are two situations now. Either AC – B square is strictly bigger than 0, or AC – B square is strictly less than 0. We will deal with these two situations separately, okay.

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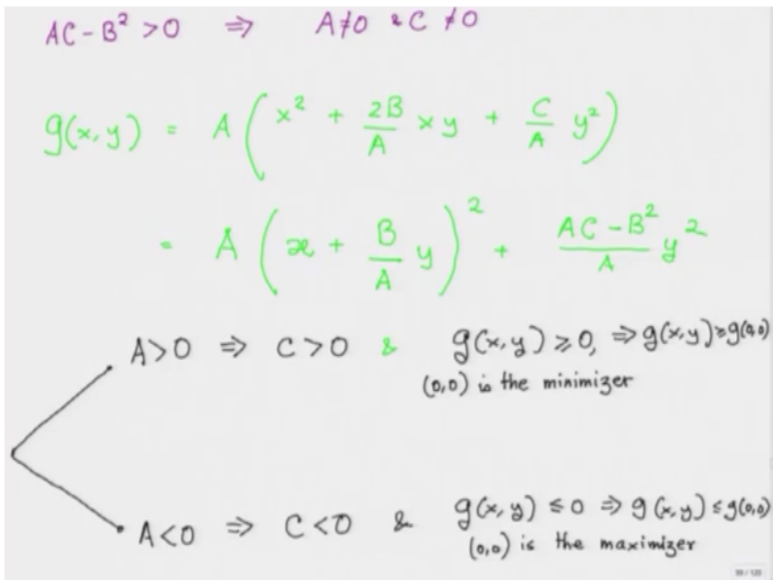


So first we will go to the case where AC – B square is greater than 0. So it is basically kind of this Mussaddi Lal's office, the television serial which came long ago in Doordharshan. So basically when AC – B square is strictly greater than 0 (FL: 28:50), one is A greater than 0 and one is B greater than 0 A less than 0. You have to observe that if this is greater than 0, it would imply that A is not equal to 0 and C is not equal to 0.

Because if A or C anyone of them is 0, then it tells me that –B square is strictly greater than 0, which cannot be, because is not negative so –B square cannot be strictly greater than 0. It has to be negative, less than equal to 0. There will be a contradiction. So once I have this condition AC – B square is strictly bigger than 0, A is not equal to 0 and B is not equal to 0 follows immediately that these two quantities cannot be 0.

Now one I know this they are not 0, I will make a small manipulation of gxy knowing that A is not equal to 0 and C is not equal to 0. I will write this as, I will break this up as, I write this as x square + 2BAxy + C/A whole square. Now you can do all these square completions and all these subtract something, add something to this, subtract something to that, which I am not going to do, because I think you should be able to do this yourself.

I am sure all of you know this much of math. Those who are observing this course should have little bit of inkling of what is going on. So this is what I have at the end after doing the little manipulations, add for example B/Ax square here, B/Ay square here and subtract B/Ay square again and then you have this. Here is the stuff. Now, A is not equal to 0, but there can be two cases one A is strictly bigger than 0 and one A is strictly negative.

So our AC – B square strictly bigger than 0, I thought (FL: 31:36) one is A greater than 0, strictly greater than 0 or A is strictly less than 0. So if A is strictly less than 0, again in a much lighter form (FL: 31:48). So what is that? When A is strictly bigger than 0 and C also strictly bigger than 0, because if C is negative, then AC becomes negative, then the whole thing becomes negative. So it shows that AC – B square is less than 0. So that cannot happen.

And when I have C strictly bigger than, sorry, when I have C strictly bigger than 0 and what do you have, so AC – B square is strictly positive, A is strictly positive, A is strictly positive, you have gxy, right, this whole thing is greater than equal to 00. This whole thing gxy is greater than equal to 0, which implies gxy is greater than equal to g00. So 00 is the minimizer. Now A is strictly less than 0 it is also (FL: 33:08), so if C is strictly less than 0.

Because then, if C is strictly bigger than 0, this will become negative and this will become negative, the AC – B square. So C is strictly less than 0 and so if this is positive and A is negative, then this whole thing is negative. Then A is negative, then this whole thing is negative, because this part is x + B/A whole square is positive. So which means that it will give me that gxy will become less than 0, because this whole thing is negative, which implies.

That is why all xy, do not worry, just for all xy if this happens, then for all xy, gf less than 0, also g xy is less than g00 or 00 is the maximizer. So in short if AC – B square is positive strictly and as I know as A cannot be 0, then if A is bigger than 0, then 00 is the minimizer. If A is strictly less than 0, then 00 is the maximizer. This is what we have to keep in mind. Now what happens if AC – B square is strictly less than 0.

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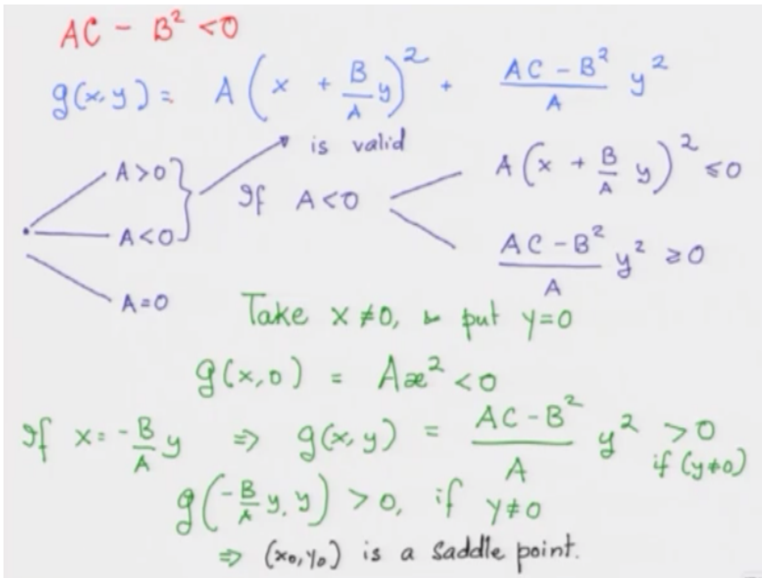


I will again reiterate the fact that AC – B square is chosen not to be equal to 0 to maintain the fact that 00 is the only critical point. This has got something related to strong convexity or strict convexity, all these convexity actually comes in. Convexity is certain in which you cannot avoid while you are maximizing or minimizing function convexity and concavity. It comes in very subtly here. It actually comes in in a very local sense.

But here we are essentially determining the quadratic function determine the concavity or convexity of the function. So convexity would give the minimum, concavity will give the maximum. This is the real thing that is the deeper thing, but we are not going to get into those deeper issues, but tell you this is very simple thing what we can do on a very simplistic approach. Now AC – B square strictly bigger than 0 is resolved.

So I have to look at what is AC – B square strictly less than 0, that has to be resolved. See the examples for this kind of this particular case and the proof of Taylor's theorem that would be put up on the, I will write down and that will be put up in the, what is that called, that will be put up in the portal, right. So let us look at that. So I know that A is not equal to 0, so anyway this expression is always valid, that we have already written in green ink.

Now let us write it in blue ink in this particular case. So if A is not equal to 0, this expression is valid. This is a valid expression. We just need A not equal to 0 for this expression to be meaningful and when AC – B square is strictly less than 0, A cannot be 0, right. If A is not equal to 0, then this is the valid expression, right. So if AC – B square is strictly less than 0, there are three situations, A is strictly bigger than 0, A is strictly less than 0, and A is equal to 0.

So when these two cases are taken, this is the valid expression. This expression is valid. This expression gxy is equal to this expression is valid because A is either positive or negative. So we will start dealing with the case if A is negative and of course A is positive and start dealing with the case A is negative. So I have just written on for A is negative. If A is negative, let us see what happens. A is negative two things are there. A is negative, then this part is negative.

And because AC – B square is negative and A is negative, that will become positive and this part becomes positive. A is negative then the two things will happen, one is that A into x + B/Ay whole square is negative and simultaneously AC – B square by Ay square is not negative. So again (FL: 38:18), if A is strictly less than 0. So now let use consider x and y, the points xy very near the point 00 in that circular disk around 00. So x0 equal to 0 and put y equal to 0.

Then g of, because this is valid, because A is strictly less than 0, g of x0 is equal to Ax square because A is strictly less than 0, this is also strictly less than 0. Now choose if x is equal to minus B by C, see the idea is that when you put y equal to 0, you take away this term and take this term out and now if I want to now just keep this term and take this term out basically what I am trying to show that there are some points around x, they are near to x, near to 00, where the function value is negative.

There are also certain points around 00 where the function value is positive, so when this condition occurs, AC – B square is strictly less than 0, what I have is the saddle point. So let me put x is equal to minus BA by y, so this part of gxy will vanish, that will imply that gxy is equal to AC – B square by Ay square, which is positive for all y0 equal to 0 or greater than equal to 0, whichever if y0 equal to 0. If y is 0, then this is strictly greater than or equal to 0.

Because we are taking points around 00, so we are not taking y to be 0 either, because is y0 only at 00 point, near small neighbourhood, if we take a point very near y and x 00, but keeping y positive or y negative not 0, then this is what we will get. If y is not equal to 0, this is what we will get. So I show that, so here for any x0 I will have this, negative and basically what I have showed here is that this one g of minus BA by y is strictly bigger than 0 if y is not equal to 0.

When y is very near 0 basically, so for all y very near 0, when y value is very near 0, this is exactly what you will have. So you will, so in the neighbourhood of 00 the function can take a positive value and negative value, when AC – B square is strictly less than 0, which makes x0 y0 a saddle point. So what for the case A strictly less than 0. So here for when A is strictly less than 0, it implies that x0 y0 is a saddle point.

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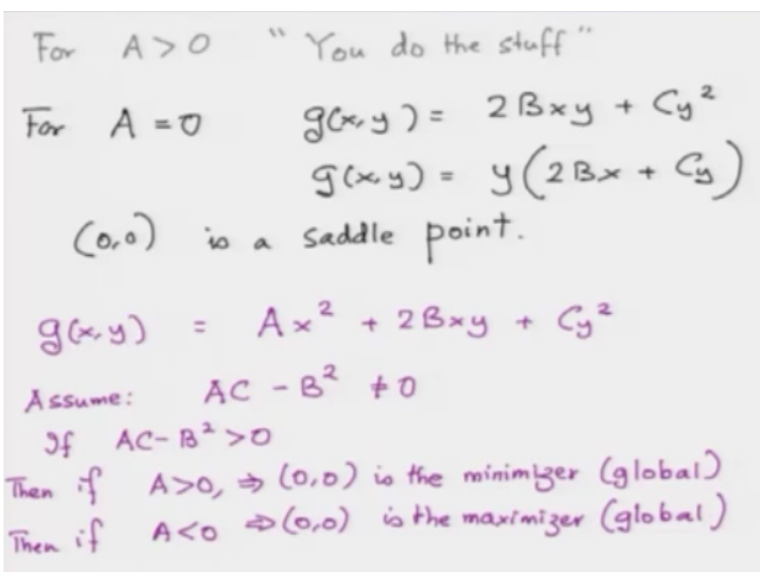


For A positive, you do the stuff absolutely, try it out, try out the argument. You do the stuff. It is fun, do not worry. Now let us look at the case A equal to 0. Now when you have seen A can be 0 in AC – B square strictly less than 0, because the whole thing is valid because then minus B square is strictly less than 0 is valid. What I want to say is that for A equal to 0, you cannot use the expression that I have used for gxy. For A equal to 0, this expression for gxy is invalid.

This does not make sense, so for A equal to 0, you have to go back to the old expression and write that gxy is 2Bxy + Cy square. So this can be written as y, so gxy is equal to y into 2Bx + Cy. So if y and 2Bx + Cy both have the same sign, it is positive, both have opposite sign it is negative. So here again I can find 00, at 00 it is 00, again at around 00, I can find points where this is positive gxy is positive and other cases gxy is negative telling that xy is a saddle point.

So again we conclude that 00, sorry 00 in this case is 00, sorry I will just write down 00 is a saddle point. So here also we conclude that if A is equal to 0 and AC – B square is strictly less than 0, 00 is the saddle point. So all the cases whatever be the value of A, 0, positive, negative 00 is always a saddle point in this case. So let us summarize, so let us make a summary. So I have gxy given to me with Ax square + 2Bxy + Cy square, assume.

So first assumption, assume AC – B square is not equal to 0. If I assume that, then what I have is the following. If AC – B square is strictly bigger than 0, then if A is strictly bigger than 0, it implies that 00 is the minimizer. So when 00 is the minimizer of gxy, it is the local minimizer of fxy because fxy is equal to fx0 y0 or f00 plus that quadratic term when this fxy value is almost equal to this part, the quadratic approximation when xy is very near 00.

So this is only valid when you are talking about a local minimizer. So whatever it is, 00 is the global minimizer for this quadratic function. It is actually a local minimizer for your original function. Basically, this condition AC – B square is strictly greater than A is strictly bigger than 0, then these two conditions tell you that Ax square + 2Bxy + Cy square is actually a convex function and here convex function which is not negative and of course 00 is the minimizer.

In a convex function and when 00 is the critical point, and this is the only critical point, so that has to be the minimizer. See every critical point of the convex function is the minimizer, that is exactly what we discuss in the last class and this is what we have. So this is actually deciding convexity, then if A is strictly less than 0, basically this happens and this happens, then 00 is the maximizer, it is the global maximizer of the quadratic function.

Please understand, this is not local, these are global. So whatever is the global maximizer or minimizer of the quadratic form, quadratic function is the local maximizer of the original function. Now we will come to the case, our expression, we had x0 y0, we did not put x0 y0 equal to 00, now we will come to what is called a general form of the quadratic expression. Now we will take a general quadratic expression. So what is a general quadratic expression? So now we will have a general quadratic expression

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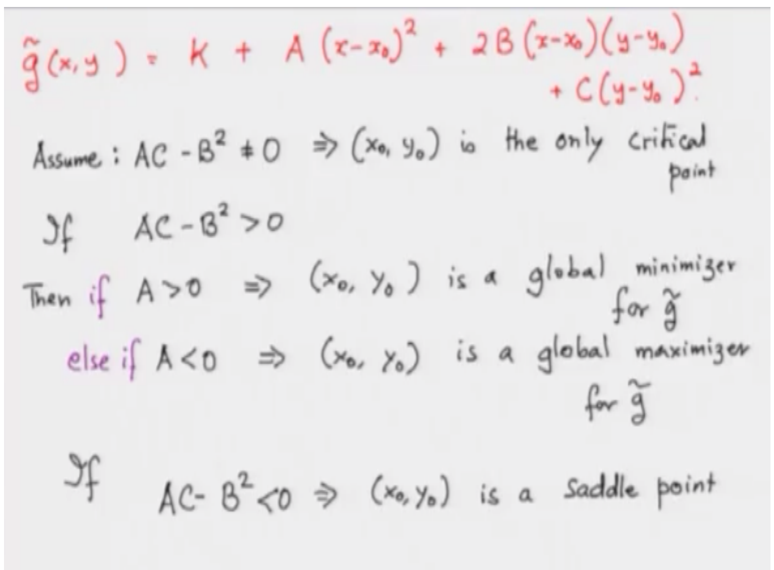


Where our g would be given as, I will write g tilde xy has some K. K in our case is fx0 y0, Ax – x0 square + 2B x – x0 into y – y0 + Cy – y0 square. Now in this case we will again assume AC – B square is not equal to 0. This is something which is important for us. This has to be maintained. This will imply that x0 y0 is the only critical point. So I have to assume this and this will imply that this is the only critical point.

This is a kind of very important thing. Actually this is the determinant, which we will not, cannot speak now in this structuring, but okay. Now those know linear algebra, they would understand we are talking about positive definite nation, positive negative define nation, hessian matrix of the functions, but we will not get into all that. That is just a little bit of naming topic, which is not a bad idea sometimes. Now what would happen in this particular case?

So we have to determine x0 y0. So again if AC – B square is strictly bigger than 0, then A greater than equal to 0 implies x0 y0 is a global minimizer for g tilde. You might ask me what about, you did not write anything about AC – B square in this particular case, the first, the AC – B square equal to, less than 0. AC – B square when it is less than 0, you have already seen, that 00 is the saddle point.

So basically it is nothing, but the saddle point that we talk about in calculus is not the saddle point in the sense of modern optimization and that has to be understood. So again, then if this, then if this else if A is less than 0, it implies that x0 y0 is a global maximizer for g tilde. If AC – B square is less than 0 as before, it would imply that x0 y0 is a saddle point. So earlier we have basically shifted x0 y0 to the origin.

Now we have said x0 y0 is an ordinary point, the same story will work. So we did our calculations in the simpler setting of x0 y0 to be 0 and then we worked with, we have just bought our intuition into the general setting. So can we translate? So this is the general quadratic form, for which how we determine how x0 y0 is a minimum. Can we translate it to a function, where we did a general quadratic form with error.

So when we will take x and y very near to x0 y0, we will forget about that error and here we will form our AC – B square. So forget the error for a little time.

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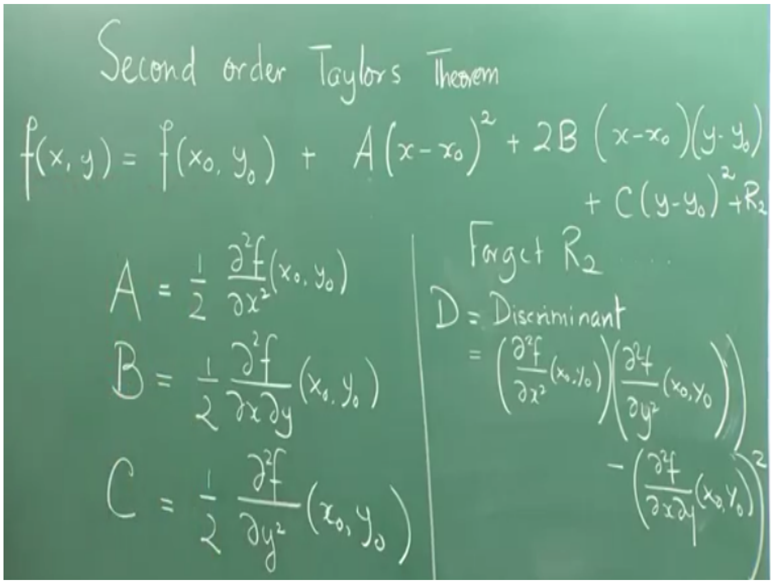


Forget R2, really should not, but for all practical purposes, let us forget R2 and then you construct what is called D. AC – B square if you write, then it will become one-fourth of this into this minus this, but this one-fourth tag you do not have to take into account, just write that, because once you are talking about something is bigger than 0, or strictly less than 0, so you multiply by another positive quantity, does not change the sign.

So what you have to do is, write the discriminant, D is equal to discriminant and that is nothing but AC – B square that is del 2f del x2 evaluated at x0 y0 into del 2f del x2 into del 2f del y2, this is C evaluated at x0 y0 minus del 2f del x del y evaluated at x0 y0, this thing whole square AC – B square. So this is your discriminant. This is called a discriminant, which you construct from this ABC. Now write the same story, which is here.

Now change your g tilde with f, because we have forgotten the R2. So your g tilde is playing the role of f, your K is playing the role of fx0 y0. So you might say that okay I am not very comfortable with forgetting R2. I understand, everybody who wants to, who is much more clear, who is more erudite will not want to forget it, but for us the same story can be written. We will write the same story, we have written for this, now which I will write down.

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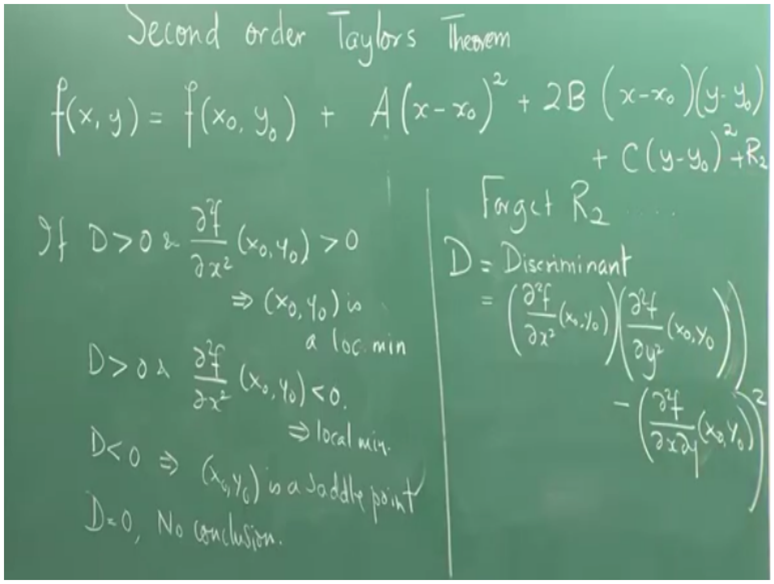


If D is greater than 0 and del 2f del x2 at x0 y0 is strictly greater than 0, it implies x, AC – B square, whatever we have written for the other one, x0 y0 is a local minimizer. If D is greater than 0 and del 2f del x2 x0 y0 is strictly less than 0, it implies local mean. If D is less than 0, it implies x0 y0 is the saddle point and D equal to 0, no conclusion and with this we conclude today’s lecture. Hope that you would go through all the calculations I have done. This is a slightly tougher thing. Thank you.