Calculus of Several Real Variables

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Module N0 # 03
Lecture No #15
Maxima and Minima for Two Variables

Welcome to the last lecture of the third week so today we are going to speak about maxima and minima that too of several variables. So essentially I would put myself down to 2 variables and I would expect that you can make the extension maximum and minimum or maxima and minima whatever you want to call it is a part of mathematics known as optimization.

So this operation of I maxima and minima is given a generic name of optima and subject is called optimization. I spend my research life trying to understand optimization problems. So it excites me to speak about maxima and minima of course you can say you could have written maximum and minimum of several variable the standard way things are written but I am taking about maxima and minima you might be wondering why I am just telling maxima and minima.

Please note that there is a very famous saying of the mathematician Leonhard Euler that nothing in this world takes place without some quantity being maximized or minimized. So optimization is everywhere nature supposedly is a big optimizer so this is a very important area of mathematics with huge which is kind of border area between theoretical math which sometimes people called pure math and applied math which is has huge applications.

A very very big mathematician have been involved in trying to understand how to figure out a maximum and minimum of function. It is not so simple one might look it might look simple on the facet of it here because we will do simple examples gives make the try to make the theory simple but whatever it is. It is not so simple as it is and to start with a little lighter version of the thing.

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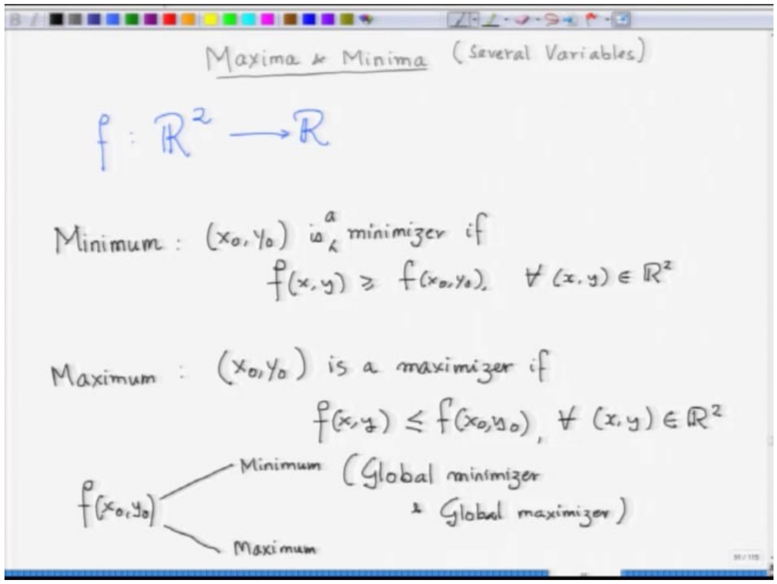

Once in New Delhi I met a family which had 2 lovely little girls and when I asked them their names they said maxima and minima. They one was called maxima other was called minima I was very surprised as an optimizer that parents could give such innovative names mathematical names. So here we would consider functions of 2 variables and we will seek the definition of minimum and maximum so point x naught y naught is minimum or minimizer is a minimizer.

If f of xy is bigger than equal to f of x naught y naught for all axis inverted A is a symbol for the word the phrase for all x y e naught 2. So I am trying to minimize this function over the whole of R2 so it is called unconstrained minimization and maximization you have the same kind of story. So x naught y naught is a maximizer if f of xy is less than equal to f of x naught y naught for all xy in R2 ok.

So this point this value f of x naught y naught whatever we x naught y naught is called the minimum or maximum is called the minimum value or maximum value as is the nature of x naught y naught. This kind this definition actually tells us that these are global maximizers global minimizer and maximizer and global minimum value and that is what we really seek when we talk about minimization and maximization. But are they is it always possible to detect them the answer surprisingly is no and that is why we look for what is for local minimizers and local maximizers.

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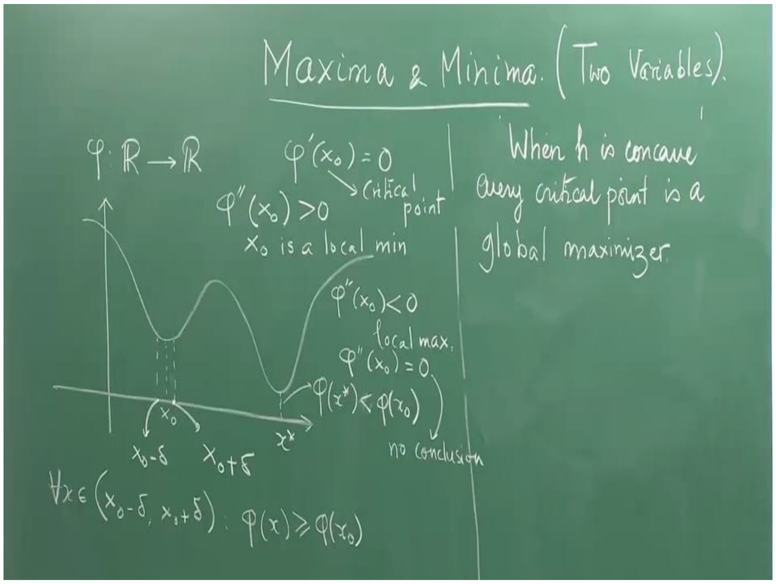

For example you have learnt in your functions of one variable suppose phi is a function from R to R then a local minimization local minimizer is something like this that okay look at let the function be like this. So here you see locally for example if I take this point x naught and take a very small delta that is maybe. So basically I take so I consider an interval x naught + delta and x naught – delta where delta is positive.

So I am considering interval x naught – delta and x naught + delta and in that interval you observe that is for all x in this interval you have that phi of x is bigger than phi of x phi value at x naught. So that is what is called a local minimizer because you come to this point say x star so this x star value at this point the function phi value the phi of x star is strictly less and the value of phi of x naught.

So there is of x star for which the function value decreases further so this is only minimum or minimizer in the local sensor not in the global sense so global means it has to this thing has to hold for every x in R here then. Here we show that it for example what through this drawing at least that it is not the case a extra is a point where this is broken with respect to phi x naught so x naught becomes a global local minimizer.

So in most cases we will be able to figure out local minimizers and not global minimizers so what are local minimizers? For a function from R2 to R for f from R2 to R so for that we have to first talk about an open ball of radius delta which we have written earlier so at a point x open wall of radius delta centered at x is the collection of all y in R2. So these are 2 times y naught 2 such that normal y – x is strictly less than delta.

So these kind of sets are often called as open sets in mathematics because we do not have the boundary that is we do not have the points y for which this is equal right that is it is a kind of one 2 dimension basically it is a circular disk something like this. Where we have everything inside but we do not have the points on the boundary we do not include that all the nuances of optimization cannot be said in such a small one lecture.

But it is kind of understanding that we want to give you about the subject that ok so the notion of this delta neighborhood is now replaced by this delta open disc. So this is called the delta which is radius I will call it a delta open disc so this simply means so if y is y1 y2 and x is x1 x2 it simply means y1 – x1 whole square + y2 – x2 whole square this sum root over has to be less than delta that is the meaning of this. This things simply means this and what does what is the meaning of a local minimizer?

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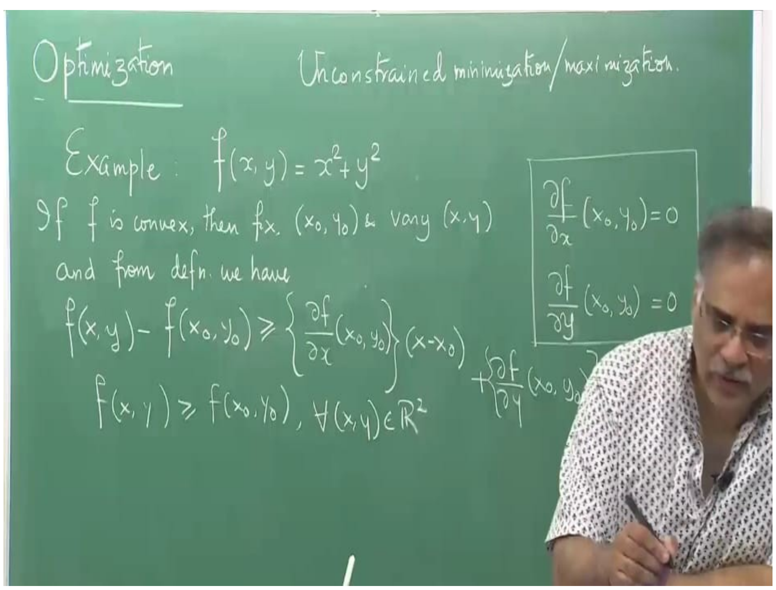

So x naught y naught is a local minimizer of f over R2. So here we are going to talk about maximization minimization over the whole space and such actions is usually termed as unconstrained minimization. There is no additional requirement on the point’s minimization maximization whatever optimization or optimization(()) (10:47). So x naught y naught is called local maximizer over R2 if there exist delta greater than 0 such that for all x in the ball delta around the point x naught y naught.

So all delta for all x which is in this ball we must have f of we have f of xy to be bigger than f or equal to f of x naught y naught. You have learnt that in order to find a local maximizer or minimizer of a function of one variable if the function is differentiable the first test is to find a point which satisfies this.

Once a point x naught which is found is called critical point then you know that ok this is the necessary condition that what you show that if x naught is a local minima then the derivative must be equal to 0. That is what that was a great one of the greatest works actually Pharma knew about this so for polynomial function but he did he one really needs to look at the way Pharma did it to see the joy of doing mathematics.

Now what happens is that if you just find a point you cannot say that it is a local minimizer the first result it says that if it is a local minimizer then this will happen. So which means a local minimizer must satisfy this condition so to find a local minimizer I should seek my local minimizer among points which satisfies phi dash x naught = 0. So once such a point is found I really have to test it if I can easily test it, it is fine if I cannot then I have to go to second order condition if that is available.

So if this is true then we declare x naught is a local mean and if this is true local max and there is no conclusion if I just have this to (()) (13:46) usually this is called a saddle point that is the shape of the curve changes here like for example y = xa cube. The derivative is 0 but the second derivative is also 0 at that point at x = 0 and you get no conclusion so from this will so second derivatives is 0 no conclusion.

Can these ideas we bought into the domain of 2 variables the answer surprisingly is yes and I do not want to get into that detailed proof which actually involves the use of Taylor's theorem. So I do not want to get into the details of that we will use Taylor's theorem little bit later but you see that what happens I am just giving a very intuitive proof this is not a rigorous proof the idea is very simple that when I say that ok.

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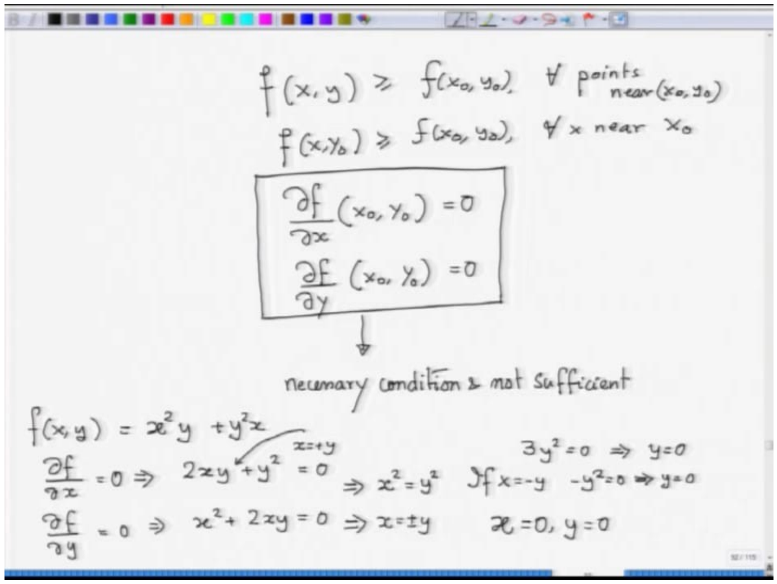

F of xy is bigger than f of x naught y naught for all points near x naught y naught I am just using I am not getting this delta stuff every time points near x naught y naught then intuitively it appears and it is true also that if I fix my y naught then f x y naught because x y naught is also point near x naught y naught. If x y is the point near x naught y naught the next y naught is also point near x naught y naught. So x or y naught is bigger than equal to f of x naught y naught so when I fixed y naught in this function if xy so it becomes a function of one variable.

So function of only one variable x and what is that the derivative must be equal to 0 is the necessary condition at x naught so delta f delta x because just the derivative of this would and tell me this thing that should be equal to 0. Similarly I can put x = 0 and tell the same story where y naught would become maximizer or minimizer of the function f of x naught y. So then that case my necessary condition from what I have written down there for 1 variable will become this I have not done this in detail.

So please forgive me for being so crude because we have so less time that sometimes we just go over giving you the ideas of the intuition. So this is a necessary condition and not sufficient we are assuming that all points are all problems are differentiable problems all functions are differentiable. We are not discussing non differentiable problems we I think we will shy away from the fact that functions are non differentiable but this is the necessary condition any x naught y naught satisfying this is not necessarily minimizer or maximizer.

An example that is given in the book which is very simple and it is a pretty well done example. So for example take f of xy = x square y + y square x. Once you have taken this take the partial derivatives delta f delta x and equate it to 0 and see what points you get. 2xy + y square = 0 and delta f delta y would give me x square + 2xy = 0 if I subtract this two I will obtain this would imply x square = y square.

So this implies to me that x = + - y now put x = +y in this first equation so it will give me when you put y here will give me 3y square. So put if I put now I put x = +y here so that gives me 3y square = 0 and that is implying y = 0 if I put x= -y also in this equation so it will become –y square = 0 this will imply y = 0. So when I put y = 0 in both these equations if I put y = 0 so x is + - y so x = 0 so my only point which satisfies these 2 equations delta f delta x = 0 and delta f delta y = 0 is x = 0 and y = 0.

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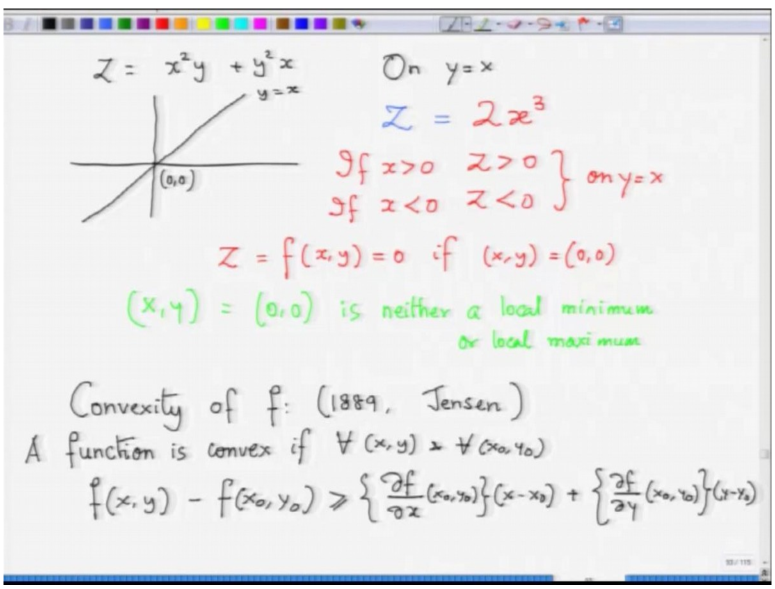

But the fun part is that this is not a minimizer because look at this equation Z= x square y + y square x. Now here I have the point 0, 0 which is a critical point here I have the point 0, 0 which is the critical point and which is not does not appear to which is a critical point and the only one is it the minimizer. So let me do a little test the test is this it is a very simple test to show that it is not even a local minimizer.

Let me consider the line y = x on that line the function value on that is on y = x Z= so I am putting y = x execute y = x execute which 2x cube. Now which means it depends on what x you choose whether you choose x positive or negative if you choose x positive if x is positive Z is positive if x is negative Z is negative on y = x but Z = fxy f evaluated xy is 0. If xy is 0 if xy is the origin the 0, 0 vector then fxy is 0.

So whatever however near you go near x on this line a 0, 0 there you can always take a positive value or all you can take a negative value the function value either becomes it becomes positive and negative in both sides of x. It is not that it is always positive or always negative so 0, 0 is neither local maxima nor local minima so 0, 0 so here is an important conclusion xy = 0, 0 is neither a local minimum or local maximum.

So this shows that this condition delta fx delta f delta y = 0 and delta f delta y delta f this one that is delta f any point which satisfies this. So if a point satisfies this it is only what possibly a point which might be a local maximizer or minimizer it does not tell me. So for maximum also we can write the same story which I am not writing so it does not tell me that it is really a local maximizer or minimizer.

So you need additional conditions what are those additional conditions one is the second order condition another is a convexity condition. Convexity is a particular nature of the function which tells us how to what would happen if I just have first order information and I have not used second derivatives here. I have used second derivatives but suppose I do not have second derivatives or do not want to compute them they are expensive to compute.

So I do not do anything so how can I get an information so that brings us to the topic of convexity of functions of 2 variables. Without going into much details about convexities history I can just tell you that this idea was known in 1889 if I am not mistaken all the dead to a Danish not Danish I think a mathematician from Holland M Jensen so it is sometimes the definition of convexity is called Jensen's inequality.

But for the case of functions of 2 variables and which is also differentiable a function is convex if I am giving an equivalent definition in terms of partial derivative. A function is convex if for all xy and for all x naught y naught these are elements in R2 of course I am not writing that f of xy – f of x naught y naught that is the definition for just take it as the definition for the time being.

Unfortunately I do not have time to teach you convex analysis though that would it be at a very different game. So this is greater than delta f delta x and evaluated at x naught y naught into x – x naught + delta f delta y evaluated at x naught y naught at y – y into y – y naught this is the definition of the convex function. I can give you an example of such a function suppose f of xy is x square + y square the paraboloide and this is an example of a convex function.

This will satisfy that definition you can try out with various points you will see that these definitions will get satisfied always. So this is an example of a convex function I do not want to get into details but what happens if the function is convex. So keep on looking at the this electronic slate where I have written down the definition. So suppose this result this fact is true that I have a critical point that delta f delta x sorry this delta f delta y delta f delta x is 0 and delta f delta y is 0.

That is the 2 necessary conditions is satisfied and the function is convex then is x naught y naught a solution the answer is yes. Because if f is convex then fix x naught y naught and vary xy and observe from what we have defined in and from definition we have f of xy – f of x naught y naught is greater than = delta f delta x at x naught y naught into x- x naught + delta f delta y evaluated at x naught y naught.

So with the point which is here on the after the minus sign we have to evaluate the derivatives at those points. Now if I put delta f delta x naught = 0 and delta f delta y naught = 0 what do I get. So here I can vary any x y naught 2 so I get f of xy because this will be 0 now this side f of xy is bigger than f of x naught y naught for all x y naught 2. So what does it say? It says that if you have a function convex and if you have its critical point then that critical point is a global maximizing value.

It is a global maximizer for example a critical point of this function is at 0, 0 and 0, 0 is the global max meaning global minimizer of this value it is not a maximizer it is minimizer. So any critical point of a convex function is a global minimizing value so for so this is a very important statement that we are going to now write down for a convex function every critical point is a global minimizer.

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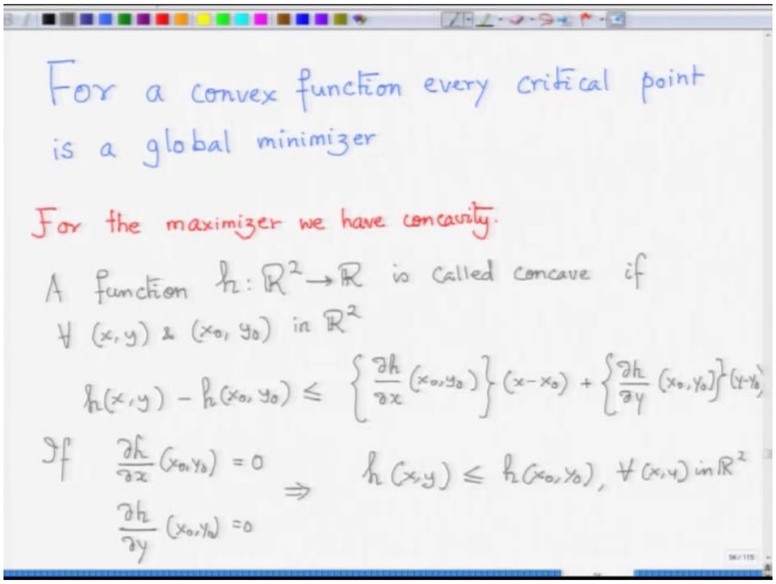

For a convex function every critical point is a global minimizer so can you tell can we tell something like this for the case when you try to find a maximizer. For the maximizer we have concavity so what is then concavity? Concavity is a function phi say a phi or shi whatever it is concave if I just reverse this inequality so let me write down a function say h from R2 to I am just changing f to h just to keep a keep consistency.

H so R2 to R is called concave ok so I am assuming differentiability please understand all functions that we talk about a differentiable there is no discussion of non differentiable functions. So function is called concave if for all xy and x naught y naught in R2 h of xy just reverse this game we reverse this game and that will give us so delta h delta x evaluated at x naught y naught x – x naught into y – y naught.

Now suppose I have a point x naught y naught if delta h delta x evaluated at x naught y naught is 0 and delta h delta y evaluated at x naught y naught is 0. It implies from this because I can vary all the xy's keeping x naught y naught fixed because these two for all xy and all x naught y naught and I can put this 0 here to obtain that h of xy is less than equal to h of x naught y naught. For all xy in R2 so what do I get when h is concave so when h is concave every critical point is a global maximizer.

So let me write that down that when h is concave every critical point is a global maximizer these are global maximizer. So this is what you will not find in normal calculus text they do not speak about convexity or concavity but I have also not done a very fair job. Because if I want to talk about concavity and convexity then it will basically take the if I go to details it will take the whole class actually whole time we will go discussing about introducing convexity. So we will end the talk here we will make a little change in our structure we will push one part into the fourth thing.

The second derivative test I will separately teach you now if f is not or h is not concave and f is not convex how do we handle or how do we use our second is there a second order derivative is a second order information. How can I use it to determine in the same way as I did for functions of one variable whether the function is concave whether the function is having a local minimizer or local maximizer or nothing a saddle point which I can say.

So that decision can be done using second order stuff but would actually entail almost nearly an hour’s lecture and I cannot copy simply waste of time constraints do it. So what I will do if you go to the course syllabus the one thing that I would have I would be talking about at the end the last part would be more about vector fields which but we can skip that because that would be little advance for you. I was thinking up a putting a advanced stuff but it is better for you to do and learn this second order how do we develop the second order condition.

How do you prove what we have so it is very important to know how to get up to get a feel of the second order condition is no to do the proof and the idea is that we can use quadratic functions that is second order Taylor’s expansion to actually do the most of our job. So because we are talking about second order we will talk about second order information we will use Taylor's expansion of order to which we did in the last class and this that will take a whole class to explain and that will do step by step with several examples.

Because that is very very important and that cannot be done in this little part so these was just about introduction of maxima and minima and just introduce you to these notions of convexity and concavity. So we will push the vector field section out I can put that as a special lecture if everybody wants you can put it on the portal we can put it as a special lecture on boron vector fields.

But instead of starting the fourth week with Lagrange multiplier I will talk about second order tests for finding maxima and minima of 2 variables of functions of 2 variables that would be what we would be discussing and then after doing that the next class would be on Lagrange multipliers and so we will shift one the last one will be shifted. So in that way the vector field would go out and so this little modification I will let the NPTEL know.

But it is very but as I walked to throughout the whole thing the yesterday night I was working out the whole thing and I realized that it is absolutely essential to have a full class on how to determine you second order information to determine whether critical point is a maximizer or minimizer or maybe nothing saddle point if you want to call it. So let it be that way that is a better measure of pedagogy I would say and let us call it quits for today and we end the fourth week here and fifth week we start with the second order test thank you very much.