Calculus of Several Real Variables
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Module No # 03
Lecture No # 14
Taylor's Theorem

So welcome to the fourth lecture today we are going to speak about Taylor's theorem for function of more than one variable. It is a very boring stuff actually to just say I will do Taylor's theorem and blah. Those who have done calculus of one real variable has definitely heard of the name of Taylor's theorem not Tailor who does your clothes. It is a English title Taylor and possible people do not bother about who this Taylor was his name was though I write think I if I have not I remember correctly I have written his name in my one variable lecture also his name was Brooks Taylor.

And Brook Taylor was the student of Newton in Cambridge and Newton was interested in the binomial theorem and he had actually used binomial theorem. He should teach this to his students just lecture his students and from there on Brooks got an idea that for a general function how the binomial theorem expansion of the binomial theorem can be used to in such a way that the coefficient get related with the binomial coefficients. That was the idea of Brooks Taylor. You try to expand the function x take the function x to the power N and just do the Taylor's theorem that you know perhaps from calculus of one variable.

I do not remember at this moment what I did in calculus of one variable it is very difficult to actually go back and really make a check. But it is not a bad idea that if I start telling you what or Taylor's theorem is. See Taylor's theorem is used to approximate the value of function which might be difficult to compute.

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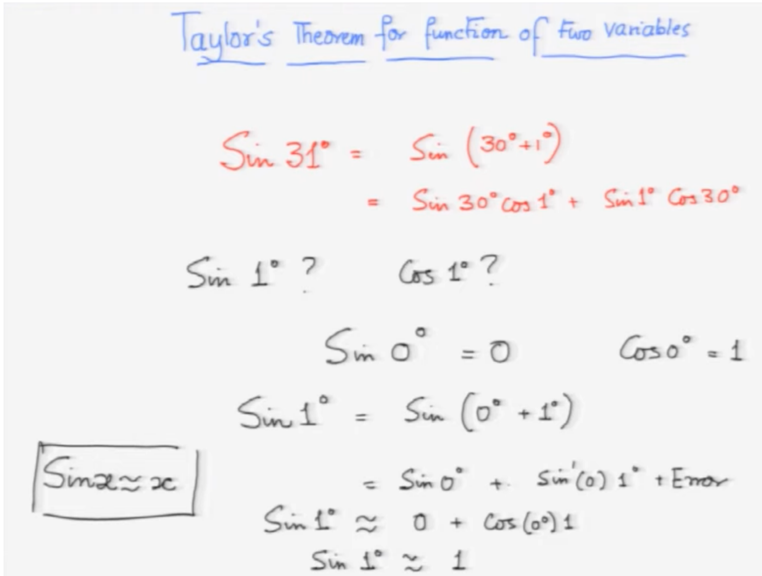

First of all, I say ok compute sin of 31 degree (()) (02:44) it is not a problem. I know Sin 30 degree so I can write this as so you can write this as Sin 30 Cos 1 Sin A Cos B + Cos A Sin B supporting Sin 30 and Cos 30 value is all right not a problem. But what is what about Sin 1 and Cos 1 that is not so easy to find out. You tell me what is Sin 1 degree if I tell ask you unless you have a trigonometric table in front of you, you cannot tell me what is Sin 1 degree right away right. I cannot tell you let me do very frank.

How can I find Sin Cos 1 degree? How can I find Cos 1 degree or say Sin 1 degree? That was the question. Computing functional values which are difficult to compute so what I do is that I can immediately get a good approximation by using the Taylor's theorem why because the Taylor's theorem depends on only 4 fundamental arithmetic operations of addition, subtraction, multiplication and division. Only with these 4 operations you can take a known function value and estimate very fast and unknown function value.

For example, I know Sin 0 degree is 0 and I know that Cos 0 degree is 1. And then I can compute the derivatives of Sin at 0 Sin first derivative second derivative and all and keep on giving a better and better estimate of the function. So I can make so those who know some those who remember their calculus of one variable Sin 1 degree can be written as Sin 0 degree + 1 degree. So 1 degree is the difference so it is Sin of 0 degree which is f of x + h which already know.

Sin of 0 degree + Sin dash at 0 into 1 degree + some error so roughly if you want to estimate Sin 1 degree or maybe point 001 degree if you want is Sin of 0 degree which is 0. So roughly and what is Cos of very rough estimate is Sin 0 is 0 + Sin of derivative of Sin is Cos. So Cos of 0 degree into 1 and we are not taking the error for the moment. So Sin of 1 degree has a rough estimate of Cos 0 which is 1.

And those who remember their trigonometry class or those remember that Taylor's expansion and those who remember the physics class you will see when x is very small facet too often use the fact that Sin x is can be approximated by x here you see that is exactly what is happening. So a very fast estimate of what you can say I do not like this idea of Sin 1 degree we know one does not look good ok just move ahead a bit.

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$$\begin{aligned} \sin 1^\circ &= \sin(0^\circ + 1^\circ) \\ &= \sin 0^\circ + \sin'(0^\circ)\,1 + \tfrac{1}{2}\sin''(0)(1)^2 + \text{Error} \\ &= 0 + \cos 0^\circ\,1 + \tfrac{1}{2!}(-\sin(0))\,1 + \text{Error} \end{aligned}$$

$$\sin 1^\circ \approx 1^\circ$$

$$\begin{aligned} \sin 1^\circ &= \sin 0^\circ + \sin'(0^\circ)\,1 + \tfrac{1}{2!}\sin''(0)(1)^2 + \tfrac{1}{3!}\sin'''(0)\,1^3 + \text{Error} \\ &= 0 + \cos(0)\,1 + \tfrac{1}{2!}(-\sin(0))(1) + \tfrac{1}{3!}(-\cos(0)).1 + \text{Error} \\ &= 0 + 1 + 0 + \tfrac{1}{6}(-1)(1) + \text{Error} \end{aligned}$$

$$\boxed{\sin 1^\circ \approx 1 - \tfrac{1}{6} \approx \tfrac{5}{6}}$$

And I have Sin of 1 degree I will just increase one more step is 0 degree + 1 degree. So I am using what I know is Sin 0 degree + Sin dash 0 degree into 1 + half Sin double dash 0 into 1 square plus error. Now let us see as so I am not bothered about the error at this moment as we keep on increasing more and more terms you know in Taylor's theorem the error becomes smaller and smaller that is the key idea.

So I am making a better and better approximation. So Sin of 0 is 0 so it 0 + Cos 0 is 1 + half of derivative of Cos is minus of Sin ok. Cosine derivative of Cos is minus Sin Theta, so it is minus of Sin 0 which is 0 into 1 + error. Oh, my goodness it is 1 again why that is not the very good idea let me take the third round of approximation okay. Sin 1 degree so Sin 1 is 1 1 so second order approximation is also 1**.** So this is not the very bad estimate of Sin 1 is 1.

So Sin 1 degree if I can write in the form of Taylor's theorem which you know is Sin of 0 degree I am just playing along do not worry Sin dash of 0 degree into 1 + half of now 1/2 factorial actually if you go by Taylor's theorem. So Sin double dash 0 into 1 whole square + one third 1/3 factorial Sin of triple dash 0 into 1 cube plus error. So it is now Sin 0 + Cos 0 into 1 minus sorry + 1/2 factorial into minus Sin of 0 + 1/3 factorial which is 1/6 into Cosine now it is minus of Sin theta.

Now you take a derivative again it will become Cos Theta. So minus of Cos of 0 into 1 so it is becoming 0 + 1 + 0 Cos 0 is -1 + 1/3 of cos 0 is 1. So one third into -1 into 1. So it is becoming 1- one third so it is two third. So see now Sin 1 degree is 2/3 it is not so I have a better approximation now plus error. Sorry I am making a mistake not 2/3 it is 1/6 this is 1/6 I am making a mistake please correct sorry I am very bad at calculations I really admit that and the so it is so approximately so from an approximation point of view when I forget their so my rough estimate is Sin 1 degree is 1- one sixth.

It is 6 - 1 5/6 so it is not very less than 1 so it is a rough idea Sin 1 degree is 5/6 not a 6/6 got 5/6 little less than 1. But this approximation vanishes when we go very near 0 this approximation becomes much better it will keep on giving you x. So I am improving a approximation of course the value has to be very quiet near 0 because that is what it is. Now you see I have just used addition, subtraction, multiplication and division to get my result knowing one value I have found the other value and that is the key idea about the Taylor's theorem. So what happens when I want to write down something in terms of derivative I am writing it down and I am trying to explain things there in the board.

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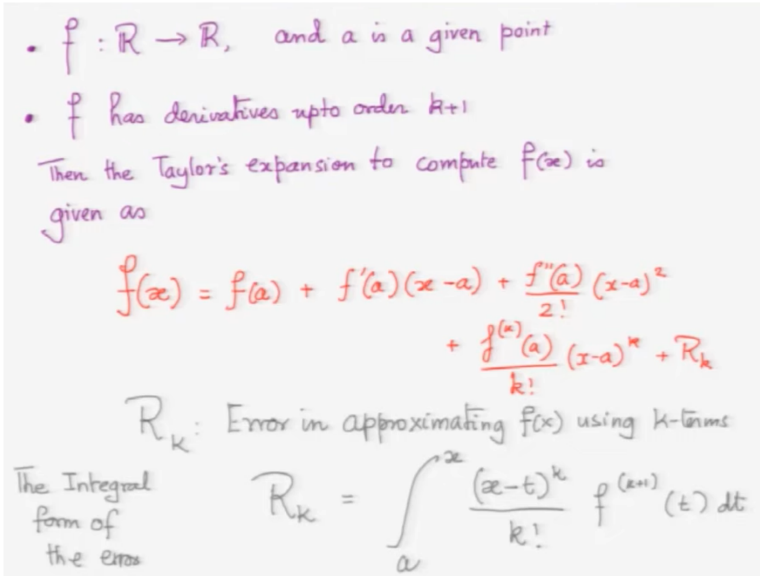

So let us now let us have function f from say R to R and a is a given point f has derivatives up to order k + 1. Then Taylor's expansion to compute fx is given as so a is known to means f of a + f dash of a into x minus a + f double dash of a / factorial 2 into x minus a whole square + the kth derivative of a / the factorial k into x minus a k plus error term. The error term always includes the error term the many way of writing the error but one of the standard ways is writing it in the integral form.

The error term always includes the k + 1 derivative. Of course, you can write 1 / k+1 factorial and all those sort of stuff. But you can write in terms of so Rk, Rk actually means error in approximating effects up to k terms fx using k terms using k terms only. So if I use k terms there is an approximation of fx f of x at x if if value at x and there is an associated error. So how is that error given Rk this is called a usual the most popular one is the it is why this is (()) (15:05) given by Cauchy the integral form of the remainder.

Integral form of the I will always call it error the people call it the remainder I will always like to call it the error because that reflects the error that you have in between this approximation and fx. Because fx minus this approximation is Rk and that is given by a to x integral x - t to the power k the t is varying between a to x. So you k factorial k t dt so this called the integral form of the error and this integral form has a very important property so what is that property? Rk / x - a to the power k this goes to 0 as x goes to a. This is an important property of the Taylor's theorem ok a pretty important property.

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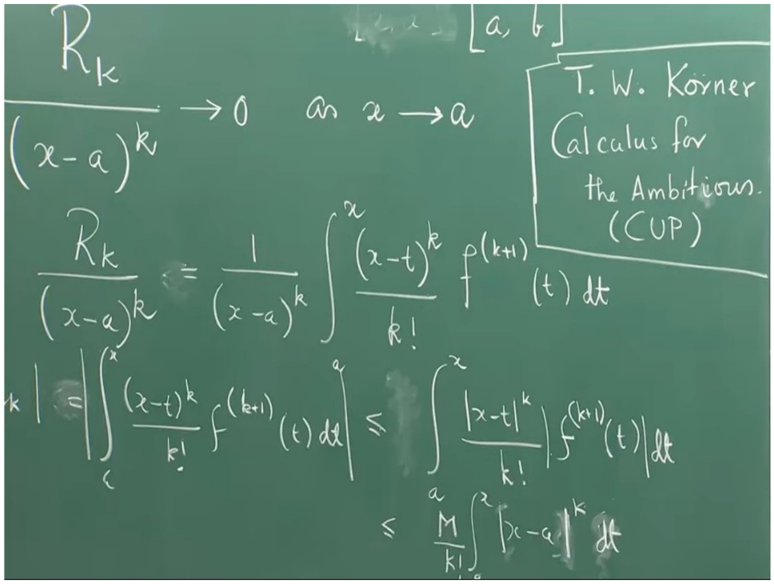

How do we get to this form so let us look at what is this? RK in so you will get the idea what is happening x - a k is sorry = 1 / x - a k integral a to x, x - t to the power k by factorial k f of kth derivative at t the k + sorry k + 1 at derivative of t dt. So what you do basically you look at Rk mod Rk x is bigger than a here of course mod Rk and look at this is equal to integral of a to mod of a to x, x - tk / k factorial f of k + 1 t dt mod whole thing.

And this is less that equal to as you know the modulus of the integral is less that equal to integral of the mod or the function. Mod of integral of mod of integral fx is less than equal to integral mod fx. So integral a to x mod of x - t to the power k this t is language of a and x by k factorial f of k + 1 / t dt. Now what happens most of the cases we consider this expansion suppose we consider this expansion on this interval x, a or in interval say sorry in the interval a to b.

So if the k + 1 is derivative we assume it is continuous so it is bounded in this interval by standard (())(19:25) theorem. So basically you will get from here is integral a to x sum M on this x - a to the power k / factorial k into dt. This strictly less than a maybe this is less than equal to M because this is the bounded function ok this is less than M.

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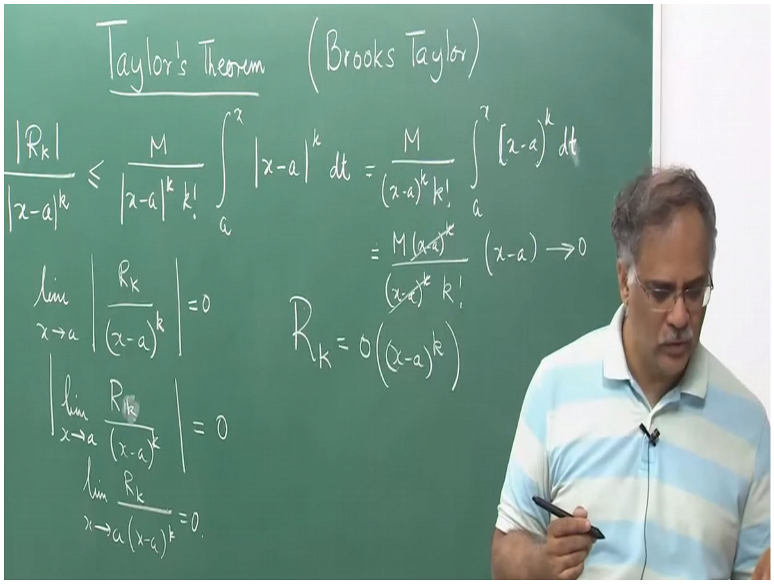

Now you divide by divide this is what you do is you take Rk mod x - a k and this is less than M / mod x - a k k dt. If you put that is equal to z dz would be dt same amount of mod x when x - a is z or whatever mod x- when x - a as z. Now because sorry because x is bigger than a this is nothing, but this integral is nothing but M / x is bigger than a so I can just replace this sorry dt sorry. So now this is free of the whole thing free of so this is nothing but M into x - a to the power k / x - a to the power k by k factorial integral a to x dt so it will become x - a.

So this cancels as x tends to a this tends to 0 so what happens so limit of limit as x tends to a mod of Rk / x - a to the power k this goes to 0. And I know that I can because this is a continuous function assuming that I can I Rk this integral because of this being continuous as a function of x is continuous. So then I can actually take this inside so it is enough to even prove mod of Rk. If I put this to be 0 that is fine right then this because I can limit and the modulus can be swapped because the mod is a continuous function and this is this is going to**.**

So mod of something is 0 see limit on the mod can be swapped the Rk so mod is the continuous function so I am using mod as continuous function Rk itself can be thought of as of function of t. So here we are not bothered here we are not bothered with the function of t we are bothered with the function of x. Actually R as a function of t can be under some condition face simple condition to be continuous but that is beside the point.

So here I am now looking at a function of x so mod itself is a continuous function. So the limit and the function can be interchange that is the meaning of continuity. That whenever x k goes to x and f is continuous then limit of f x k goes to f x. That is you can interchange of function and the limit. So which means if modulus of some quantity is 0 the quantity itself is 0 and that is what it is so this is the key idea that the error term.

So this Rk in the (()) (24:36) of mathematics RK is written as o of small o of x - a to the power k. That is if I divide this by x - a to the power k and as x goes to a I will get the resulting limit to be 0. We simply says that the error can the error as x goes to a the error vanishes faster the Rk the error vanishes faster than x - a. That is RK goes to 0 faster than x - a goes to 0. So Rk is an error which will keep on vanishing very fast when x is going towards a so that is that is the key idea.

So the same idea comes in of function of 2 variables I will not here give a proof of the function of 2 variable what does it means. We will do only Taylor's theorem up to second order because we have not used higher order derivatives other than second order because using higher order derivatives who will put us into much more technical domain which we were not going to get into viewing the audience that I am talking too I cannot get into all kinds of stuff.

But I do only function of 2 variables and up to second order. We will not give a proof here we will give some examples of what to do. Instead of giving the proof I will write a very interesting proof found in a book by T W Korner. I think this book I wanted to mention sometimes I will bring in the class and show it. I have a personal copy and I think this book is something you should all those who it is not very costly now.

Now I hope it is not costly and I Cambridge university press if they have some sense, I am sure they can get it done for the Indian audience it will be really helpful. It is called calculus for the ambitious. I like the title also. What is the spelling he is correct ambitious. So Cambridge university press CUP so this book has it is has a lovely way of looking at calculus lovely. And it is so exciting even for us for me it is so exciting, and this book has a beautiful proof for Taylor's theorem in 2 variables.

I will work out the proof in much more detail and now I will ask my TA to put it up on the portal. So let us look at function of 2 variables. So let me first make a first order expansion and then make a second order expansion.

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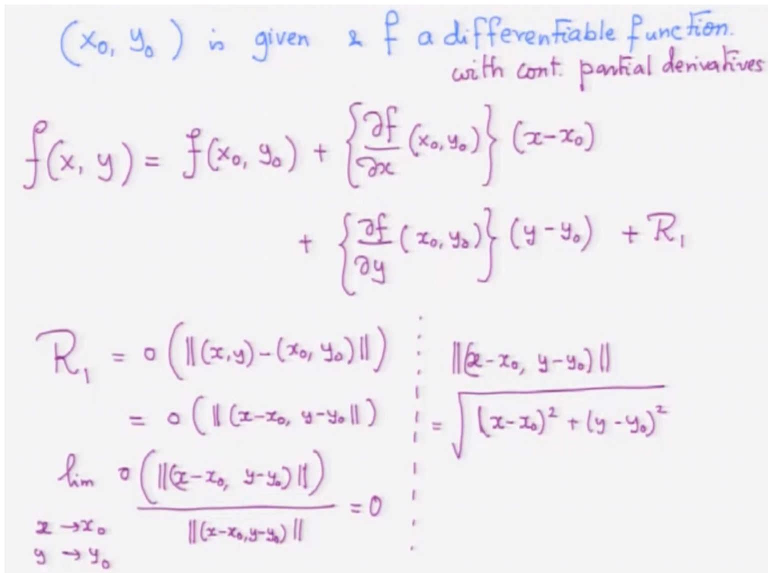

So here I have x naught y naught a point given to me and f a differentiable function. Right now this role of a that we wrote there is now played by x naught y naught and I want to figure out the value at any f at x naught x y and that is equal to f of x naught y naught + del f del x evaluated at x naught y naught multiply this thing multiplied with x - x naught + del f del y this is actually outcome of chain rule.

Del f del y x naught y naught y - y naught plus error R1. I am not writing the error in detail but R1 the error has to be a o of norm of here basically we need the length. Norm of the vector x y - x naught y naught. So this is your x this is the x, x y is the x in the one variable and x naught y naught is playing the role of a. So this is simply means o of norm of x - x naught y - y naught. So this means limit o of norm x small o it is always called small o.

Hey these vectors x - x naught and y - y naught divided by the length this x - x naught length of this norm of this vector. When x goes to x naught and y goes to y naught this is 0. And of course, you know what is the length this is a Euclidean using the Pythagoras theorem you know very well what is x - x naught what is the norm of this. It is nothing but x- x naught square + y - y naught square and square root of this. So that is the first order expansion.

First so we are going to be more concern about the second order expansion, so this is the same thing. So we will assume that the second order derivative is a mixed partial derivatives everything exist and it is continuous. Here we have assumed that del f del x del y are continuous. So one thing we now had is a differential function with continuous partial derivatives first partial derivatives actually.

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$$f(x,y) = f(x_0,y_0) + \left\{\frac{\partial f}{\partial x}(x_0,y_0)\right\}(x-x_0) + \left\{\frac{\partial f}{\partial y}(x_0,y_0)\right\}(y-y_0)$$
$$+ \frac{1}{2!}\left\{\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\right\}(x-x_0)^2$$
$$+ \left\{\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\right\}(x-x_0)(y-y_0)$$
$$+ \frac{1}{2!}\left\{\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\right\}(y-y_0)^2$$
$$+ R_2$$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{R_2}{\|(x-x_0, y-y_0)\|^2} = 0 \qquad R_2 = o\left(\|(x-x_0, y-y_0)\|^2\right)$$

Second order expansion of the same thing here I will write little smaller because we have to accommodate many more things here. So second order derivatives root comes second order expansion means second order derivatives must come. You want to ask me about the third order of derivatives but that will complicate things much farther. So we will not get into the third order issues.

So del f del f x evaluated at x naught y naught x - x naught + del f del y evaluated at x naught this is the first part done y naught + 1/2 factorial. Del 2 f del x 2 evaluated at the same point x naught y naught. Now I am assuming that all of these partial derivatives first partial, second partial means for all are continuous. So the mix partial del f del x y and del2 f del2 f del x del y and del 2 f del y del x are equal by Euler's result that we have learned which is also called Schwartz theorem or (()) (33:34) theorem when if you look at higher dimensions.

So let us have a more detail look so this into x- sorry here I have missed y - y naught x - x naught whole square + del2 f del x del y evaluated at x naught y naught multiplied with x - x naught y - y naught + half factorial del 2 f del y2 x naught y naught y - y naught whole square + R2 the second order error term. Where R2 has a same behavior R2 divided by the norm of x- sorry / the norm of x - x naught y - y naught whole square.

What we can show is that here are things slightly changes. This whole square this limit has x goes to x naught and y goes to y naught that is x y goes to x naught y naught this goes to 0. R2 vanishes faster than even the square. I know that a square root comes towards 0 faster than x because between 1 and 0 x square is less than x. So it will come faster towards 0 but here that is exactly one wants to make what did a approximation your error vanishes faster.

So that is exactly what is what they are trying to say that when that is when x is becoming coming x and y are coming more near and near close more near to x 0 y now y 0 the error value is when going down faster and faster that is the meaning. So R2 is called the o of norm x - x naught y - y naught whole square. So let us now take an example and let us try to write down the first order and the second order expansion. Let me take one from the exercise there were is a extra simple problem where are then taking it from the.

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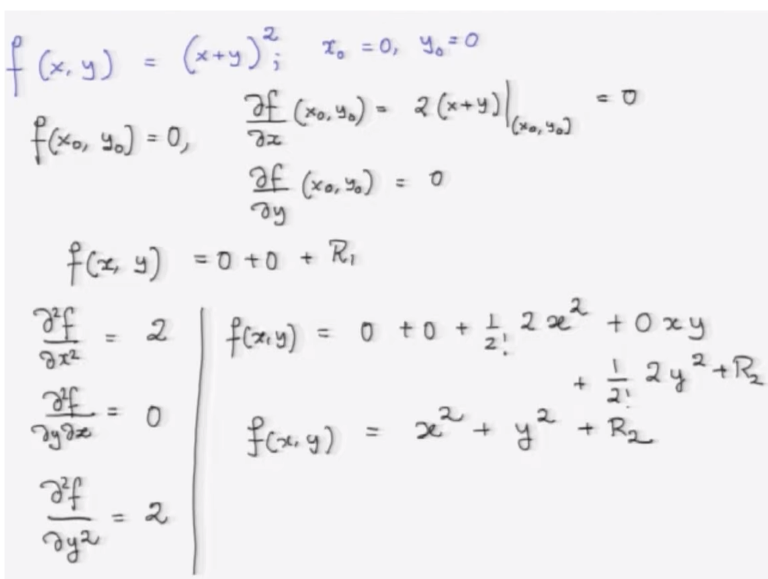

So let me have f of x y is x + y whole square by the way this Taylor's expansion of second order first thing was not really due to Taylor. The thing these things developed much later during by

Cauchy and other people. So it is not really Taylor's job. The name is given as Taylor's expansion because it is actually Taylor's expansion can be used to get this as I will show you in the proof which is which will come on the portal.

So I take 0 = 0 and y0 = 0 find the value of this. Let us first find it first order expansion. So to find the first order of expansion I have to first find f of x naught y naught f of y x naught y naught is 0 in this case. And then I have to evaluate del f del x at x naught y naught at 0 0 del f del x is 2 of del x + y at 0 0. This is 2 of x + y at x naught 0 y naught 0 so it is 2 of x + y evaluated at x naught y naught that is 0 similarly, del f del y is 0.

So my first order expansion f of x y in this is a very typical case is 0 + 0 + R1. So the value at any x y first order if you take a first order approximation is the error itself that is interesting. Now let us come to the second order expansion for that we have to do del2 f del x2 and what is that. Del 2 f del x2 in this case is 2 is y there is no y del2 f del y del x then take y of that is 0. So now I take del y of this 2x + 2y is 2 and del 2 f del y2 is also 2.

So now let us go to the second order expansion. So f of x y so first order part f is 0, 0, 0 this part is 0 + 1 factorial by 2, 2 into x is square +0 into x into y + 1/2 factorial 2 into y square + R2. So f x y is x as square + y square + R2. So this is the second order approximation right here it is x - x0 when I see x0 is 0, y0 is 0 it is like that. Second derivative is 2 because just check through. So for this 2x + 2y so you would take a derivative with respect to x you will get 2 here and this 2y will become will be 0 with respect to x will be 2.

Del f del 2 f del y2 would be 2 while del2 f del y del x would be 0. So you see this is the way we keep on working and this is the way we should you should express this is enough for you to now get an idea of what just you have an idea how we go up to say an order Taylor the approximation. But proof will come in the portal just have a look at that that is a elegant one. Thank you very much it is it is very interesting to speak to you I am sure lot of young people would be listening to this I have not done much I do not want to put you into too much of you know issues at this moment.

On next thing is are much more exciting thing my own area of research it is optimization it is maximization and minimization of function of 2 variables. And there are certain issues which as an optimizer I would like to tell you, but I would not tell you but rather go by the book. This is the kind of issue which many people have no idea and who are not into the subject of optimization what are the real issues involved in a fun problem of minimizing and maximizing.

But we will go by the book, but I will try to show you something interesting. We just do some interesting problems with you right. So that will be our so in tomorrow's lecture there are a lot of thing would be done and so that is that is what it is. Thank you.