Calculus of Several Real Variables
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Module No # 03
Lecture No #13
Higher order partial derivatives

So welcome to the third lecture of the third week so now we are progressing in a sense that just like we have spoken about second derivatives in calculus of one variable. Third derivative nth derivative we are now going to speak about higher derivatives in this setting that is higher partial higher order partial derivatives.

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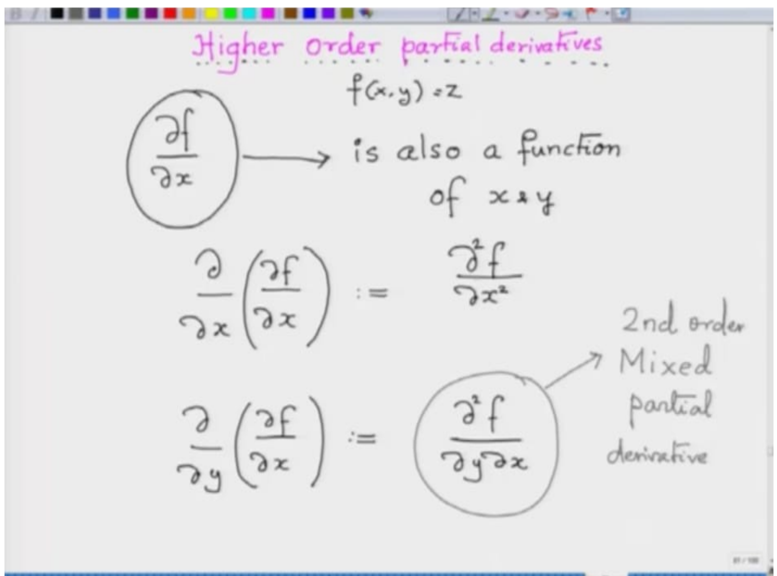

So for example suppose you have del f of del x suppose you consider a function f to be a function of 2 variables for the time being f is xy and you consider del f del x. This del f del x is also a function of x, y x and y and this is what you know you do not have to be told this because we have seen in our example that is what happens. Now suppose I say that ok now this is the function of x, y which I want to differentiate again and find the partial derivatives and yes ok.

I can find the partial derivative with respect to x as it with respect to y we can do this it is absolutely a legitimate operation. Then the resulting partial derivatives are partial derivatives of higher order or second order partial derivative that is del-del x of del f del x is denoted as del 2 f of del x 2 or del 2 del x 2 of f. whichever you want to speaking about this is matter of choice similarly here you have del 2 f del y del x.

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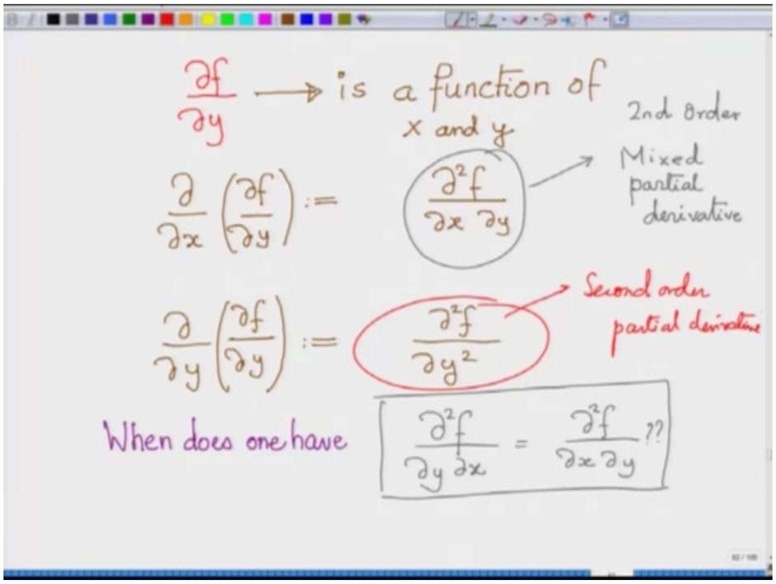

You know need not stop at this point cause you can always also talk about del f del y when you talk about del f del y itself is a function of x and y. So you can write del del x of del f del y and del del y of del f del y and it is and both of these operations are absolutely legitimate. So they are just finding partial derivatives of a function of 2 variables which happens to be itself a partial derivative of another function.

So this is written as del 2 f del x del y and del 2 f del y 2 so these derivatives delta 2 f del x del y and del 2 f del y del x these are called mixed partial derivatives. So first with respect to x and then with respect to y mixed partial second order mixed partial derivative. If I want to be very erudite I should say second order mixed partial derivative.

Similarly here this one is called again second order mixed partial derivative an interesting question an intriguing one is when is when does one have del 2 f del y del x = del 2 f del x del y. Let us see through some examples whether this happens or not or it is always different. So this is the question when does this happen when can we say that these two are equal because that has its own advantages. I will choose example from the exercise in this book and then let me try to figure that exercise out.

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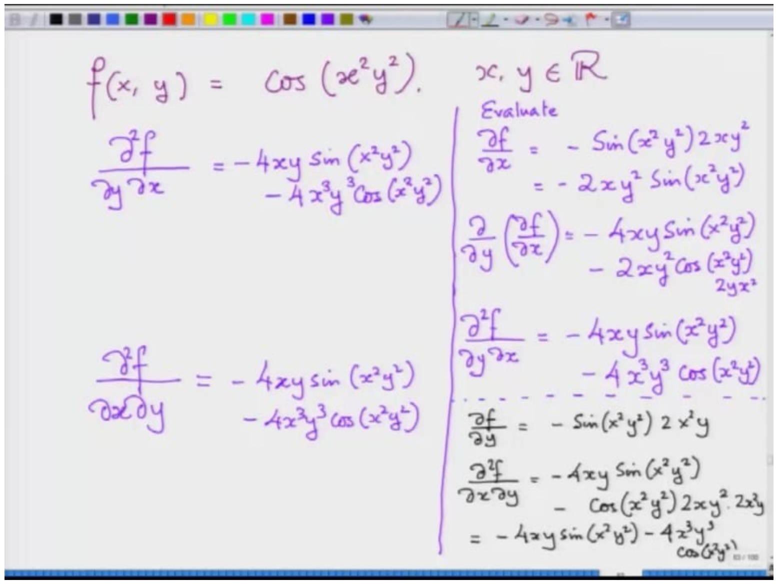

So let this exercise the function is f of xy is cosine of x square x y square. I intriguing looking function now so fx so it is function of 2 variables when x and y are of course in real line their elements are. So I want to first find del 2 f del y del x the first case and I also want to find del 2 f del x del y second case. In this case first we evaluate so let us do the side rough work so you have to first evaluate del f and del x and see it is a function of x and y.

So what is delta f del x? It is minus sin of x square y square into 2x y square so I can write this as -2x y square sin of x square y square because the first this is the chain rule it is an application of the chain rule just for the variable x. Now once I do it I now you see it is a function of x and y now I want to differentiate it with respect to the variable y. So I want del del y of del f del x but if I want to do this evaluation I have to observe here I do have a product form of two wise function wise.

So first I say I will take this one so it is -4xy into sin x square y square + of minus so it is minus again 2x y square into cosine of x square y square into 2y x square. So finally what you get is del 2 f del y del x = -4xy sin x square y square ok again if I multiply this I will have x square here coming out. So it is -4y into so x cube y cube cosine x square y square so that is what you get as an expression. So let me write it down as -4xy sin of x square y square -4 x cube y cube into cosine of x square y square.

Let us see whether this matches or not if it does not match maybe my calculation is incorrect or it does not match it may not always match. So let me do the other part here so what is del f del y here del f del y is now again do the calculation faster the minus sin x square y square would continue to remain with the fact that I will now do the same thing but I will have 2x square y right. Because now I have taken derivative with respect to y so now if I want to del 2 f del x del y now I am taking with respect to x so when I am taking with respect to x first I am taking with respect to this.

So I will have -4xy sin x square y square then I do the opposite then I have there is a symmetry if you observe then I take it with respect to the sin would become Cos and negative sin would remain. So minus cosine of x square y square into now I am taking the derivative with respect to x so it is 2x y square into 2x square y so what I finally getting is -4xy sin x square y square minus if I multiply this I have 4x cube y cube into cosine x square y square and that is it. So you see that finally if I did the calculation if I write down what is it what was it then I have this to be also -4xy sin of x square y square -4x cube y cube cosine of x square y square.

You see so this actually works that ok we have a case where the equality holds the question will be when will such inequality always hold that idea was first given by Euler when he was trying to look for some results in fluid mechanics the continuity of partial derivatives right.

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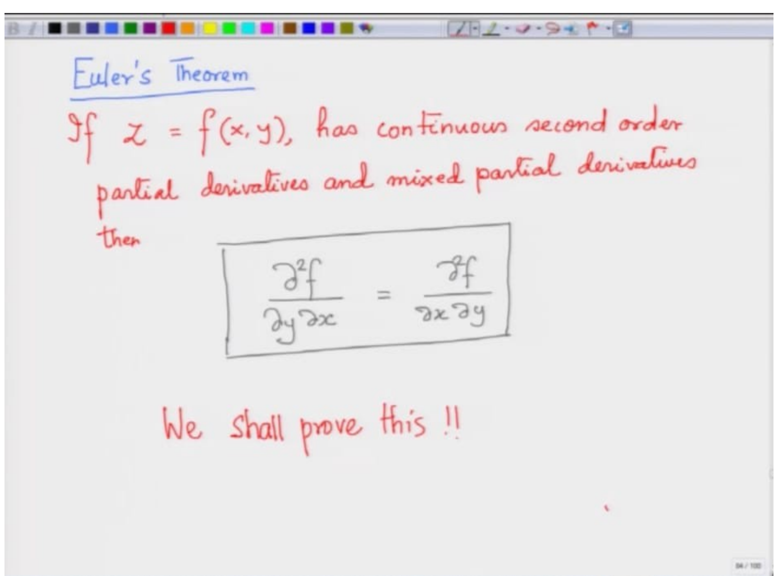

Let us write down Euler's theorem for high dimension this is the result that we give here for high dimension it was given by Schwarz and Young it is sometimes called Schwarz's theorem and sometimes Young's theorem. I have always remembered it has Young's theorem so let me just for a function of 2 variables it just it is Euler's theorem. So what is Euler's theorem? If Z= f xy has continuous second order partial derivatives and mixed partial derivatives.

So the partial derivatives is del 2 f del x 2 they are just called partial derivatives of second order while this is just called a second order partial derivative just to put in some terminology. You know second order partial derivatives why we are learning it because they will be helpful in detecting maxima and minima which will come in the next discussion so we are trying to get ready for that.

So it has continuous second order partial derivatives and mixed partial derivatives then these functions the derivatives themselves are functional control continuous functions of xy then del 2 f del y del x = del 2 f del x del y. Though I intend to but I would like to go to the proof of this step by step so what I will do is that I will first write down the proof on the board step by step and then try to explain to you what is happened.

So I will ask my cameraman to just hold on for a while I write the proof on the board and then explain to you step by step because the proof if I just keep on explaining to you may not find it very comfortable. So one needs to understand the proof of these to see how arguments are made in functions of higher from functions of more than one variable.

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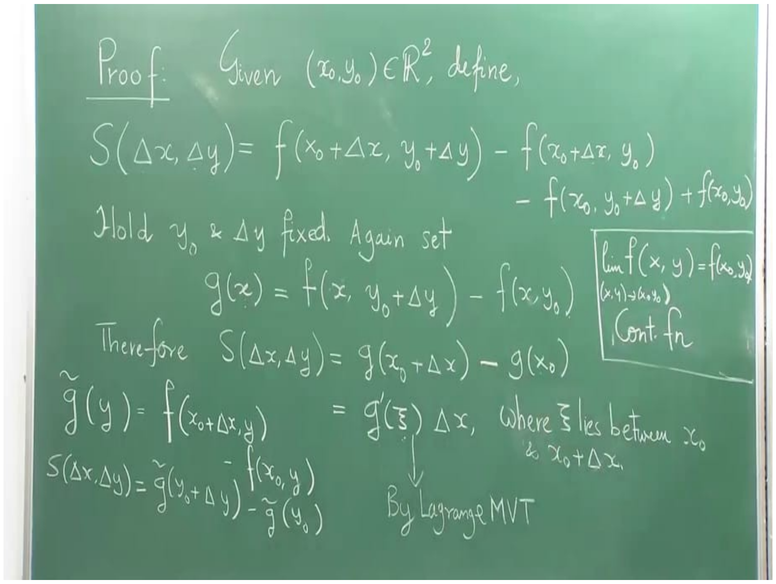

So (()) (16:34) so now you see look at the board I have already written down the proof beforehand the idea is to tell you that if I just try to write down the proof and try to explain to you might get confused. So we will start from the left hand of the board and then come down to this end hoping that you are able to see it. I would also personally go ahead and try to see that you can see this thing.

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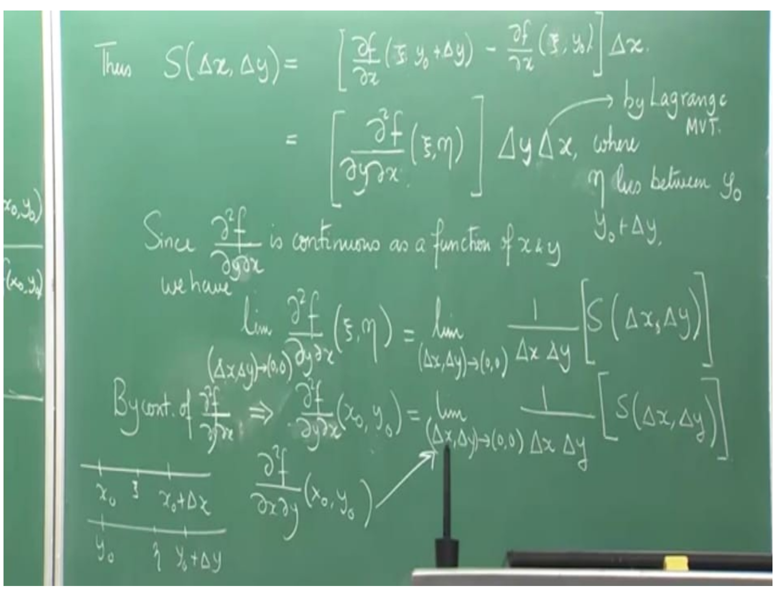

Now what we do is that we show that at any given point x naught y naught in R2 any point that I give you if these partial derivatives these mixed partial derivatives and second order partial derivatives. Hence all first order partial derivatives are continuous then at every point if I compute del 2 f del x del y and delt2 f del y del x these 2 values must coincide. They must be equal and that is exactly what I want to say.

So I start with the point which is any arbitrary point and show that for that this will be true and once I can show that this is true for a given point and for I can do the same logic for any point and hence it will be true for any point and that gives you the general result. So whenever I am writing this kind of general result del 2 f del x delta y del x del 2 f del x del y means they are evaluated at every point. Whatever point you evaluate these two will be equal that is the meaning and that is the meaning that I want to carry forward here.

So here our job would be today to do this proof and in need of course I can talk to you about partial differential equations etc., but since this course none of my calculus courses involve differential equations. Differential equation is a completely different ball game and you need to study them separately and we will not involve differential equations in this course. So what I will do is the following let me start to explain to you this proof but it will be very counter intuitive.

This proof is from the book which I am following the book by again I if those who are coming late into the course I can I show you this book very clearly it is basic multivariable calculus by Marsden Trumbo and Weinstein published by Springer a very good book for a first course. What I am trying to say is this proof was done by Euler to study some issues in fluid mechanics but this proof has a counter intuitive feel.

Given this problem to you it will be difficult for you to really figure out why I should start this proof in this way see the idea here behind the proof is that we will show that both sorry there is a kind of mistake here this will be del x del y sorry. I am just saying some corrections sorry there is a mistake writing mistake so if you look at the proof what I am going to show that del 2 f del y del x at x naught y naught is equal to some quantity and del 2 f del x del y at the same point is equal to that same quantity and hence they must be equal.

The proof actually realize completely on the Lagrange mean value theorem, the Lagrange mean value theorem which you know for one variable calculus which you have been taught in one variable calculus and you can go back and see my lectures on Lagrange mean value theorem in one variable calculus and that itself gives you a quite detailed understanding of Lagrange mean value theorem.

So you can actually apply it here so what I do is first start defining a quantity called S of del x delta y so I fix this x naught y naught for the time being and I define this quantity in this way. So what I do with x naught I add this del x that is an addition I have changed y naught I added this then I do this, then I do this and then I do this. So del x and del y is my variables here I can keep on changing them so I keep on changing them this because x naught and y naught is fixed.

So the only thing that this function can only depend on this del x and del y and that is how I have defined if I can hold y naught and delta y fixed for the moment. Then if I look at this quantity if I define this quantity g x is f of x y naught + del y f of x y naught then this quantity because I have fixed y naught and delta y just becomes a function of y sorry a function of x. Because y naught and y are fixed this quantity becomes a function of x.

Now if I write instead of x I can put any value of x if I put x naught + delta x I will get this part if I put just x naught here g of x naught here I will get this part this minus this I will put it in the bracket. And so S of delta x delta y can be represented as a difference of 2 functions difference of values of difference of the 2 difference of values of function g at 2 different points. So when you can write the whole thing the function of 2 different when a difference of one variable function at 2 different points then I can apply the Lagrange mean value theorem.

Because here because this partial derivatives exists and its continuous I can take the derivatives of this which is nothing but the partial derivative of this minus the partial derivative of this with respect to x. Because y naught and delta y are fixed you can put any number there so they will be fixed so do not change them. So if you do so then here I can apply the Lagrange mean value theorem so which will say that the difference between this and this nothing but delta x.

So g dash of sum xi x delta x where xi is number lying between x naught and x naught + del x this is exactly what Lagrange mean value theorem say so here we have applied Lagrange MVT. Please take a careful note of this that we have here applied the Lagrange mean value theorem. Now what is g dash xi which is nothing g dash x is del f del x at that x – del f del this the partial derivative of this function with respect to x.

So it is del f del x so g dash xi is del f del x evaluated at x xi here del f del x evaluated at xi here and that is what you get so whatever y naught and delta y you figure out right you can always write it like this. Now what is happening now you have xi and you have now this y naught and delta y are fixed but now we have also fix xi. Now I can look at this function del f del x as a function of 2 variables.

But once I fix xi I can think of these part as a function of the variable y evaluated at the point y naught + delta y and y naught give I actually kept them fixed but these are 2 points where I have evaluated these 2 functions but xi is now fixed. Xi is already known so it is fixed I fix the x spot here this is also fixed but I am not bothered now about the x spot I am looking at it just as a function of y and I have evaluated it at 2 points.

Then I can apply on this the mean value theorem again so my derivative of this would be in terms of the del y so delta or del-del y of del f del x. So del 2 f del y del x evaluated at some point eta which lies between y naught and y naught + delta y into delta y. so this is here I applied the Lagrange mean value theorem here and into delta x which is already there. Delta y del 2 del y del x is continuous as the function of xy.

So you have if you have a continuous function I am just writing it so if x fxy is continuous then f xy limit xy tending to x naught y naught. This is this simply means f of x naught y naught that is the meaning of continuity we already studied continuity of these functions of 2 variables and that is exactly the meaning of continuity that whenever xy is going towards x naught y naught it should give me f of x naught y naught.

Nothing matter whatever be the path now here what is happening is that this is continue which means what is happening xy is going through x naught y naught which means when I am taking the limit I can actually push the limit inside. So I can write this whole thing as f of limit of xy = f of x naught y naught which is natural because limit of xy is what when limit of xy when xy is going towards x naught y naught is x naught y naught and that will give you x naught y naught.

So when you to have a continuity operation the limit of this was will take you to f of x naught y naught that is the meaning of the continuous operation that the function and the limit can be exchanged ok. So this is what you know note from continuity continuous function so these are very important property which is these are very basic definition it comes from the basic definition and that is what it is we have studied this earlier.

Now I know that this is the continuous function because that is given to me has a statement of theorem right that has been given already. Now once that is given to me what I will do I have it like this and I will take this because this del x del y and non-zero quantities to begin with. So we will mention we have not mentioned it but tacitly we are taking that these are non-zero quantities.

If it is 0 quantities then it is nothing then it is just everything will cancel out it will become 0 right. So we will not bother about that we let taking about non-zero del x del y so I am dividing this side I am bringing del x del y to this side dividing both sides by del x del y I will get this is this. Now if I take the now what I will do I will take limit on both sides as del x del y tends to 0, 0.

Now if you observe if del x del y tends to sorry this is del 2 f del y del x as del y where does xi and eta go when del x del y is pushed to 0, 0. Be basically if you just take x naught to be positive for the time x naught or y naught both positive does not just to for an explanation you have y naught and y naught + delta y. You have xi you found somewhere here eta you found somewhere here.

Now you when you push make delta y 0 y naught and y naught + delta y both coincide because what we have brought xi and eta or x, x naught and x naught + delta x both coincide. Because xi is lying between them it will be sandwiched between them and did they will also approach x naught. Because when delta x goes to 0 if they are made smaller and smaller because this story this will be divided by 1/ del x del y.

Del 2 f del y delta x evaluated at a point xi eta where xi is a point lying between x naught and these and eta is a point lying between y naught and y naught + delta know this will be true for whatever del x del y non-zero del x del y you choose. Now if you make them go towards 0 make them smaller and smaller then xi and eta would be also because they sandwiched between these 2 quantities will be forced to go to y naught.

Because x naught + delta x goes to x naught as delta x goes to 0 and y naught + delta y also goes to y naught as well delta y goes to 0. So xi and eta means as they are sandwiched between these two they will be also pushed towards x naught and y naught. So as delta x delta y goes to 0, 0 xi eta going to x naught and y naught by the continuity of this is nothing of this one of the second partial derivative mixed partial derivative.

We have delta 2 f del y del x is x naught y naught why you needed continuity of all the kinds of partial derivatives because see because if the second partial derivatives or first partial or second partial derivative continuous of first partial derivatives are continuous. Because of that you can actually compute this thing and you going to write down g dash xi and because of the continuity of the second partial derivatives you can write down this right.

So without the continuity because once this is continuous the others are continuous without the continuity of this derivative you cannot put this x naught y naught in this limit you cannot say that this limit is this. Because this when it goes to 0, 0 xi eta is going to x naught y naught and that by continuity by this thing you will have this and this is the limit. So these been a finite quantity this also has to be a finite quantity because both the limits are equal.

Now what you do? So you get you prove that del 2 f del y del x evaluated at x naught y naught is something like this quantity. Now you change the role of x and y so in fact you define g tilde y as f of instead of x f of x naught + delta x y – f of x naught y. If we define it then what would happen then S of delta x delta y that would be written as if I put g tilde of y naught + delta y so g / y y naught + delta y.

So this minus this plus this minus this so –g tilde of y naught so what is g tilde y naught + delta y here you will have y naught + delta y here you will have y naught + delta y. So these will club these two together and will club these two together and you will have the same story been told and you will be able to show that del 2 f del x del y at x naught y naught it is the homework for you.

You can try this by the same logic is exactly equal to this and in that way these two are equal because these 2 finite quantities are equal to the same limit of the limit the same limit which has to be finite. So that is about it and we end our talk here hoping that you get an idea this is a very important concept that the equality of mix partial derivatives. This plays an important role when we will when we do optimization or minimization or maximization functions and then that is what we are going to start in start tomorrow thank you very much.